2. WAVE MOTION

Mechanical waves originate in the displacement of some portion of an elastic medium from its normal position, causing it to oscillate about an equilibrium position. Because of the elastic properties of the medium, the disturbance is transmitted from one layer to the next. This disturbance, or wave, consequently progresses through the medium. Note that the medium itself does not move as a whole along with the wave motion. The various parts of the medium oscillate only in limited paths.

It is necessary to have a material medium to transmit mechanical waves. The properties of the medium determines the speed of a wave through that medium. Its elasticity gives rise to the restoring forces on any part of the medium displaced from its equilibrium position. Its inertia tells us how this displaced portion of the medium will respond to these restoring forces.

Mechanical waves can be distinguished by considering how the motions of the particles of matter are related to the direction of propagation of the waves themselves. If the motions of the matter particles conveying the wave are perpendicular to the direction of propagation of the wave itself, we have a **transverse wave**. If, however, the motion of the particles conveying a mechanical wave is back and forth along the direction of propagation, we have **a longitudinal wave.**

Waves can also be classified as one-, two-, and three- dimensional waves, according to the number of dimensions in which they propagate energy. For example, waves moving along the string are one-dimensional,. waves on water are two-dimensional and sound waves are three-dimensional

A surface formed by points undergoing a similar disturbance at a given instant is called **wavefront**. All of whose points are in the same phase of motion. As time goes on, this surface moves in direction of the wave propagation. If the medium is homogeneous and isotropic, the direction of propagation is always at right angles to the wavefront. A line normal to the wavefronts indicates the direction of motion of the waves and it is called a ray.

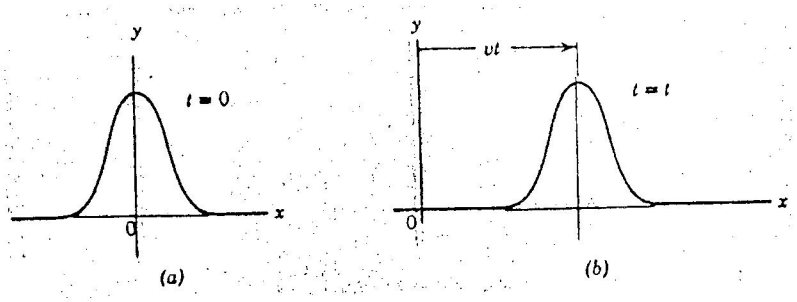
Wavefronts can have many shapes. If the disturbances are propagated in a single direction, the waves are called **plane waves**. At a given instant conditions are the same everywhere on any plane perpendicular to the direction of propagation. The wavefronts are plane and the rays are straight lines. Another case is that of **spherical waves**. Here the disturbance is propagated out in all directions from a point source of waves. The wavefronts are spheres and the rays are radial lines leaving the point source in all directions. Far from the source the spherical wavefronts have very small curvature, and over a limited region they can often be regarded as plane.

2.1 Traveling Waves

Let us consider a long string stretched in the x-direction along which a transverse wave is traveling. At $t = 0$ let the shape of the string be represented by

$$y = f(x) \qquad \text{for } t = 0 ,$$

where y is the transverse displacement of the string at the position x. The Fig. shows a possible waveform (in this case a pulse) on the string at $t = 0$. At some time later the wave has traveled a distance $x = vt$ to the right, where v is magnitude of the wave velocity.



The equation of the curve at the time t is therefore

$$y = f(x - vt) \quad \text{for } t = t .$$

This gives the same waveform about the point x = vt at time t as we had about x = 0 at time t = 0. This is the **general equation** representing a wave of any shape traveling to the right. To describe a particular shape we must specify exactly what the function f is. Mathematically this means we look at how x changes with t when (x - tv) has some particular fixed value. It is seen at once that as t increases x must increase in order to keep (x - tv) fixed. Hence, the equation does in fact represent a wave traveling to right - increasing x as time goes on.

To represent a wave traveling to the left we would write $y = f(x + vt)$, for here the position x of some fixed phase (x + vt) of the wave decreases as time goes on.

Let us require that

$$x - vt = \text{constant}$$

for a particular phase of a wave traveling to the right. Then differentiation with respect to time gives

$$\frac{dx}{dt} - v = 0 \quad or \quad \frac{dx}{dt} = v ,$$

so that v is really the **phase velocity** of the wave.

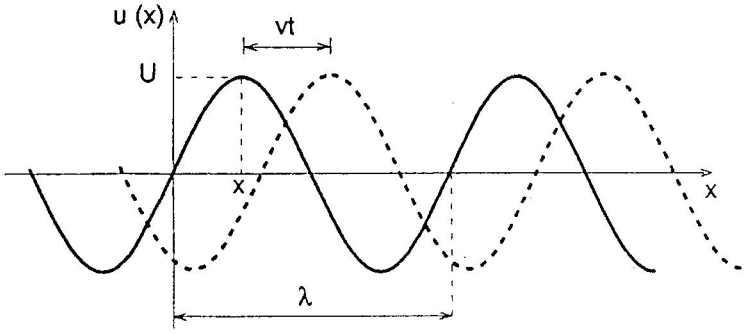
Note that for any fixed value of the time t the equation representing a traveling wave gives y as a function of x. This defines a curve that represents the actual shape of the string at this chosen time. On the other hand if we focus on one point of the string of a fixed value of x, then this equation gives y as a function of the time t and this describes how the transverse position of this point on the string changes with time.

The equation representing a traveling wave just presented holds for longitudinal waves as well as for transverse waves.

