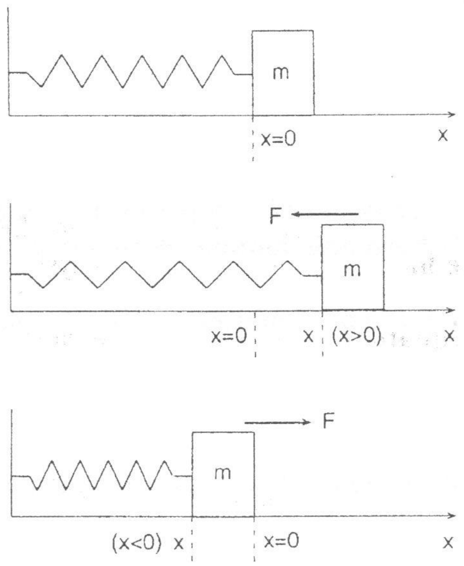
1. OSCILLATIONS

1.1 The Simple Harmonic Oscillator

When we speak of a **vibration** or an **oscillation**, we mean the motion of an object that repeats itself, back and forth, over the same path. Such a motion is **periodic.**



The simplest form of periodic motion can be represented by an object oscillating on the end of a spring. The mass of the spring is assumed to be ignored. We assume that the spring is mounted horizontally (as shown in Fig.), so that the object of mass m slides without friction on the horizontal surface. Any spring has a natural length at which it exerts no force on the mass m, and this is called the **equilibrium position**. If the mass is moved either to the left, which compresses the spring, or to the right, which stretches it, the spring exerts a force on the mass which acts in the direction of returning it to the equilibrium position; it is called a **restoring force**. The **magnitude** of the restoring force F is directly proportional to the distance x the spring has been stretched or compressed (see Fig.):

$$F = -kx\ .$$

This equation is accurate as long as the spring is not stretched or compressed beyond the elastic region. The minus sign indicates that the restoring force is always in the direction opposite to the displacement x. The proportionality constant k is called the **force constant**. The greater the value of k, the greater the force needed to stretch or to compress a spring a given distance. That is, the stiffer the spring, the greater the force constant k.

We shall define a few terms. The distance x of the mass from the equilibrium point at any moment is called the **displacement**. The maximum displacement - the greatest distance from the equilibrium - is called the **amplitude**. One **cycle** refers to the complete motion from some initial point back to that same point (say from x=A to x= -A back to x=A). The **period**, T, is defined as the time required for one complete cycle. The **frequency**, f, is the number of complete cycles per second. Frequency is usually specified in **hertz** (Hz) where 1 Hz = 1 cycle per second. It is evident that

$$f = \frac{1}{T} \quad and \quad T = \frac{1}{f}\ .$$

Problem 1-1. A spring stretches 0.150 m when 0.300-kg mass is hung from it. The spring is then stretched an additional 0.100 m from this equilibrium point and released. Determine: a) the value of the force constant

b) the amplitude of its oscillation

Solution: a) Since the spring stretches 0.150 m when 0.300 kg is hung from it, we may find the force constant k from the equation for a restoring force:

$$k = \frac{F}{x} = \frac{mg}{x} = 19{,}6\ N/m.$$

b) Since the spring is stretched 0.100 m from equilibrium and is given no initial speed, the amplitude of its oscillation has to be A = 0.100 m.

1.2 Simple Harmonic Motion

Any vibrating system for which the restoring force is directly proportional to the negative of the displacement is said to exhibit **simple harmonic motion**. Such a system is often called a **simple harmonic oscillator**.

From Newton's second law we may state the **equation of motion** for the simple harmonic oscillator:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0,$$

where *m* is the mass which is oscillating. The general solution of the equation of motion equals

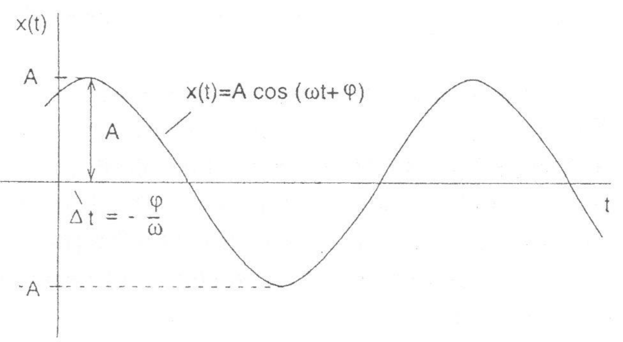
$$x = a\ \cos \omega t + b\ \sin \omega t$$

where *a* and *b* are arbitrary constants determined by the initial conditions. The constant ω is called the **angular frequency** and it equals

$$\omega = \sqrt{\frac{k}{m}}.$$

This general solution can be written in equivalent and more convenient form:

$$x = A\cos(\omega t + \varphi).$$



The physical interpretation of this solution is simpler. As shown in Fig., A is simply the amplitude (which occurs when the cosine has its maximum value of 1); and φ, called the **phase angle**, says how long after or before t=0 the peak at x=A is reached. For φ=0, we have x = A cos ωt and for φ= -π/2 we have x = A sin ωt.

Since the oscillating mass repeats its motion after a time equal to its period T, it must be at the same position and moving in the same direction at t=T as it was at t=0. And since a cosine function repeats itself after every 2π radians, then we have

