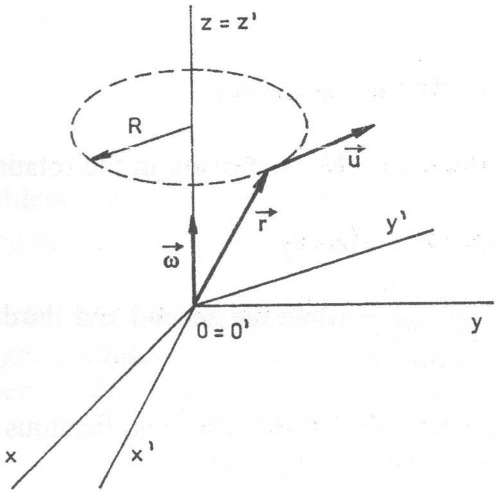
1.3 MOTION IN A ROTATING SYSTEM

We shall consider two co-ordinate systems S(O,x,y,z) and S'(O', x', y', z'). Both the systems have their origins in common and they also have their z-axes in common.



The S'-system (called the primed or dashed system) rotates with angular velocity ω around the vertical axis. Thus, the vector $\boldsymbol{\omega}$ is oriented to the vertical direction.

Let us now consider a particle resting with respect to the S'-system. The particle, in this case, makes a circular motion with angular velocity ω. Its linear velocity with respect to the S-system will be given in terms of a vector product

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$$

where **r** is the position vector of the particle (we call the velocity **u** a driving velocity).

If the particle starts to move in the S'-system with velocity **v'** measured with respect to the S'-system, its velocity with respect to the S-system will be equal to

$$\mathbf{v} = \mathbf{v'} + \mathbf{u}$$

The instantaneous position of the particle will be described by the position vector **r** which is the same in both the systems (since the origins of both the systems are identical).

$$\mathbf{r} = \mathbf{r'} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x'\mathbf{i'} + y'\mathbf{j'} + z'\mathbf{k'}$$

Let us express the velocity **v'** in terms of the position vector

$$\mathbf{v'} = \mathbf{v} - \mathbf{u}$$

or

$$\frac{d'\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt} - \omega \times \mathbf{r}$$

where the dash on the left side indicates the derivative of the position vector when the position vector is expressed in terms of coordinates in the S'-system. This transformation relation is held for a time derivative of any vector.

Thus, for an acceleration vector **a'** of the particle in the S'-system we may write

$$\mathbf{a}' = \frac{d'\mathbf{v}'}{dt} = \frac{d\mathbf{v}'}{dt} - \omega \times \mathbf{v}' = \frac{d}{dt}(\mathbf{v} - \omega \times \mathbf{r}) - \omega \times \mathbf{v}'$$

Since vector ω is a constant vector we may make the arrangement

$$\mathbf{a}' = \frac{d\mathbf{v}}{dt} - \omega \times \frac{d\mathbf{r}}{dt} - \omega \times \mathbf{v}' =$$

$$= \mathbf{a} - \omega\times(\mathbf{v}' + \omega\times\mathbf{r}) - \omega\times\mathbf{v}' =$$
$$= \mathbf{a} - \omega\times\mathbf{v}' - \omega\times(\omega\times\mathbf{r}) - \omega\times\mathbf{v}' =$$
$$= \mathbf{a} - 2\omega\times\mathbf{v}' - \omega\times(\omega\times\mathbf{r}) = \mathbf{a} + 2\mathbf{v}'\times\omega - \omega\times(\omega\times\mathbf{r})$$

Now, for the total force exerted on the particle of mass m moving in the rotating S'-system we have

$$\mathbf{F}' = m\mathbf{a}' = m\mathbf{a} + 2m\,\mathbf{v}'\times\omega - m\,\omega\times(\omega\times\mathbf{r})$$

where m**a** = **F** represents the actual (or real) force while the second and third terms represent fictitious forces.