Let us now consider a particular waveform and let us suppose that at the time t = 0 a wavetrain along the string is given by

$$y = y_m \sin \frac{2\pi}{\lambda} x .$$

The wave shape is a sine curve. The maximum displacement y_m is the **amplitude** of the wave. The value of the transverse displacement y is the same at x as it is at $x+\lambda$, $x+2\lambda$. etc., where λ is called the **wavelength** of the wave and it represents the distance between two adjacent points in the wave having the same phase. As time goes on the wave travel to right with a phase velocity v. Hence, the equation of the wave at the time t is



$$y = y_m \sin\frac{2\pi}{\lambda}(x - vt)$$

The **period T** is the time required for the wave to travel a distance of one wavelength λ, so that

$$\lambda = v\,T \qquad (*)$$

Putting this relation into the equation of the wave yields

$$y = y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

From this it is clear that y, at any given time, has the same value at x+λ, x+2λ, etc., as it does at x, and that y, at any given position has the same value at the time t+T, t+2T, etc., as it does at the time t.

To reduce the last equation to a more compact form, we define two quantities, the **wave number k** and the **angular frequency** ω:

$$k = \frac{2\pi}{\lambda} \quad and \quad \omega = \frac{2\pi}{T} \qquad (**)$$

By comparing (*) and (**) the phase velocity v of the wave is given

$$v = \frac{\lambda}{T} = \frac{\omega}{k}$$

In terms of these quantities, the equation of a sine wave traveling to the right (positive x-direction) is

$$y = y_m \sin(kx - \omega t)$$

and for a sine wave traveling to the left (negative x-direction)

$$y = y_m \sin(kx + \omega t)$$

In these traveling waves it is assumed that the displacement y is zero at the position x=0 at the time t=0. The general expression for a sine wave traveling to the right is

$$y = y_m \sin(kx - \omega t - \phi)\ ,$$

where ϕ is called the phase constant.

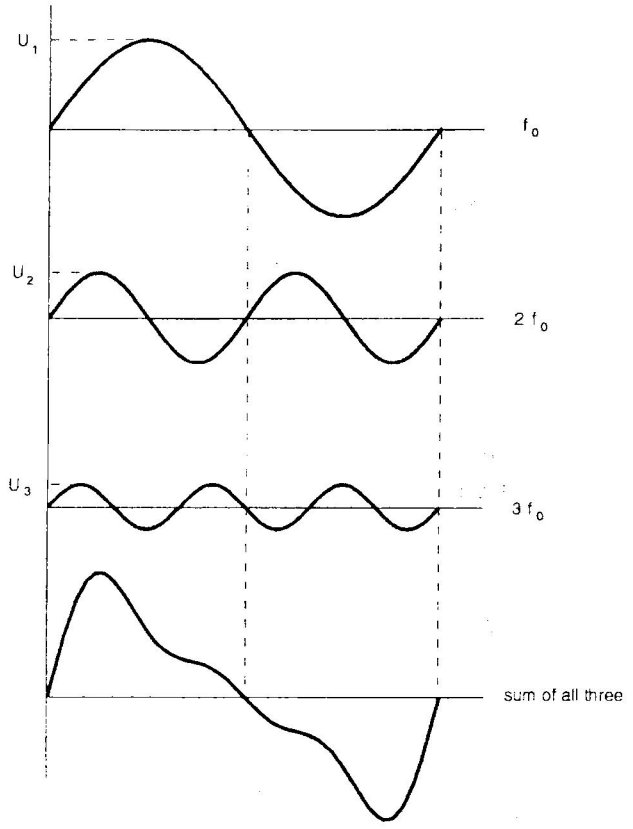
If we choose any point of the string, say x = π/k, the displacement y at that point can be written as [using that sin(π - α) = sin α]

$$y = y_m \sin(\omega t - \phi)$$

This is similar to the equation for simple harmonic motion. Hence, any particular element of the string undergoes simple harmonic motion about its equilibrium position as this wave travels along the string.

2.2 The Principle of Superposition

When two or more waves pass through the same region of space at the same time, it is found that for many waves the actual displacement is the vector or algebraic sum of the separate displacements. This is called the principle of superposition. It is valid for mechanical waves as long as the displacements are not too large and there is a linear relationship between the displacement and the restoring force of the oscillating medium. If the amplitude of a mechanical wave, is so large that it goes beyond the elastic region of the medium, and Hook's law is no longer operative, the superposition principle is no longer accurate.



The figure shows an example of the superposition principle. In this case there are three sinusoidal waves present, on a stretched string, each of different amplitude and frequency. At any time the actual amplitude at any position is the algebraic sum of the amplitude of the three waves at that position. The actual wave is not a simple sinusodial wave and is called a composite or complex wave.

It can be shown that any complex wave whatever can be considered as composed of many simple sinusoidal waves of different amplitudes, wavelengths, and frequencies. This is known as Fourier's theorem. A complex periodic wave of period T can be represented as a sum of pure sinusoidal terms whose frequencies are integral multiples of $f = 1/T$. If the wave is not periodic, the sum becomes an integral, called a Fourier integral.

When the restoring force is not precisely proportional to the displacement for mechanical waves in some continuous medium, the speed of sinusoidal waves depends on the frequency. This is called **dispersion**. The different sinusoidal waves that compose a complex wave will travel with slightly different speeds in such a case. Consequently, a complex wave will change shape as it travels if the medium is dispersive.

2.3 The Wave Velocity

The wave velocity is often referred to as the phase velocity, and it must be distinguished from the velocity of a particle of the medium itself. For example, for a wave traveling along a string the wave velocity points along the length of the string, whereas the velocity of a particle of the string is perpendicular to it.

The velocity of wave depends on the properties of the medium in which it travels, especially it depends on the elasticity of the medium and on its inertia. For a stretched string the elasticity is measured by the tension F in the string. The greater the tension the greater will be the elastic restoring force in an element of the string. The inertia characteristic is measured by μ, the mass per unit length of the string. Then, the wave velocity is derived as

$$v = \sqrt{\frac{F}{\mu}}$$

This relation, however, holds only for relatively small displacements of the string (compared to the length of the string) - which case, however, is widely applicable in practice.