$$\omega T = 2\pi.$$

Hence

$$\omega = \frac{2\pi}{T} = 2\pi f,$$

where f is the frequency of the motion. Thus the solution can be written as

$$x = A\cos\left(\frac{2\pi}{T}t + \varphi\right)$$

or

$$x = A\cos(2\pi f t + \varphi),$$

where

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}},$$

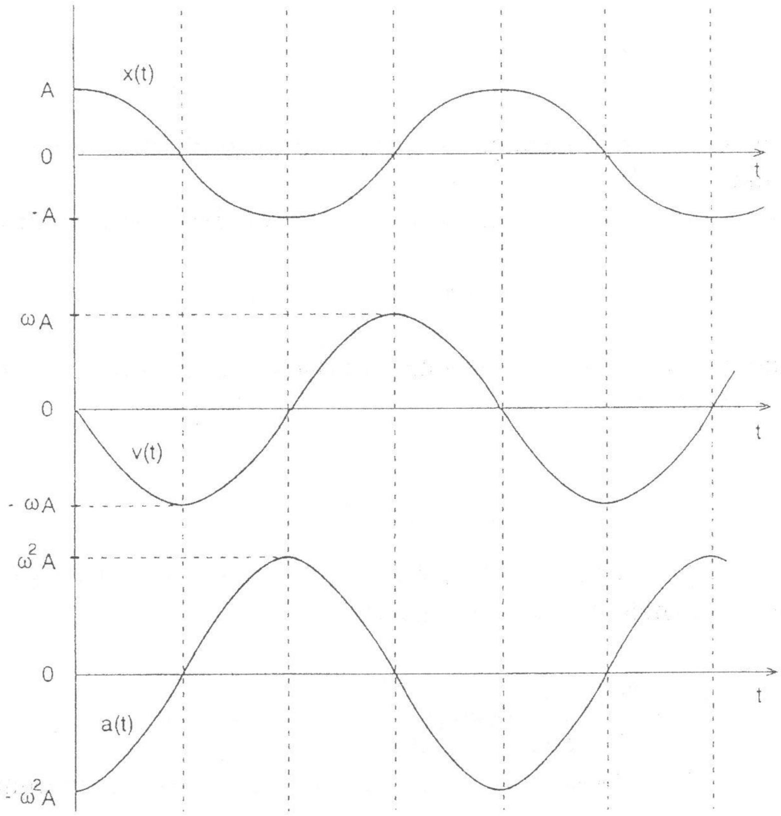
$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Note that the **frequency and period do not depend on the amplitude**.

The velocity and acceleration of the oscillating mass can be obtained by differentiation of the function x(t):

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \varphi)$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \varphi).$$



The displacement, velocity and acceleration as a function of time for the case when φ=0 are plotted in Fig.
The speed reaches its maximum

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A$$

when the oscillating object is passing through its equilibrium point, x=0; and it is zero at points of maximum displacement, $x = \pm A$.
The acceleration has its maximum

$$a_{max} = \omega^2 A = \frac{k}{m}A,$$

where $x = \pm A$, and is zero at x=0.

Problem 1-2. A vertical spring stretches $y_0 = 0.3m$ when m=0.6 kg mass is hung from it. The spring is then stretched an additional $y_1 = 0.2m$ from its equilibrium point and released with no initial speed.

Determine: a) the value of the force constant k

b) the angular frequency of the oscillation

c) the maximum velocity

d) the maximum acceleration

e) the period and frequency

Solution: a) $k = \frac{F}{y} = \frac{mg}{y_0} = 19.6\,N/m$

b) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{y_0}} = 5.72\,s^{-1}$

c) $v_{max} = \omega A = \sqrt{\frac{k}{m}}A = \sqrt{\frac{g}{y_0}}y_1 = 1.14\,m/s$

d) $a_{max} = \omega^2 A = \frac{k}{m}A = \frac{g}{y_0}y_1 = 6.53\,m/s^2$

e) $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}} = 1.1s,\quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{y_0}} = 0.91\,Hz$

Problem 1-3. For the simple harmonic motion given in the previous problem determine:

a) the displacement y as a function of time

b) the velocity of the motion as a function of time and then calculate the velocity at time t = 0.3 s

c) the acceleration of the motion as a function of time

Solution: a) The motion begins at a point of maximum displacement downward. If we take y positive upward, then x= -A at t=0 and so $\varphi=\pi$.

Hence

$$y(t) = -A\cos\omega t = -y_1\cos\sqrt{\frac{g}{y_0}}\,t.$$

Putting in numbers yields y(t) = - 0.2 cos 5.72 t ; this describes the motion, where t is in seconds, y is in meters and the angle 5.72 t is in radians.

b) The velocity at any time is

$$v(t) = \frac{dy}{dt} = \sqrt{\frac{g}{y_0}}\,y_1\sin\sqrt{\frac{g}{y_1}}\,t.$$

Putting in numbers yields v(t) = 1.14 sin 5.72 t . The velocity at t=0.3s is now equal v = 1.13 m/s.

c) The acceleration at any time is

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{g}{y_0}y_1\cos\sqrt{\frac{g}{y_0}}\,t.$$

Putting in numbers yields a(t) = 6.53 cos 5.7 t .

Problem 1-4. a) A mass starts the simple harmonic motion at its maximum displacement x=A and it is released without a push (v=0 at t=0).

b) In the second case, the mass was at t=0 at x=0 and it was struck, giving it an initial velocity v_0 towards increasing values of x.

For both cases determine the constants a and b in the general solution

$$x = a\cos\omega t + b\sin\omega t\ .$$

Solution: a) Applying the initial conditions, x=A and v=0 at t=0 we can write

$$x = a\cos 0 + b\sin 0 = A$$

and $$v = \frac{dx}{dt} = -a\omega\sin 0 + b\omega\cos 0 = b\omega\ .$$

Thus a=A and bω=0, so b=0, and the motion is a pure cosine curve $x = A\cos\omega t$.

b) Applying new initial conditions x=0 and $v = v_0$ at t=0 we write

$$x = a\cos 0 + b\sin 0 = 0$$

$$v = -a\omega\sin 0 + b\omega\cos 0 = v_0\ .$$

From the first equality it is evident that a=0 and the second equality yields $b = \frac{v_0}{\omega}$. This motion is a pure sine curve $x = \frac{v_0}{\omega}\sin\omega t$ with the amplitude equal $\frac{v_0}{\omega}$.

1.3 Energy in the Simple Harmonic Oscillator

For a simple harmonic oscillator the restoring force is given by F = - k x . Thus the **potential energy** function is given by

$$U = -\int F\,dx = \frac{1}{2}kx^2,$$

where the constant of integration is set equal to zero so U=0 at x=0.

Then the **total energy** equals

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2,$$

where v is the velocity of the mass m when it is a distance x from the equilibrium position. If there is no friction the **total energy must remains constant**. Since at the extreme points, x = A and x = -A , the velocity v=0, so all the energy is potential energy and we have

$$E = \frac{1}{2}kA^2.$$

Thus, the **total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion**.