We may state the result. In addition to the actual force **F** there are two fictitious forces acting on the particle m when it moves in the rotating S'-system:

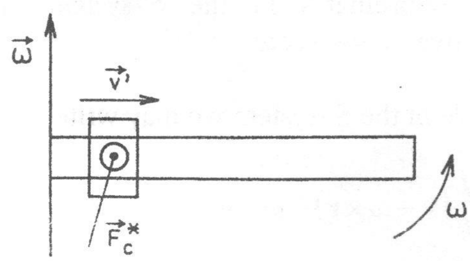
The force $\mathbf{F}_c = 2m\mathbf{v}'\times\omega$ is called **Coriolis' force**. This force is zero only if the particle moves in the same direction as that of ω. In this case the vectors **v'** and ω make a zero angle and the cross product equals zero. **v'** is the velocity vector of a particle with respect to the rotating S'-system. ω is an angular velocity vector of the rotating S'-system.

The force $\mathbf{F}_o = -m\omega\times(\omega\times\mathbf{r})$ is called **centrifugal force**. It is directed from the axis of rotation.
Since the driving velocity $\mathbf{u} = \omega\times\mathbf{r}$ has the magnitude $u = \omega r \sin\alpha$, the centrifugal force has the magnitude

$$F_o = m\omega^2 R = m\frac{u^2}{R}$$

where u is the linear velocity of the rotating S'-system with respect to the rest S-system. R is the perpendicular distance from the axis of rotation.

Problem 1-55. A horizontal bar rotates around the vertical axis that passes through the bar end. At the same time a sliding weight of mass m=1.6 kg slides with constant velocity $v' = 25 cm s^{-1}$ along the bar away from the axis of rotation. The sliding weight acts on the bar by a side force of 15 N. Calculate the frequency of rotation of the bar



Solution: The side force represents the Coriolis' force. In our case the vectors **v'** and **ω** are perpendicular to each other for all times. Thus, the magnitude of this side force will be equal to

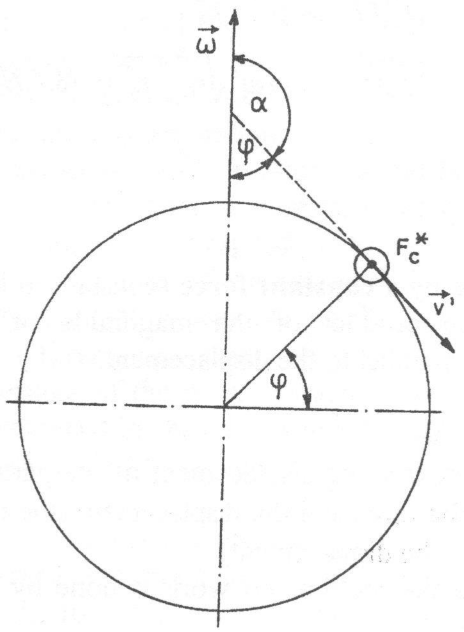
$$F_c = 2mv'\omega = 2mv'2\pi f$$

and from here

$$f = \frac{F_c}{4\pi m v'} = 2.98\, s^{-1}$$

Problem 1-56. A train of mass m = $5\times10^5\ kg$ is moving with a velocity $72\, km\, h^{-1}$ along the meridian from north to south.

Calculate the Coriolis' force acting on the rail due to the Earth's rotation at the place of the geographical latitude φ on the northern hemisphere.
Which rail does the Coriolis force act on ?



Solution: The train is moving in a system that rotates with angular velocity ω. This ω is equal to the angular velocity of the Earth, so

$$\omega = \frac{2\pi}{T}$$

where T = 24×3600 s is the period of the Earth's rotation.