The velocity of a longitudinal wave has a similar form. In particular, for a longitudinal wave traveling down a long solid rod

$$v = \sqrt{\frac{E}{\rho}} ,$$

where E is the elastic modulus of the material and ρ is its density.

For a longitudinal wave traveling in a liquid or gas

$$v = \sqrt{\frac{B}{\rho}} ,$$

where B is the bulk modulus and ρ again the density.

Problem 2-1. A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar which moves the end up and down through a distance of 0.50 cm. The motion is continuous and is repeated regularly 120 times per second.

(a) If the string has a linear density of 0.25 kg/m and is kept under a tension of 90 N, find the speed, amplitude, frequency, wavelength, wave number and angular frequency of the wave motion.

(b) Assuming the wave moves in the +x-direction and that, at t = 0, the end of the string described by x = 0 is in its equilibrium position y = 0, write the equation of the wave.

(c) Find the velocity and acceleration of a particle 62 cm from the end.

Solution:

(a) The end moves 0.25 cm away from the equilibrium position, first above it, then below it, therefore, the amplitude y_m is 0.25 cm.

The motion is repeated 120 times each second so that the frequency f = 120 Hz.

The wave speed is given

$$v = \sqrt{\frac{F}{\mu}} = 19\ m/s \quad .$$

The wavelength is given

$$\lambda = \frac{v}{f} = 16\ cm \quad .$$

The wave number equals

$$k = \frac{2\pi}{\lambda} = 0.39\ cm^{-1} \quad .$$

The angular frequency is

$$\omega = vk = 740\ s^{-1} \quad .$$

(b) The general expression for a transverse sinusoidal wave moving in the +x-direction is

$$y = y_m \sin(kx - \omega t - \phi) \quad .$$

Requiring that y = 0 for x = 0 and t = 0 yields

$$0 = y_m \sin(-\phi) \quad ,$$

which means that the phase constant ϕ may be taken to be zero.
Hence for this wave we write

$$y = y_m \sin(kx - \omega t) \quad ,$$

and with the values found above we obtain the equation for the wave

$$y = 0.25 \sin(0.39\ x - 740\ t)$$

where x and y are in centimeters and t is in seconds.

(c) We write again the general form of this wave

$$y = y_m \sin(kx - \omega t) = y_m \sin k(x - vt) \quad .$$

The v in this equation is the constant phase velocity in horizontal direction (the velocity of the wavetrain).Each particle of the string moves vertically - in the y-direction. In order to determine the particle velocity, which we shall designate by the symbol u, let us consider a particle at a particular position x (x is now a constant in this equation) and ask how the particle displacement y changes with time.
With x constant we obtain

$$u = \frac{\partial y}{\partial t} = -y_m \omega \cos(kx - \omega t) \quad ,$$

in which the partial derivative reminds us that although in general y is a function of both x and t, we here assume that x remains constant so that t becomes the only variable.
The acceleration of the particle at this (constant) value of x is

$$a = \frac{\partial^2 y}{\partial t^2} = \frac{\partial u}{\partial t} = -y_m \omega^2 \sin(kx - \omega t) = -\omega^2 y \quad .$$

With the values found above we obtain

$$u = -185\cos(24 - 740t)$$

and

$$a = -13.7 \times 10^4 \sin(24 - 740t)$$

where t is expressed in seconds u in cm/s and a in cm/s^2.

Problem 2-2. The left end of a long horizontal string oscillates transversely in simple harmonic motion with frequency f = 250 Hz and amplitude 2.6 cm. At the beginning this end of the string is displaced upward of 1.6 cm and is rising. The string is under tension of 140 N and has linear density $\mu = 0.12$ kg/m.
State the equation for the traveling wave along the string.

Solution: The general form is $y(x,t) = y_m \sin(kx - \omega t + \phi)$.
In accordance with initial conditions we may write $1.6 = 2.6 \sin\phi$.
So that $\phi = 0.66$ rad and $\omega = 2\pi f = 1570\ s^{-1}$.
To calculate k we need to know the wave velocity

$$v = \sqrt{\frac{F}{\mu}} = 34\ ms^{-1},$$

then $$\lambda = \frac{v}{f} = 0.14\ m\ ,$$

and now $$k = \frac{2\pi}{\lambda} = 45\ m^{-1}$$

Hence, the equation for the wave equals

$$y = 0.026 \sin(45\,x - 1570\,t - 0.66)\ ,$$

where y and x are in meters and t in seconds.

Problem 2-3. A wave whose wavelength is 0.30 m is traveling down a 300-m long wire whose total mass is 30 kg. If the wire is under a tension of 4000 N, what is the velocity and frequency of this wave?

[v = 63 m/s; f = 210 Hz]

Problem 2-4. You can often hear an approaching train by putting your ear to the track. How long does it take for the wave to travel down the steel track if the train is 1.0 km away?

[t = 0.20 s]

2.4 Energy Transmitted by Waves

As wave travels through a medium, the energy is transmitted from particle to particle of the medium. For a sinusoidal wave of frequency f, the particles move in simple harmonic motion as a wave passes, so each particle has an energy $E = \frac{1}{2} k y_m^2$, where y_m

is the maximum displacement (amplitude) of its motion, either transversely or longitudinally.
Using $k = m\omega^2 = 4\pi^2 mf^2$ we can write

$$E = 2\pi^2 mf^2 y_m^2 .$$

If the wave is three-dimensional wave traveling in an elastic medium, we can write
m = ρV, where ρ is the density of the medium and V is its volume. The volume V can be expressed as V = S L , where S is the cross-sectional area through which the wave travels and L = vt is the distance the wave travels in a time t, where v is the speed of the wave.
Thus m = ρV = ρSL = ρSvt and now

$$E = 2\pi^2 \rho S v t f^2 y_m^2$$

represents the average energy that enters the volume V in a time t.
We obtain the important result that the **energy transported by a wave is proportional to the square of the amplitude**. This fact holds for all types of waves.