At the equilibrium point, x=0, all the energy is kinetic

$$E = \frac{1}{2}mv_{max}^2,$$

where v_{max} represents the maximum velocity during the motion. At intermediate points the energy E is part kinetic and part potential. Since the total energy E is conserved we have

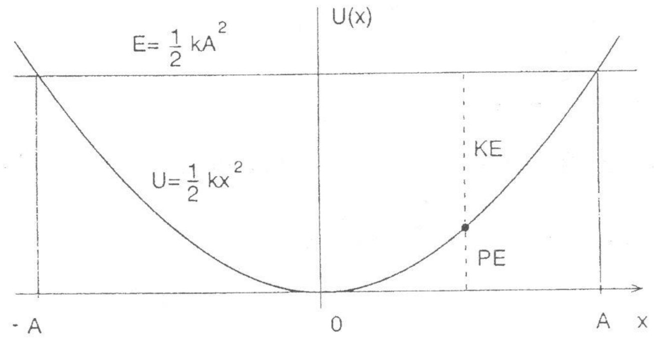
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv^2_{max}.$$

From this equation it can be obtained a useful equation for the velocity v as a function of x:

$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$

or, since $v_{max} = A\sqrt{\frac{k}{m}}$, $\qquad v = v_{max}\sqrt{1 - \frac{x^2}{A^2}}$

Again it is seen that v is a maximum at x=0, and is zero at $x = \pm A$.



The potential energy U is plotted in Fig. The horizontal line represents the total energy E. The distance between this line and the U curve represents the kinetic energy, and the motion is restricted to x values between -A and +A.

Problem 1-5. A spring stretches $y_0 = 0.15$ m when m=0.3 kg mass is hung from it. The spring is then stretched an additional $y_1 = 0.1$ m from its equilibrium point and released.

Determine: a) the total energy

b) the kinetic and potential energies as a function of time

c) the velocity when the mass is y = 0.05 m from equilibrium

d) the kinetic and potential energies at half amplitude

<u>Solution</u>: a) First we need to know the force constant k and the amplitude of the oscillation:

$$k = \frac{F}{y_0} = \frac{mg}{y_0} = 19.6\ N/m.$$

Since the spring is stretched 0.1 m from equilibrium and is given no initial speed, the amplitude of the oscillation equals $A = y_1 = 0.1\,m$.

Thus the total energy E is

$$E = \frac{1}{2}kA^2 = 9.8 \times 10^{-2}\ J.$$

b) To calculate the kinetic and potential energies as a function of time we need to know the displacement x as a function of time and the velocity as a function of time as well. The motion begins at a point of maximum displacement downward. If we take x positive upward, then x = - A at t=0 and so the phase angle equals $\varphi=\pi$.

Hence

$$x = -A\cos\omega t,$$

where the angular frequency ω is given by

$$\omega = \sqrt{\frac{k}{m}} = 8.08\, s^{-1}\,.$$

Putting in numbers yields the displacement x as a function of time

$$x = -\,0.1 \cos 8.08\, t\,,$$

where t is in seconds and x is in meters.

The velocity as a function of time is given by

$$v = \frac{dy}{dt} = 0.808 \sin 8.08t\,.$$

So, the kinetic energy as a function of time is

$$KE = \frac{1}{2}mv^2 = 9.8\times10^{-2}\sin^2 8.08\,t$$

and the potential energy as a function of time is

$$PE = \frac{1}{2}kx^2 = 9.8\times10^{-2}\cos^2 8.08\,t\,.$$

c) To calculate the velocity when the mass is $y=0.05$ m from equilibrium we first need to know its maximum velocity v_{max}:

$$v_{max} = \omega A = 0.808\, m/s\,,$$

and the velocity at the place of $y = 0.05$ m from equilibrium is given by

$$v = v_{max}\sqrt{1-\frac{y^2}{A^2}} = 0.7\, m/s\,.$$

d) At $y = A/2 = 0.05$ m the potential and kinetic energies have values:

$$PE = \frac{1}{2}kx^2 = 2.5\times10^{-2}\, J\,,$$

and the kinetic energy can be calculated as the total energy minus the potential energy:

$$KE = E - PE = 7.3\times10^{-2}\, J\,.$$

Problem 1-6. Solve generally the previous problem and express all the results in terms of the given quantities $m,\ y_0\ and\ y_1$.

Solution: a)

$$k = \frac{F}{y_0} = \frac{mg}{y_0}\,,\quad A = y_1\,,\quad \varphi = \pi\,,\quad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{y_0}}\,,$$

$$E = \frac{1}{2}kA^2 = \frac{1}{2}\frac{mg}{y_0}y_1^2$$

b) $y = -y_1\cos\omega t = -y_1\cos\sqrt{\frac{g}{y_0}}\,t$

$$v = -y_1\sqrt{\frac{g}{y_0}}\sin\sqrt{\frac{g}{y_0}}\,t$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\frac{mg}{y_0}y_1^2\left(\sin\sqrt{\frac{g}{y_0}}t\right)^2$$

$$PE = \frac{1}{2}ky^2 = \frac{1}{2}\frac{mg}{y_0}y_1^2\left(\cos\sqrt{\frac{g}{y_0}}t\right)^2$$

c) $v(y) = \sqrt{\frac{k}{m}\left(A^2 - y^2\right)} = \sqrt{\frac{g}{y_0}\left(y_1^2 - y^2\right)}$

d) at $y = y_1/2$

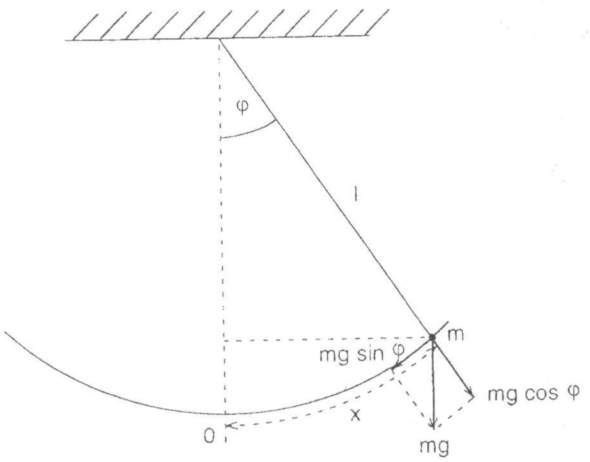
$$PE = \frac{1}{2}ky^2 = \frac{1}{8}\frac{mg}{y_0}y_1^2$$

$$KE = E - PE = \frac{3}{8}\frac{mg}{y_0}y_1^2$$

Problem 1-7. Discuss the forces acting on the simple pendulum and answer the following questions:

a) Is the simple pendulum really undergoing the simple harmonic motion?
b) Is the restoring force proportional to its displacement?
c) State the approximate formula for its period.