We know that $\mathbf{F}_c = 2m\mathbf{v}'\times\omega$, where **v'** is the velocity vector of the train with respect to the Earth.
So, this Coriolis' force acts on the right rail from the view of the train motion and its magnitude is equal to

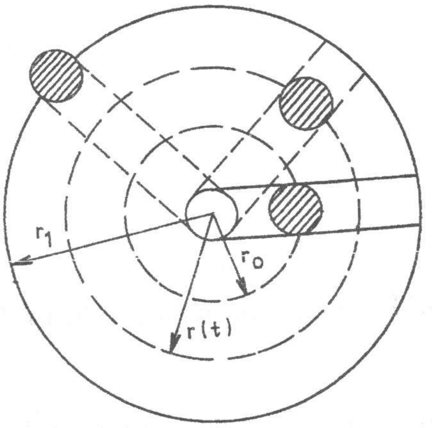
$$F_c = 2mv'\omega\sin\alpha = 2mv'\omega\sin(\pi-\varphi) = 2mv'\omega\sin\varphi = 2mv'\frac{2\pi}{T}\sin\varphi$$

If we substitute $v' = 72\, km\, h^{-1} = 20\, m\, s^{-1}$ and T = 24×3600 = 86400 s we obtain the magnitude of the Coriolis force

$$F_c = 1450\sin\varphi \quad [\text{N}]$$

We see that this force depends on the train's position. It is zero at the equator and it has its maximum value $F_c^{max} = 1450\, N$ at the pole.

Problem 1-57. A circular disk of radius $r_1 = 1.3\ m$ rotates around the vertical axis with constant angular velocity $\omega = 8\ s^{-1}$. At a distance $r_0 = 0.5\ m$ from the axis there is a sphere of mass m = 0.25 kg inside a radial groove on the disk.



Determine: a) The time dependence of the distance of the sphere from the disk's centre.

b) The dependence of the sphere's relative velocity v' on the distance from the disk's centre.

c) The magnitude of the Coriolis' force acting on the sphere at the disk's periphery.

$\left[r(t) = r_0 \cosh \omega t\right]$

$\left[v'(r) = \omega\sqrt{r^2 - r_0^2}\right]$

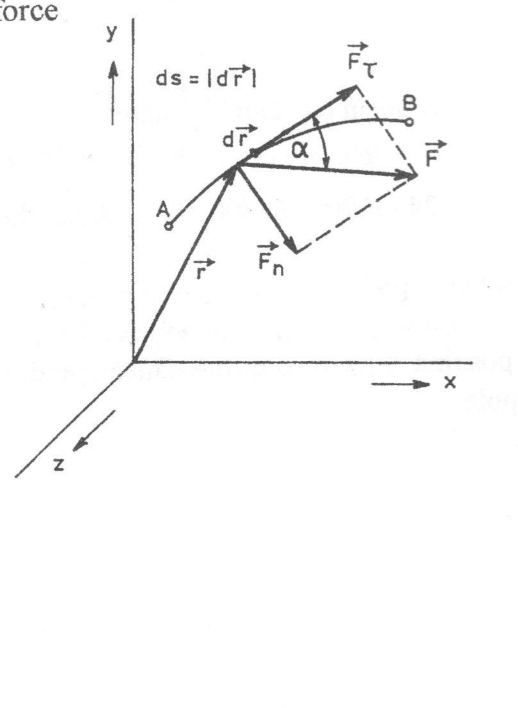
$\left[F_c(r_1) = 2m\omega^2\sqrt{r_1^2 - r_0^2} = 38.4\ N\right]$

1.4 WORK AND ENERGY

Specifically, the **work** done on a particle by a **constant force** (constant in both magnitude and direction) is defined as the product of the magnitude of the displacement times the component of the force parallel to the displacement.
In equation form this can be written as

$$W = F\ d\ \cos\varphi$$

where F is the magnitude of the constant force, *d* is the displacement of the particle, and φ is the angle between the directions of the force and the displacement (the term F $\cos\varphi$ is the component of the force parallel to the displacement).
Note that, when the force is perpendicular to the motion, no work is done by that force



In many cases a **force varies** in magnitude or direction during a process. In this case the exact result for the work done equals

$$W = \int_A^B F ds \cos\varphi$$

or, using dot-product notation, we can write

$$W = \int_A^B \mathbf{F}.d\mathbf{r}$$

This is the most general definition of work.