The average rate of energy is the **average power**

$$\overline{P} = \frac{E}{t} = 2\pi^2 \rho S v f^2 y_m^2 .$$

The **intensity** of a wave is defined as the average power transferred across a unit area perpendicular to the direction in which the wave is traveling

$$I = \frac{\overline{P}}{S} = 2\pi^2 v \rho f^2 y_m^2 .$$

The intensity of wave is also proportional to the square of its amplitude.

The note: if the wave flows out from the source in all directions it is a three-dimensional wave. If the medium is isotropic then the wave is spherical in shape and is said to be a **spherical wave**.

Problem 2-5. Spherical waves travel from a source of waves whose power output P is assumed to be constant. Find how the wave intensity depends on the distance from the source. The medium is assumed to be isotropic and the source is assumed to radiate uniformly in all directions.

Solution: The intensity of a three-dimensional wave is the power transmitted across a unit area normal to the direction of propagation. As the wavefront expands from a distance r_1 to a distance r_2 , its surface area increases from $4\pi r_1^2$ *to* $4\pi r_2^2$. If there is no absorption of energy, the total energy transported per second by the wave remains constant (it is conserved) at the value P.
So that

$$P = 4\pi r_1^2 I_1 = 4\pi r_2^2 I_2$$

where I_1 *and* I_2 are the wave intensities at r_1 *and* r_2 respectively.

Hence,

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

The wave intensity varies inversely as the square of its distance from the source.

Problem 2-6. Assuming the previous problem find how the amplitude of the wave depends on the distance from the source.
The hint: since the intensity is proportional to the square of the amplitude, the amplitude of the wave must vary inversely as the distance from the source.

Problem 2-7. Find how the intensity and the amplitude of a one-dimensional wave depends on the distance from the source.
The hint: in this case the area S remains constant so the amplitude also remains constant. Thus the intensity as well as the amplitude do not vary with distance.

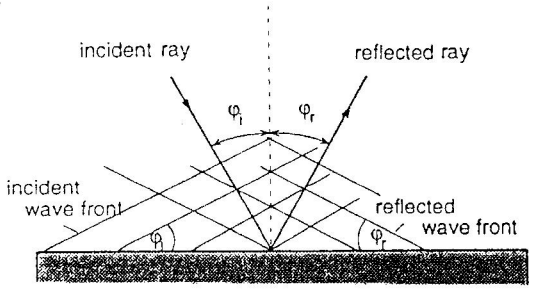
The note: in practice, frictional damping is generally present and some of the energy is transformed into thermal energy. Thus the amplitude and intensity of a one-dimensional wave decrease with distance from the source. For a three-dimensional wave the decrease will be greater than that found above.

Problem 2-8. If the intensity of a earthquake wave is $1.0 \times 10^6\ W/m^2$ at 100 km from the source, what is the intensity 400 km from the source?

$$\left[6.2 \times 10^4\ W/m^2\right]$$

2.5 Reflection and Refraction of Waves

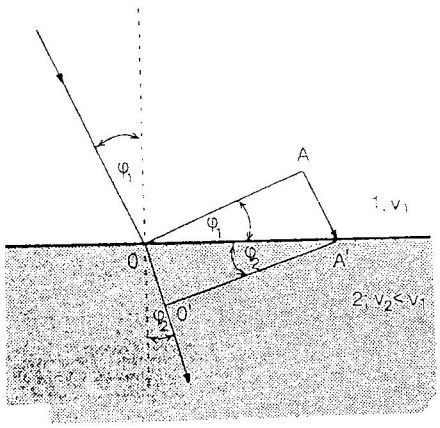
When a wave strikes an obstacle, or comes to end of the medium, at least a part of its energy is reflected. In the case of a one-dimensional wave the reflected wave is inverted if the end of the rope is fixed and returns right side up if the end is free. The reflected wave will thus be 180^0 out of phase with the incident wave if the end of the rope is fixed. When the end of the rope is free the reflected wave is not inverted and thus it has no phase change.



For a two- or three- dimensional wave, we are concerned with wave fronts, by which we mean the line or surface of all points having the same phase. A line drawn in the direction of the motion, perpendicular to the wave front, is called a ray. The Fig. illustrates the **law of reflection**: the angle that

incident wave makes with the reflecting surface is equal to the angle made by the reflected wave. Or, if we define the angle of incidence as the angle the incident ray makes with the normal to the reflecting surface and the angle of reflection is the corresponding angle for the reflected ray, the law of reflection can be said in the short form: the angle of reflection equals the angle of incidence.
When the wave reaches the obstacle, not all of the energy is reflected. Some of it is absorbed by the obstacle. Part of the absorbed energy is transformed into thermal energy and part continues to propagate through the material of the obstacle. If this second medium has a greater linear density than the first, the reflected wave undergoes a phase change of 180^0 . If the second medium has a lower density, the reflected wave does not change phase.

When any wave strikes a boundary, some of the energy is reflected and some is transmitted or absorbed. When a two- or three-dimensional wave traveling in one medium crosses a boundary into a medium where its velocity is different, the transmitted wave may move in a different direction than the incident wave. This is known as **refraction.** In Fig., the velocity of the wave in medium 2 is less than in medium 1. In this case the direction off the wave changes so that it moves more nearly perpendicular to the boundary. That is, the angle of refraction φ_2 is less than the angle of incidence φ_1.