Solution: A simple pendulum is an idealised body consisting of a point mass, suspended by a light inextensible cord. When pulled of its equilibrium and released, the pendulum swings in a vertical plane under the influence of gravity. The motion is periodic and oscillatory.



The figure shows a pendulum of length *l*, particle mass *m*, making an angle φ with the vertical. The **forces acting** on *m* are *mg*, the gravitational force, and T, the tension in the cord. Let's resolve the force *mg* into a radial component of magnitude mg cosφ , and a tangential component of magnitude mg sinφ . The radial component supplies the necessary centripetal acceleration to keep the particle moving on a circular arc. The tangential component is the restoring force acting on *m* tending to return it to the equilibrium position. Hence, the restoring force is

$$F = - mg \sin\varphi .$$

Notice that the restoring force is not proportional to the angular displacement φ but to sinφ instead. The resulting motion cannot be, therefore, simple harmonic. However, if the angle φ is small, sinφ is very nearly equal to φ in radians. The displacement along the arc is x = l φ , and for small angles this is nearly straight-line motion.
Hence, assuming

$$\sin \varphi \cong \varphi ,$$

we obtain

$$F = -mg\varphi = -\frac{mg}{l}x ,$$

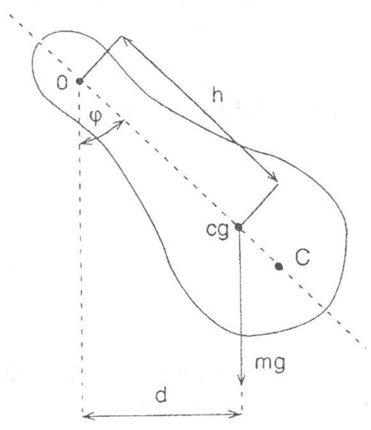
and for small displacements, therefore, the restoring force is proportional to the displacement and is oppositely directed. The constant mg/l represents the force constant k and thus the period of a simple pendulum when its amplitude is small is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} .$$

It is seen that the **period is independent of the mass** of the suspended particle.

Problem 1-8. Derive the formula for the period of the physical pendulum for small angular displacements.

Solution: Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a physical pendulum. In the figure a body of irregular shape is pivoted about a horizontal friction less axis through 0 and displaced from the equilibrium position by an angle φ. The distance from pivot to centre of gravity is h, the moment of inertia of the body about an axis through the pivot is I and the mass of the body is m. The restoring torque for an angular displacement φ is



$$\tau = -\,mgh \sin\varphi .$$

This is due to the tangential component of the force of gravity. Since τ is proportional to sinφ, rather than φ, the condition for simple harmonic motion does not hold here. For small angular displacements, however, the relation sinφ = ≅ is a good approximation, so that for small amplitudes,

$$\tau = -\,mgh\varphi .$$

Newton's second law for rotational motion states that

$$\tau = I\frac{d^2\varphi}{dt^2} ,$$

where I is the moment of inertia of the body calculated about an axis through point 0, and the second derivative represents the angular acceleration.
Thus the equation of motion for the physical pendulum has the form

$$\frac{d^2\varphi}{dt^2} + \frac{mgh}{I}\varphi = 0 .$$

It is clear that for small angular displacement a physical pendulum undergoes the simple harmonic motion and the term mgh/I replaces k/m.
Hence, the angular frequency of a physical pendulum oscillating with small amplitude is

$$\omega = \sqrt{\frac{mgh}{I}}$$

and the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgh}} .$$

(At larger amplitude the physical pendulum still has a harmonic motion, but not a simple harmonic one.)
From the last formula the physical pendulum can be used for determinations of g.

Problem 1-9. As a special case of the physical pendulum consider a point mass *m* suspended at the end of a weightless string of length l and calculate the period of its swings.

Solution: Putting $I = ml^2$, h =l in the expression for the physical pendulum yields

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{l}{g}} .$$

Problem 1-10. Find the length of a simple pendulum whose period is equal to that of a particular physical pendulum.

Solution: Equating the period of a simple pendulum to that of a physical pendulum, we obtain

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{I}{mgh}}$$

$$or \qquad l = \frac{I}{mg} .$$

Hence, as far as its period of oscillation is concerned, the mass of a physical pendulum may be considered to be concentrated at a point whose distance from the pivot is l = I/mh. This point is called the **centre of oscillation** of the physical pendulum. It depends on the location of the pivot. If C represents the centre o f oscillation when 0 is the pivot point, then when C is the pivot point, 0 is the centre of oscillation and the period is the same.

Problem 1-11. An easy way to measure the moment of inertia of an object about any axis is to measure the period of its oscillation about that axis. Suppose a nonuniform 1-kg stick be balanced at a point 42 cm from one end. If it is pivoted about that end it oscillates at a frequency of 2.5 Hz. Find its moment of inertia about this end and its moment of inertia about an axis perpendicular to the stick through its centre of mass.

Solution: From the formula for the period of the physical pendulum we have for I

$$I = \frac{T^2 mgh}{4\pi^2}$$

Putting T = 1/f = 0.4 s , h = 0.42 m , m = 1 kg yields $I = 0.27\,kg\,m^2$.

To answer the second question we use the parallel-axis theorem. The centre of mass is where the stick balanced, 42 cm from the end, so

$$I_{cm} = I - mh^2 = 0.09\,kg\,m^2 .$$

Notice that since an object does not oscillate about its centre of mass, the parallel-axis theorem provides a convenient method to determine I_{cm}.

Problem 1-12. A straight uniform rod of length l = 1 m and mass m = 160 kg hangs from a pivot at one end.

Determine: a) its period for small amplitude oscillations,

b) the length of a simple pendulum that will have the same period.

Solution: a) The moment of inertia of a this rod about an axis through one end is $I = \frac{1}{3}ml^2$. Since the centre of mass is at its centre, h = l/2 and then the period is

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{2l}{3g}} = 1.64\,s.$$

b) To have the same period, a simple pendulum must have a length L that can be calculated from the equality

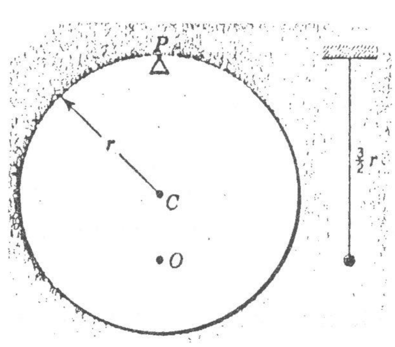
$$2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{mgh}}$$

$$or \quad L = \frac{I}{mh}$$

which, for our rod pivoted at one end , is

$$L = \frac{\frac{1}{3}ml^2}{m\frac{l}{2}} = \frac{2}{3}l = 0.67\,m.$$

Problem 1-13. A disk is pivoted at its rim (the point P in Fig.). Find its period for small oscillations and the length of the equivalent simple pendulum. Where is the centre of oscillation of the disk? What is the period if the disk is pivoted at a point midway between the rim and its centre?



Solution: The moment of inertia of a disk about an axis through its centre is $\frac{1}{2}mr^2$, where r is the radius and m is the mass of the disk.

Hence, the moment of inertia about the pivot at the rim is

$$I = \frac{1}{2} mr^2 + mr^2 = \frac{3}{2} mr^2 .$$

The period then is (h = r)

$$T = 2\pi \sqrt{\frac{I}{mgh}} = 2\pi \sqrt{\frac{3}{2}\frac{r}{g}} .$$

Notice that the period is independent of the mass of the disk.
Simple pendulum having the same period will have a length calculated from the equality

$$2\pi \sqrt{\frac{l}{g}} = 2\pi \sqrt{\frac{I}{mgh}}$$

or $$l = \frac{I}{mh} = \frac{\frac{3}{2} mr^2}{mr} = \frac{3}{2} r ,$$

or three-fourths the diameter of the disk. The centre of oscillation of the disk pivoted at P is, therefore, at 0, a distance $\frac{3}{2} r$ below the point of support.
If the disk will be pivoted at a point 0 midway between the rim and the centre its moment of inertia changes in value $I = \frac{3}{4} mr^2$ and h = r/2.
The period of its oscillations now is

$$T = 2\pi \sqrt{\frac{I}{mgh}} = 2\pi \sqrt{\frac{\frac{3}{4} mr^2}{mg\frac{r}{2}}} = 2\pi \sqrt{\frac{3}{2}\frac{r}{g}}$$

just as before. This illustrates again a general property of the centre of oscillation.

Problem 1-14. The period of a disk of radius 10.2 cm executing small oscillations about a pivot at its rim was measured to be 0.784 s. Find the value of g, the acceleration due to gravity at that location.

$\left[g = 9.82\ ms^{-2}\right]$

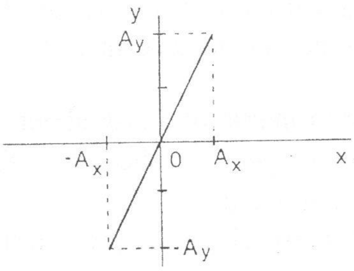
Problem 1-15. Examine the combination of two simple harmonic motions equal in frequencies along two perpendicular directions, say the x and y axes.
Deal with the cases: a) equal phases,

b) the phases differ by π/2 and the amplitudes are equal,

c) the phases differ by π/2 but the amplitudes are not equal.

Solution: a) The x-motion is described by

$$x = A_x \cos(\omega t + \varphi)$$

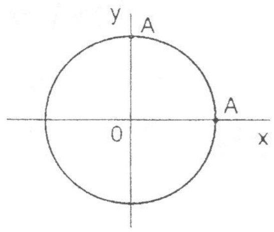


then the y-motion can be described by

$$y = A_y \cos(\omega t + \varphi) = \frac{A_y}{A_x} x ,$$

which is the equation of a straight line whose slope is (A_y / A_x).
Hence, the resultant motion will be a straight line in the xy plane, of slope (A_y / A_x). In Fig. we have the

case of $A_y / A_x = 2$. Both the x- and y- displacements reach a maximum at the same time and reach a minimum at the same time.



b) Here the phases differ by $\pi/2$ and the two motions can be described

$$x = A\cos(\omega t + \varphi)$$

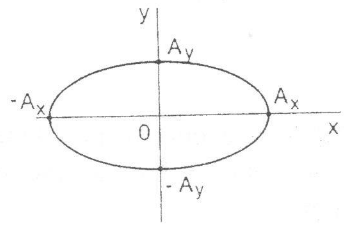
$$y = A\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = A\sin(\omega t + \varphi)$$

and we have

$$x^2 + y^2 = A^2\cos^2(\omega t + \varphi) + A^2\sin^2(\omega t + \varphi) = A^2,$$

which is the equation of a circle of radius A in the xy plane (shown in Fig.).

c) Here the phases again differ by $\pi/2$ but the amplitudes are not equal. The two motions are now described



$$x = A_x\cos(\omega t + \varphi)$$

$$y = A_y\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = A_y\sin(\omega t + \varphi)$$

$$or \quad \frac{x}{A_x} = \cos(\omega t + \varphi)$$

$$\frac{y}{A_y} = \sin(\omega t + \varphi)$$

and thus we obtain

$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1,$$

which is the equation of the ellipse with major and minor axes equal to $2A_x$ *and* $2A_y$ (shown in Fig.).

Problem 1-16. A piece of rubber is 145 cm long when a weight of 18 N hangs from it and is 168 cm long when a weight of 22.5 N hangs from it. What is the force constant of this piece of rubber?