Generally, it holds

$$\frac{\sin \varphi_2}{\sin \varphi_1} = \frac{v_2}{v_1}$$

Of course, if the wave were going in the opposite direction, the angles φ_1 and φ_2 would change roles. Thus, if the wave travels into a medium where it can move faster, it will bend from the normal.

Problem 2-9. An earthquake wave passes across a boundary in rock where its velocity increases from 6.5 km/s to 8.0 km/s. If it strikes this boundary at 30^0 what is the angle of refraction?

$[38^0]$

2.6 Interference of Waves

Interference refers to what happens when two waves pass through the same region of space at the same time; it is an example of the superposition principle.
The next problem makes this physical effect clear.

Problem 2-10. Let us consider two one-dimensional waves of equal frequency ω, different amplitude $Y_1 \neq Y_2$ and different phase constant $\varphi_1 \neq \varphi_2$. We shall suppose that the displacement directions of the both waves are the same and the waves travel down a positive direction of x-axis with the speed v.
The equation of the two waves will be

$$y_1 = Y_1 \sin\left[\omega\left(\frac{x}{v} - t\right) + \varphi_1\right]$$

$$y_2 = Y_2 \sin\left[\omega\left(\frac{x}{v} - t\right) + \varphi_2\right]$$

Now let us find the resultant wave, which, on the assumption that superposition occurs, is the sum

$$y = y_1 + y_2 = (Y_1 \cos\varphi_1 + Y_2 \cos\varphi_2)\sin\omega\left(\frac{x}{v} - t\right) + (Y_1 \sin\varphi_1 + Y_2 \sin\varphi_2)\cos\omega\left(\frac{x}{v} - t\right) \quad (1)$$

The terms in parentheses are constants and we see that the resultant wave is also harmonic with the same frequency. To find the amplitude and phase angle of this wave we express it in standard form

$$y = Y \sin\left[\omega\left(\frac{x}{v} - t\right) + \varphi\right]$$

or (2)

$$y = Y\cos\varphi \sin\omega\left(\frac{x}{v} - t\right) + Y\sin\varphi\cos\omega\left(\frac{x}{v} - t\right)$$

where Y and φ represent the amplitude and the phase angle of the resultant wave.
As Eqs. (1) and (2) must be identically equal for any x and any t, we obtain the following expressions

$$\begin{aligned} Y\cos\varphi &= Y_1\cos\varphi_1 + Y_2\cos\varphi_2 \\ Y\sin\varphi &= Y_1\sin\varphi_1 + Y_2\sin\varphi_2 \end{aligned} \quad (3)$$

Dividing these two equations yields the phase angle φ

$$tg\varphi = \frac{Y_1\sin\varphi_1 + Y_2\sin\varphi_2}{Y_1\cos\varphi_1 + Y_2\cos\varphi_2}$$

To find the amplitude Y we square equations (3) and then we add them:

$$Y^2 = Y_1^2 + Y_2^2 + 2Y_1Y_2\cos(\varphi_2 - \varphi_1) \quad . \quad (4)$$

From this result it is clear that the amplitude Y depends on phase difference between the waves.

In practice, interference effects are obtained from waves which originate in the same source (or in sources having a fixes phase relationship to one another) but which follow different paths to the point of interference. The phase difference φ between the waves arriving at a point can be calculated by finding the difference between the paths traveled by them from the source to the point of interference.

We shall now find the relationship between the phase difference and the path difference.

The displacement of the one-dimensional wave of the sine course with the angular frequency ω at any point x and at any time t is described by the function y(x,t) of two variables x and t

$$y(x,t) = Y \sin \omega\left(\frac{x}{v} - t\right) ,$$

where Y and v are the amplitude and velocity of the wave, respectively.
If we choose the fixed point x_0, the function $y(x_0,t)$ describes the motion at this point x_0, thus for $x = x_0$ we can write

$$y(t) = Y \sin\left(\omega\frac{x_0}{v} - \omega t\right) = Y \sin(\varphi - \omega t) ,$$

where $\varphi = \omega\frac{x_0}{v}$ is constant and it represents the phase delay of the point x_0 with respect to the point in origin. The greater distance of x_0 is the greater delay φ.

Thus, for any point x we have the relationship

$$\varphi = \omega\frac{x}{v} = \frac{2\pi}{T}\frac{x}{v} = 2\pi\frac{x}{\lambda} .$$

The phase difference $\varphi_2 - \varphi_1$ of two waves of different phase angle is now equal

$$\varphi_2 - \varphi_1 = \frac{2\pi}{\lambda}(x_2 - x_1) ,$$

or

$$x_2 - x_1 = \frac{\lambda}{2\pi}(\varphi_2 - \varphi_1) ,$$

where $(x_2 - x_1)$ represents the path difference $d = (x_2 - x_1)$.

Let us now return to the equation (4) in more detail. If the phase difference

$$\varphi_2 - \varphi_1 = 2k\pi , \qquad \text{where } k = 0, 1, 2, \ldots$$

or the path difference is equal to the even multiple of the half-wavelength

$$d = 2k\frac{\lambda}{2} , \qquad k = 0, 1, 2, \ldots$$

the amplitude of the resultant wave will have its **maximum** $Y = Y_1 + Y_2$.
This case is called **constructive interference**. In constructive interference the wave are in phase.