[19.6 N/m]

Problem 1-17. What is the equation describing the motion of a spring that is stretched 20 cm from equilibrium and then released, and whose period is 0.75 s? What will be its displacement after 1.8 s?

$$\left[x = 20\cos\left(\frac{2\pi t}{0.75}\right) \quad , \quad -16cm\right]$$

Problem 1-18. A block of mass m is supported by two identical parallel vertical springs, each with force constant k. What will be the frequency of vibration?

$$\left[\frac{1}{2\pi}\sqrt{\frac{2k}{m}}\right]$$

Problem 1-19. The position of a SHM as a function of time is given by

$$x = 3.8\cos(7\pi t/4 + \pi/6)$$

where t is in second and x in meters.

Find (a) the period and frequency, (b) the position and velocity at t = 0, and (c) the velocity and acceleration at t = 2 s.

[7/8 s, 8/7 Hz, 3.3 m, -10.4 m/s, 18.1 m/s, -57.4 m/s^2]

Problem 1-20. A mass m is placed on the end of a freely hanging spring. The mass then falls 30 cm before it stops and begins to rise. What is the frequency of the motion?

[1.3 Hz]

Problem 1-21. At what displacement of a SHM is the energy half kinetic and half potential? What fraction of the total energy of a SHM is kinetic and what fraction potential when the displacement is half the amplitude?

$$\left[\pm A\sqrt{2};\ PE = \frac{1}{4};\ KE = \frac{3}{4}\right]$$

Problem 1-22. If one vibration has 10 times the energy of a second one of equal frequency, but the first's force constant k is twice as large as the second's, how do their amplitudes compare?

$$\left[A_1 = A_2\sqrt{5}\right]$$

Problem 1-23. At t = 0, a 650-g mass at rest on the end of a horizontal spring (k = 84 N/m) is struck by a hammer which gives it an initial speed of 1.26 m/s.

Determine (a) the period and frequency of the motion, (b) the amplitude, (c) the maximum acceleration, (d) the position as a function of time .

[0.55 s; 1.81 Hz; 0.111 m; 14.3 m/s^2; x = 0.111 cos(11.4 t)]

Problem 1-24. What is the period of a simple pendulum on Mars, where the acceleration of gravity is about 0.37 that on Earth, if the pendulum has a period of 0.8 s on Earth?

[0.49 s]

Problem 1-25. What is the period of a simple pendulum 60 cm long (a) on the earth, and (b) when it is in a freely falling elevator?

[1.6 s; infinite]

Problem 1-26. Derive a formula for the maximum speed of a simple pendulum bob in terms of g, the length L, and the angle of swing φ.

$$\left[v = \sqrt{2gL(1-\cos\varphi)}\right]$$

Problem 1-27. The pendulum of an accurate clock oscillates with an amplitude of $\pm 12^0$. If, due to a faulty mechanism, the amplitude is instead maintained at $\pm 1^0$, what will be the clock error per day? Does it gain or lose?

[clock will gain 3 min 54 s]

Problem 1-28. A 4-kg block extends a spring 16 cm from its unstretched position. The block is removed and a 0.5-kg body is hung from the same spring. If the spring is then stretched and released, what is its period of oscillation?

[0.28 s]

Problem 1-29. A particle executes linear harmonic motion about the point x = 0 . At t = 0 it has displacement x = 0.37 cm and zero velocity. The frequency of the motion is 0.25 Hz.

Determine (a) the period, (b) the angular frequency, (c) the amplitude, (d) the displacement at time t, (e) the velocity at time t, (f) the maximum speed, (g) the maximum acceleration, (h) the displacement at t = 3 s, and (i) the speed at t = 3 s.

Answer: [4 s; π/2 rad/s; 0.37 cm; 0.37 cos(πt/2) in centimeters; -0.58 sin(πt/2) in centimeters per second; 0.58 cm/s; 0.91 cm/s^2; zero; 0.58 cm/s]

Problem 1-30. A body oscillates with simple harmonic motion according to the equation

$$x = 6\cos(3\pi t + \pi/3)$$

where x is in meters, t is in seconds, and the numbers in the parentheses are in radians. Determine the displacement, the velocity, the acceleration, and the phase at the time t = 2 s. Find also the frequency and the period of the motion.

Answer: [3 m; -49 m/s; -270 m/s^2; 20 rad; 1.5 Hz; 0.67 s]

Problem 1-31. A simple pendulum of length 1 m makes 100 complete oscillations in 204 s at a certain location. What is the acceleration due to gravity at this point?

[9.49 m/s^2]

1.4 Damped Harmonic Motion

The amplitude of any real oscillation gradually decreases to zero as a result of friction. The motion is said to be damped and is called **damped harmonic motion**. The damping is generally due to the resistance of air and to internal friction within the oscillatory system. The energy that is thus dissipated to thermal energy is reflected in a decreased amplitude of oscillation. In most cases of interest the frictional force is proportional to the velocity of the motion but directed opposite to it.

The equation of motion of the damped harmonic oscillator is given by the second law of motion, F = ma , in which F is the sum of the restoring force -kx and the damping force -Bv, where B is a constant and *m* is the mass of the system.
Thus, we obtain

$$-kx - Bv = ma$$

or

$$m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = 0$$

or

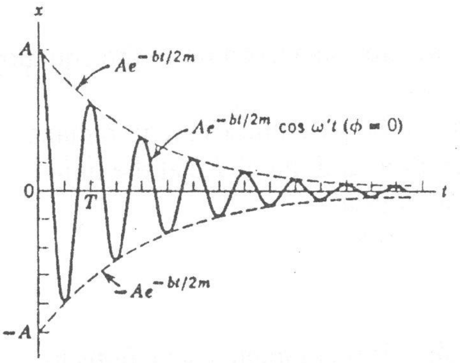
$$\frac{d^2x}{dt^2} + \frac{B}{m}\frac{dx}{dt} + \frac{k}{m}x = 0..$$

Putting $\frac{B}{m} = 2\delta$ *and* $\frac{k}{m} = \omega^2$ yields the **final form of the equation of motion for damped oscillations**

$$\frac{d^2x}{dt^2} + 2\delta\frac{dx}{dt} + \omega^2 x = 0$$

where the new constant δ is called the **damping constant** $([\delta] = s^{-1})$. The second constant ω represents the well-known angular frequency of non damped oscillations when damping force is equal zero.
If B and thus δ are not too large, the solution of this differential equation (given without proof) is

$$x = Ae^{-\delta t}\cos(\omega' t + \varphi),$$



where

$$\omega' = 2\pi f' = \sqrt{\omega^2 - \delta^2}.$$

The period T_1 of damped oscillations is

$$T_1 = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega^2 - \delta^2}}.$$

In Fig. the displacement x as a function of the time t for the damped harmonic motion with small damping is plotted. The motion is oscillatory with ever decreasing amplitude. The amplitude (---) is seen to start with value A and decay exponentially to zero as $t \to \infty$.
The solution can be interpreted as follows. First, the frequency ω' is less and the period is longer than for non damped motion. (In most practical cases of light damping, however, ω' differs only slightly from ω). If no friction were present, δ would equal zero

and ω' would equal the angular frequency ω of non damped motion. Second, the amplitude of the motion gradually decreases to zero. The constant δ is a measure of how quickly the oscillations decrease toward zero. The time $t_1 = 1/\delta$ is the time taken for the oscillations to drop to 1/e of the original amplitude; t_1 is called the mean lifetime of the oscillations. Note that the larger δ is, the more quickly the oscillations die away.

The ratio of two successive maxima of displacements on the same side from the equilibrium is

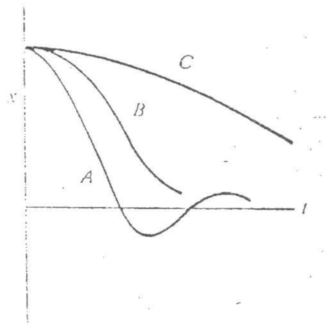
$$\frac{e^{-\delta t}}{e^{-\delta(t-T_1)}} = \frac{1}{e^{-\delta T_1}} = e^{\delta T_1} = b$$

where the constant b is called the **damping**. The natural logarithm of the damping b is then called the **logarithmic decrement of damping** Λ :

$$\Lambda = \ln b = \delta T_1 .$$

Note that Λ is dimensionless.

If the force of friction is great enough, δ becomes too large. The solution x is no longer a valid solution of the equation of motion if δ is so large that $\delta^2 > \omega^2$ since then ω' would become imaginary. In this case the system does not oscillate at all but returns directly to its equilibrium position when released from its initial displacement. Three common cases of damped systems are shown in Fig. Curve C represents the situation when the damping is so large $(\delta^2 \gg \omega^2)$ it takes a long time to reach equilibrium - the system is **over damped**. Curve A represents an **under damped** situation $(\delta^2 < \omega^2)$ in which the system makes several swings before coming to rest and corresponds to the solution for x(t). Curve B represents **critical damping** $(\delta^2 = \omega^2)$. In this case equilibrium is reached in the shortest time.



Problem 1-32. The initial displacement of damped oscillations is $U_0 = 3\,cm$ (for t = 0). At time $t_1 = 10\,s$ the maximum of displacement equals $U_1 = 1\,cm$. Calculate time t_2 for which the maximum of displacement will be $U_2 = 0.3\,cm$.

<u>Solution</u>: the maximum of displacement at time t_1 is described by

$$U_1 = U_0 e^{-\delta t_1} .$$

From here we can calculate the damping constant

$$\delta = \frac{1}{t_1} \ln \frac{U_0}{U_1} .$$

To calculate time t_2 we express

$$U_2 = U_0 e^{-\delta t_2}$$

and from here

$$t_2 = \frac{1}{\delta}\ln\frac{U_0}{U_2} = t_1\frac{\ln\frac{U_0}{U_2}}{\ln\frac{U_0}{U_1}} = 21\,s\,.$$

Problem 1-33. Calculate the time in which the energy of damped oscillations with frequency f = 600 Hz will decrease 10^6 times, if the logarithmic decrement $\Lambda = 0.0008$.

Solution: at time t_1 the amplitude is $A_1 \approx e^{-\delta t_1}$ and the energy is $E_1 \approx A_1^2$;
at time $t_2 = t_1 + t_x$ (t_x is the time to be calculated) the amplitude will be $A_2 \approx e^{-\delta t_2}$ and the energy $E_2 \approx A_2^2$;

We know that

$$\frac{E_1}{E_2} = n \quad (where = 10^6)\,.$$

Hence,

$$\frac{E_1}{E_2} = \frac{A_1^2}{A_2^2} = n$$

or

$$\frac{A_1}{A_2} = \frac{e^{-\delta t_1}}{e^{-\delta(t_1+t_x)}} = e^{\delta t_x}\,.$$

Thus

$$e^{\delta t_x} = \sqrt{n}$$

and

$$t_x = \frac{\ln\sqrt{n}}{\delta}\,.$$

The unknown damping constant can be calculated from the logarithmic decrement $\Lambda = \delta\, T_1$, where T_1 is the period of damped oscillations.
So that

$$\delta = \frac{\Lambda}{T_1} = \Lambda\, f_1\,.$$

Now

$$t_x = \frac{\ln\sqrt{n}}{\Lambda\, f_1} = 14.4\,s\,.$$

Problem 1-34. Calculate the logarithmic decrement of the simple pendulum whose length L = 0.8 m if its initial amplitude $\alpha_0 = 5^0$ and after 5 minutes it will be $\alpha = 0.5^0$.

Solution: the logarithmic decrement of the damped oscillatory system is defined

$$\Lambda = \delta\, T_1$$

where δ is the damping constant and T_1 is the period of the damped oscillations.
For maxima of displacements

$$\alpha(t) = \alpha_0 e^{-\delta t}$$

where α_0 is the initial displacement.
From here

$$\delta = \frac{1}{t}\ln\frac{\alpha_0}{\alpha} .$$

The period of the damped pendulum equals

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega^2 - \delta^2}}$$

where $\omega = \frac{2\pi}{T}$ is the angular frequency of the non damped pendulum, when

$T = 2\pi\sqrt{\frac{L}{g}}$ is its period.