If the phase difference

$$\varphi_2 - \varphi_1 = (2k+1)\pi , \qquad k = 0, 1, 2, \ldots$$

or the path difference is equal to the odd multiple of the half-wavelength

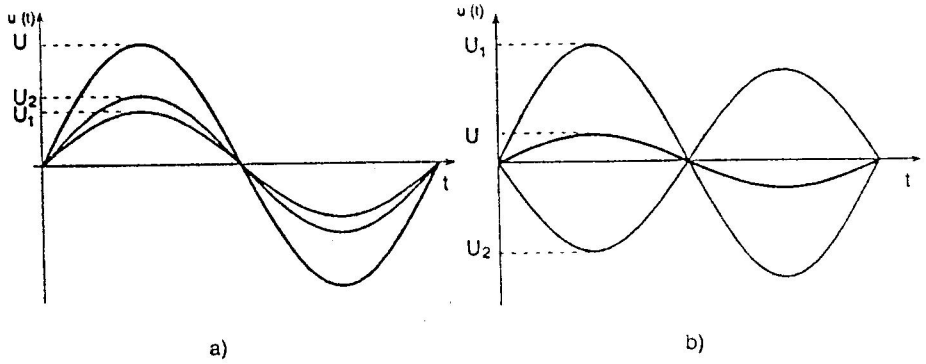
$$d = (2k+1)\frac{\lambda}{2} , \qquad k = 0, 1, 2, \ldots$$

the amplitude of the resultant wave will have its **minimum**

$$Y = Y_1 - Y_2 \quad \textit{if } Y_1 \textit{ greater than } Y_2$$
$$\textit{or } Y = Y_2 - Y_1 \quad \textit{if } Y_2 \textit{ greater than } Y_1 .$$

This case is called **destructive interference** . In destructive interference the waves are out of phase by one-half wavelength or 180^0 . For the special case of $Y_1 = Y_2$ the resultant amplitude Y = 0 and such the point will be at rest, that is, there are no oscillations at this point.

In Fig. there are plotted constructive and destructive interference versus time at a given point in space.

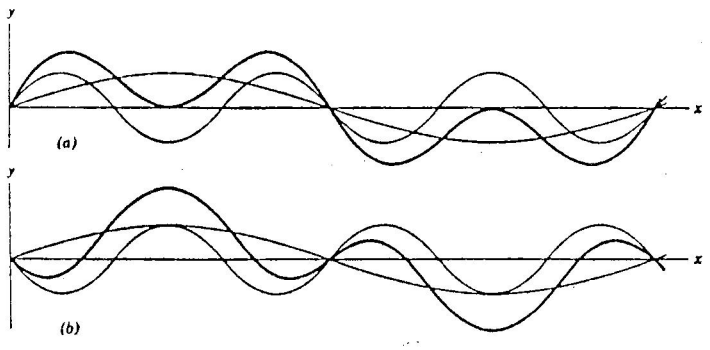


a) b)

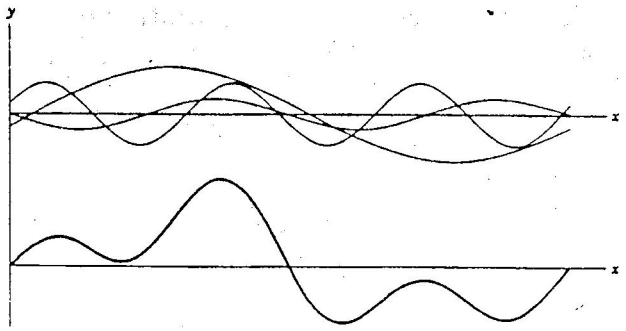
Problem 2-11. The waves we have considered thus far have been of the simple harmonic type, in which the displacements at any time are represented by a sine curve. We have seen that superposition of any number of such waves having the same frequency and velocity, but arbitrary amplitudes and phases, still gives rise to a resultant wave of this simple type.

If, however, we superimpose waves that have different frequencies, the resulting wave is complex. In a complex wave the motion of a particle is no longer simple harmonic motion, and the wave shape is no longer a sine curve.

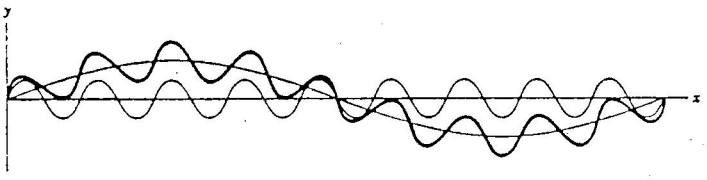
In this problem we shall consider only the qualitative aspects of complex waves and we will look at the results of adding graphically waves traveling with the same speed in the same direction but having various frequencies, amplitudes, and phases.



In this figure the addition of two waves with a frequency ratio 3:1 (light lines) yields a wave whose shape (heavy line) depends on the phase relationship of the components. Compare (a) and (b).



The following figure shows the addition of three waves (top Fig.) of different frequencies and amplitudes. The resultant complex wave (bottom Fig.) is quite different from a simple periodic wave.



In this figure we see the result of the addition (heavy line) of two waves of widely differing frequency (light lines). A wave of very high frequency is added to one of very low frequency.

Note that in all these figures the resultant wave is obtained under the assumption that the principle of superposition holds, by simply adding the displacements caused by the individual waves at every point.

2.7 Standing Waves and Resonance

Problem 2-12. Consider two waves of the same frequency, speed, and amplitude which are traveling in **opposite directions** along a string and derive the equation of a resultant wave.

Solution: Two such waves may be represented by the equations

$$y_1 = y_m \sin(kx - \omega t),$$
$$y_2 = y_m \sin(kx + \omega t).$$

Hence, the resultant may be written as

$$y = y_1 + y_2 = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

$$or, \quad as \quad y = 2y_m \sin kx \cos \omega t \quad .$$

The last equation is the **equation of a standing wave**.

Notice that a particle at any particular point x executes simple harmonic motion as time goes on, and that all particles oscillate with the same frequency. We know that in a traveling wave each particle of the string oscillates with the same amplitude

Characteristic of a standing wave, however, is the fact that the amplitude is not the same for different particles but varies with the location x of the particle.

In fact, the amplitude, $2y_m \sin kx$, has a **maximum value** of $2y_m$ at positions where

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \; etc.$$

$$or \quad x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, etc.$$

These points are called **antinodes** and are spaced one-half wavelength apart.

The amplitude has a **minimum value** of zero at positions where

$$kx = \pi, 2\pi, 3\pi, etc.$$

$$or \quad x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda, etc.$$

These points are called **nodes** and are spaced one-half wavelength apart. The separation between a node an adjacent antinode is one-quarter wavelength.

It is clear that energy is not transported along the string to the right or to the left, for energy cannot flow past the nodal points in the string which are permanently at rest. Hence, the energy remains "standing" in the string.

If two waves producing the standing wave have different amplitudes, we talk about **partial standing wave**. The important feature of it is that its nodes are not motionless. The amplitude of the oscillations of nodes is non-zero and depends on the difference between the amplitude of original waves.

Problem 2-13. Examine the reflection of a wave at a fixed as well as at a free end of a stretched string.

Suppose a wave travels down a stretched string which is fixed at one end. When arriving at that end the wave exerts a force on the rigid support. By Newton's third law the support exerts an equal but oppositely directed force on the string. This reaction force generates a wave at the fixed endpoint which travels back along the string in a direction opposite to that of the incident wave. This reflected wave returns with its transverse displacement reversed. Since endpoint is fixed, these two waves (incident and reflected) must always interfere destructively at the fixed endpoint. Hence, the reflected wave is always 180^0 out of phase with the incident wave at a fixed boundary. **On reflection from a fixed end a wave undergoes a phase change of** 180^0 **.**

If we consider reflection of a wave at a free end of a stretched string this generates a wave which travels back along the string in a direction opposite to that of the incident wave. This reflected wave is always in phase with the incident wave at that point. Hence, **at a free end a wave is reflected without change of phase.**

Hence, when we have a standing wave in a string, there will be a node at a fixed end and an antinode at a free end.

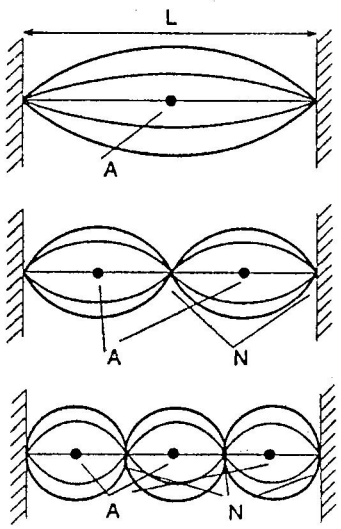
Problem 2-14. Explain the phenomenon of change of wavelength as a wave passes from one medium to another.

In the treatment given in previous problem we assumed that there is total reflection at the boundary. In general, at a boundary there is partial reflection and partial transmission. The amplitude of the reflected wave will be less than that of the incident wave because a transmitted wave continues along the second medium and carries away some of the incident energy. If the second medium has a greater linear density than the first, the wave reflected back will suffer a phase shift of 180^0 on reflection. But because its amplitude is less than the incident wave, the boundary point will not be a node and will move. If the second medium has a smaller linear density than the first, partial reflected wave occurs without change of phase.

Note that the transmitted wave travels with a different speed than the incident and reflected wave. The speed is determined by the relation $v = \sqrt{F/\mu}$. Hence, the wave travels slowly in the denser medium. The frequency of the transmitted wave is the same as that of the incident and reflected waves. Waves having the same frequency but traveling with different speed have different wavelength. Hence, from the relation $\lambda = v/f$ we conclude that in the denser medium, where the speed is less, the wavelength is shorter.

Problem 2-15. Find the natural frequencies of oscillation of a string fixed at both ends.

Solution: Let us consider a string fixed at both ends. Oscillations or standing waves can be established in the string. The only requirement we have to satisfy is that the end points be nodes. There may be any number of nodes in between or none at all, so that the wavelength associated with the standing waves can take on many different values. The distance between adjacent nodes is $\lambda/2$, so that in a string of length L there must be exactly an integral number n of half wavelengths, $\lambda/2$.



A-antinodes
N-nodes

That is

$$n\frac{\lambda}{2} = L$$

$$or \quad \lambda = \frac{2L}{n}, \quad n = 1,2,3,\ldots$$

But $\lambda = v/f$ and $v = \sqrt{F/\mu}$, so that the natural frequencies are

$$f = \frac{n}{2L}\sqrt{\frac{F}{\mu}}, \quad n = 1,2,3,\ldots$$

If the string is set oscillating and left to itself, the oscillations gradually die out. The motion is damped by dissipation of energy. We can pump energy into the system by applying a driving force. If the driving frequency is near that of any

natural frequency of the system, the string will oscillate at that frequency with a large amplitude. This is called the **resonance**.
Because the string has a large number of natural frequencies, resonance can occur at many different frequencies. The lowest frequency is called the **fundamental frequency**. The other frequencies are called **harmonics**. These are integral multiples of the fundamental. The next mode after the fundamental is called the second harmonic. The fundamental frequency is often called the first harmonic.

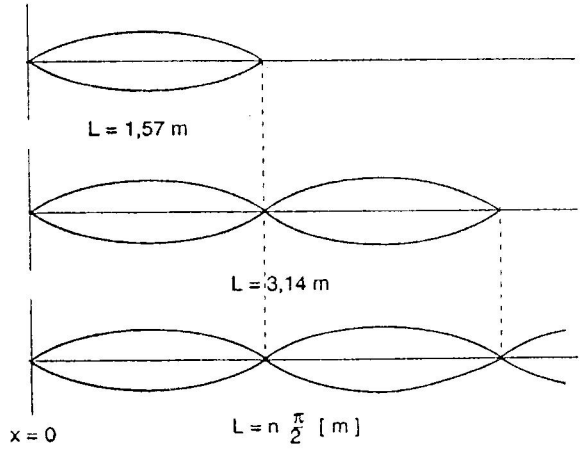
Problem 2-16. Two waves traveling in opposite directions on a string fixed at x = 0 are described by functions

$$y_1 = 0.2\sin(2x - 4t),$$

$$y_2 = 0.2\sin(2x + 4t).$$

a) Determine the function of the standing wave.
b) What is the amplitude at x = 0.45 m?
c) Where is the other end fixed?
d) What is the maximum amplitude and where is its position?