So that, $\omega = \sqrt{\frac{g}{L}}$ and for the logarithmic decrement we obtain

$$\Lambda = \delta T_1 = \frac{2\pi}{\sqrt{\omega^2 - \delta^2}} = \frac{2\pi}{\sqrt{\frac{g}{L} - \left(\frac{1}{t}\ln\frac{\alpha_0}{\alpha}\right)^2}} = 1.78 .$$

Problem 1-35. The maximum of displacement of damped oscillations decreases three times in time of one period T_1. Determine the magnitude of the ratio T_1/T, when T is the period of non damped oscillations.

Solution: Let ω_1, T_1 be the angular frequency and the period of damped oscillations, respectively. Let ω, T be the angular frequency and the period of non damped oscillations, respectively.
There is the relation between ω_1 and ω

$$\omega_1^2 = \omega^2 - \delta^2 \qquad (1)$$

where δ is the damping constant.
The ratio of two successive maxima of displacements represents the damping:

$$\frac{A_1}{A_2} = e^{\delta T_1} .$$

Hence,

$$\delta = \frac{\ln\frac{A_1}{A_2}}{T_1} .$$

We substitute this δ for (1) to obtain

$$\frac{4\pi^2}{T_1^2} = \frac{4\pi^2}{T^2} - \frac{(\ln 3)^2}{T_1^2}$$

and from here we express the ratio

$$\frac{T_1}{T} = \sqrt{1+\left(\frac{\ln 3}{2\pi}\right)^2} = 1.015 .$$

The result tells us that the period of these damped oscillations is of 1.5 % longer than that of non damped oscillations.

1.5 Forced Oscillations; Resonance

When the body is subject to an oscillatory external force the oscillations that result are called **forced oscillations**. These forced oscillations have the frequency of the external force and not the natural frequency of the body.
In a forced oscillations, the amplitude and hence the energy transferred to the oscillatory system is found to depend on the difference between the forced and the natural frequency as well as on the amount of damping.
The equation of motion of a forced oscillator follows from the second law of motion. In addition to the restoring force -kx and the damping force -Bv, we have also the applied oscillating external force. Let this external force be given by

$$F_{ext} = F_0 \cos\omega t \ ,$$

where F_0 is the maximum value of the external force and $\omega = 2\pi f$ is its angular frequency.
Then the equation of motion is

$$ma = -kx - B\frac{dx}{dt} + F_0\cos\omega t$$

or

$$m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = F_0\cos\omega t .$$

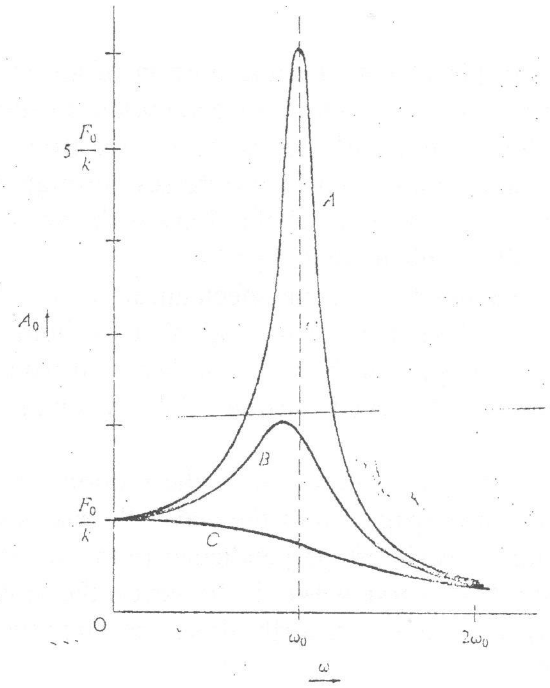
The solution of this equation is

$$x = A_0 \sin(\omega t + \phi)$$

where the amplitude A_0 equals

$$A_0 = \frac{F_0}{m\sqrt{(\omega^2 - \omega_0^2)^2 + B^2\omega^2/m^2}}$$

when $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural undamped angular frequency of the system and m is its mass. Note that the system oscillates with the frequency ω of the driving force and that the motion is undamped harmonic motion. Actually, the general solution takes another term for the damped motion. But this second term approaches zero in time.
The amplitude of forced harmonic motion, A_0, depends strongly on the difference between the applied frequency ω and the natural frequency ω_0. A plot of A_0 as a function of the applied frequency ω , is shown in Fig. for three specific values of



damping. Curve A represents light damping, Curve B fairly heavy damping, and Curve C overdamped motion.

The amplitude can become large when, $\omega \approx \omega_0$, the driving frequency ω is near the natural frequency, as long as the damping is not too large. When the damping is small, the increase in amplitude near $\omega \approx \omega_0$ is very large (and often dramatic). This is known as **resonance** and the value of ω at which resonance occurs is called the **resonant frequency**. The smaller the damping in a given system the closer is the resonant frequency to the natural undamped frequency ω_0 . Frequently the damping is small enough so that the resonant frequency can be taken to equal the natural undamped frequency ω_0 with small error. If B = 0, resonance occurs at $\omega = \omega_0$ and the amplitude A_0 becomes infinite. In such a case, energy is being continuously transferred into the system and none is dissipated. For real system, B is never precisely zero, and the amplitude is finite. The peak of the amplitude does not occur precisely at $\omega = \omega_0$, although it is quite close to ω_0 unless the damping is very large. If the damping is large, there is little or no peak as seen in curve C.

Problem 1-36. Derive the precise formula for the forced angular frequency (the resonant frequency) for which the amplitude A_0 of forced oscillations reaches its maximum value for non zero damping.

$$\left[\omega_{res} = \sqrt{\omega_0^2 - B^2 / 2m^2} = \sqrt{\omega_0^2 - 2\delta^2}\right]$$

Problem 1-37. Prove that the amplitude of the velocity for forced oscillations is given by the product of the amplitude A_0 and the forced angular frequency ω . Then prove that the extreme of this amplitude occurs when the forced angular frequency equals the natural frequency regardless the value of the damping.