Solution: a) A general equation of a standing wave was given in Problem 2-12.



For our case we have: $y_m = 0.2m$, $k = 2m^{-1}$ *and* $\omega = 4\,s^{-1}$.

Hence, the standing wave is described by the equation

$$y(x,t) = 0.4\sin 2x\cos 4t.$$

b) At x = 0.45 m the displacement of the standing wave is described by

$$y(t) = 0.4 \sin 0.9 \cos 4t$$

and the amplitude at this point equals

$$2y_m \sin kx = 0.4\sin 0.9 \quad (m)$$

and occurs when cos 4t = 1.

c) Both ends of the string must be at nodes. So that,

$$\frac{\lambda}{2} = \frac{1}{2}\frac{2\pi}{k} = \frac{\pi}{2}\ (m) \quad = 1.57\,m.$$

So, if the string includes only one loop, its length is L = 1.57 m. But it could be twice as long L = 3.14 m including two loops or any integral number times 1.57 m. (see Fig.)

d) The nodes occur at x = 0, x = 1.57 m and if the string is longer than 1.57 m at x = 3.14 m , 4.71 m and so on. The maximum amplitude (antinode) is 0.4 m and occurs midway between nodes. For L = 1.57 m, there is only one antinode at x = 0.79 m.

Problem 2-17. A piano string is 1.10 m long and has a mass of 9.00 g.
a) How much tension must the string be under if it is to oscillate at a fundamental frequency of 131 Hz?
b) What are the frequencies of the first four harmonics/

Answer: [$F = v^2\mu = 679\,N$; the frequencies of the second, third, and fourth harmonics are 2, 3, and 4 times the fundamental frequency (the first harmonic): 262, 393, and 524 Hz]

Problem 2-18. The equation of a transverse wave traveling along a very long string is given by $y = 6.0 \sin(0.02\,\pi x + 4.0\,\pi t)$ where x and y are expressed in cm and t in seconds.
Calculate the amplitude, the wavelength, the frequency, the speed, the direction of propagation of the wave, and the maximum transverse speed of a particle in the string.

Answer: [6.0 cm; 100 cm; 2.0 Hz; 200 cm/s; negative x-direction; 75 cm/s]

Problem 2-19. The linear density of an oscillating string is $1.3 \times 10^{-4}\,kg/m$. A transverse wave is propagating on the string and is described by the equation

$$y = 0.021 \sin(x + 30t),$$

where x and y are measured in meters and t in seconds.
What is the tension in the string?

[0.12 N]

Problem 2-20. Spherical waves are emitted from a 1.0-watt source in an isotropic non-absorbing medium. What is the wave intensity 1.0 m from the source?

[0.080 W/m^2]

Problem 2-21. A string oscillates according to the equation

$$y = 0.5 \sin\frac{\pi x}{3} \cos 40\pi t\,,$$

where x and y are in centimeters and t is in seconds.

a) What are the amplitude and velocity of the component waves whose superposition can give rise to this oscillation?
b) What is the distance between nodes/c) what is the velocity of a particle of the string at the position x = 1.5 cm when t = (9/8) s ?

[0.25 cm; 120 cm/s; 3.0 cm; zero]

Problem 2-22. A rope of mass 0.85 kg is stretched between two supports 30 m apart. If the tension in the rope is 120 N, how long will it take a pulse to travel from one support to the other?

[0.46 s]

Problem 2-23. Compare the intensities and the amplitudes of an earthquake wave as it passes two points 10 km and 20 km from the source.

[intensity drops to 1/4 and amplitude drops to 1/2]

Problem 2-24. The velocity of waves on a string is 480 m/s. If the frequency of standing waves is 131 Hz, how far apart are the nodes?

[1.83 m]

Problem 2-25. The displacement of a standing wave is given by

$$y = 8.6 \sin(0.6x) \cos(58\, t),$$

where x and y are in centimeters and t is in seconds.
a) What is the distance between nodes?
b) Give the amplitude, frequency, and speed of each of the component waves.
c) Find the speed of a particle of the string at $x = 3.2$ cm when $t = 2.5$ s.

[a) 5.2 cm; b) 4.3 cm, 9.2 Hz, 97 cm/s; c) 220 cm/s]

Problem 2-26. A particle violin string plays at a frequency of 294 Hz. If the tension in increased 10 percent, what will the new frequency be?

[308 Hz]

2.8 Sound Waves

Sound waves are longitudinal mechanical waves. The material particles transmitting such a wave oscillate in the direction of propagation of the wave itself. Sound cannot travel in the absence of matter. There is a large range of frequencies within which longitudinal mechanical waves can be generated. The range from about 20 Hz to about 20 kHz is called the **audible range**. A longitudinal mechanical wave whose frequency is below the audible range is called an **infrasonic wave**. This wave is usually generated by large sources, earthquake wave as an example. The wave whose frequency is above the audible range is called an **ultrasonic wave**. This wave may be produced by elastic oscillations of a quartz crystal induced by resonance with an applied alternating electric field (piezoelectric effect). In this way it is possible to produce ultrasonic frequencies as high as 6×10^8 *Hz*.

The **speed of sound** is different in different materials. The speed depends on the bulk (or elastic) modulus and the density of the material. In air and other fluids

$$v = \sqrt{\frac{B}{\rho}},$$

where B is the bulk modulus and ρ the density of the fluid.
If the medium is a solid the bulk modulus is replaced by elastic (Young's) modulus E.