2. TEMPERATURE AND HEAT

2.1 TEMPERATURE, THERMAL EXPANSION, IDEAL GAS LAW

In everyday life, **temperature** refers to how hot or cold an object is. In order to measure temperature quantitatively we use thermometers with various temperature scales. The conversion between Celsius temperature t (°C) and thermodynamic temperature T (K) is

$$t = T - 273.15$$

A change in temperature of one kelvin equals a change of one degree Celsius.
Most substances expand when heated and contract when cooled. This effect is called **thermal expansion.** The length of a solid rod is directly proportional to the temperature change $(t - t_0)$ and to its original length l_0 or

$$l = l_0[1 + \alpha(t - t_0)]$$

where α is the coefficient of linear expansion.
The change in volume of most solids is proportional to the temperature change $(t - t_0)$ and to the original volume V_0

$$V = V_0[1 + \beta(t - t_0)]$$

where the coefficient of volume expansion β is approximately equal to 3α.
Thermal stresses appear due to temperature change in materials with rigidly fixed ends. The force due to the thermal stress of a solid rod is

$$F = E.S.\alpha.\Delta t$$

where E is Young's modulus of the material.
The thermal expansion of liquids is described by the following equation

$$V = V_0\left(1 + At + Bt^2 + Ct^3\right)$$

where V_0 is the initial volume which corresponds to the temperature $0\,°C$. The values of coefficients A, B and C are found experimentally. The pressure which must be applied to compress the liquid to its original volume when the temperature of the liquid is increased from t_1 to t_2 is

$$\Delta P = K\left[A(t_2 - t_1) + B\left(t_2^2 - t_1^2\right) + C\left(t_2^3 - t_1^3\right)\right]$$

where K is the bulk modulus of the liquid.
The thermal expansion of gases is described by Gay-Lussac's law

$$\frac{P}{P_0} = \frac{T}{T_0}$$

where $T_0 = 273.15\ K$. Using the Celsius temperature scale we can rewrite this law as

$$P = P_0(1 + \gamma t) \quad \text{(volume is constant)}$$

where γ is the coefficient of thermal expansion of the gas. A similar effect is described by Charles's law

$$V = V_0(1 + \gamma t) \quad \text{(pressure is constant).}$$

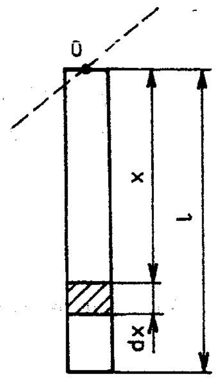
The equation of the state of a gas, called the **ideal gas law** has the form

$$PV = nRT$$

where n stands for the amount of substance and R is the universal gas constant.

Problem 2-1. A physical pendulum in the form of a thin rod changes in length from l to l_{30} owing to thermal expansion. The length of the rod at temperature t_1=20°C is l_{20}=0.6 m. Calculate the change in the period of the rod swing $\Delta\tau$ if the temperature is increased to t_2=30°C. ($\alpha = 12 \times 10^{-6}\ K^{-1}$).

Solution: The period of the physical pendulum is $\tau = \pi\sqrt{\dfrac{J}{m\,g\,l_t}}$ where J is the moment of inertia and l_t is the distance between the centre of mass and the point of suspension. We first determine the moment of inertia of the rod



$$J = \int \rho r^2 dV = \int_0^l \rho x^2 S dx = \rho S \frac{l^3}{3} = \frac{1}{3} m l^2$$

where S stands for the cross section of the rod and m for its mass. The mass of the rod is

$$m = \rho l.S$$

The change in the pendulum period due to the increase of temperature is

$$\Delta\tau = \tau_{30} - \tau_{20}$$

$$\tau_{30} = \pi\sqrt{\frac{1}{3}\frac{m\,l_{30}^2}{m\,g\,\frac{l_{30}}{2}}} = \pi\sqrt{\frac{2\,l_{30}}{3\,g}}$$

where $\dfrac{l_{30}}{2} = l_t$.

Similarly we have $\quad \tau_{20} = \pi\sqrt{\dfrac{1}{3}\dfrac{m\,l_{20}^2}{m\,g\,\frac{l_{20}}{2}}} = \pi\sqrt{\dfrac{2\,l_{20}}{3\,g}}$

Since the increase of the pendulum length due to heating is

$$l_{30} = l_{20}(1 + \alpha\Delta t)$$

we obtain for $\Delta\tau$

$$\Delta\tau = \pi\left(\sqrt{\frac{2l_{30}}{3g}} - \sqrt{\frac{2l_{20}}{3g}}\right) = \pi\left(\sqrt{\frac{2l_{20}(1+\alpha\Delta t)}{3g}} - \sqrt{\frac{2l_{20}}{3g}}\right) =$$

$$= \pi\sqrt{\frac{2l_{20}}{3g}}\left(\sqrt{1+\alpha\Delta t} - 1\right)$$

This expression may be simplified further using the first term of the binomial expansion

$$(1 \pm x)^n \approx 1 \pm nx \qquad \text{if } x \ll 1.$$

In our case we therefore have

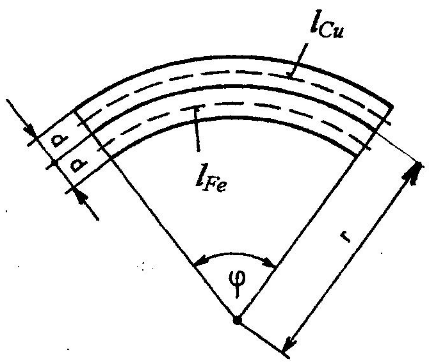
$$(1+\alpha\Delta t)^{\frac{1}{2}} \approx 1+\frac{1}{2}\alpha\Delta t$$

Finally, for $\Delta\tau$ we obtain

$$\Delta\tau=\pi\sqrt{\frac{2l_{20}}{3g}}\left(1+\frac{1}{2}\alpha\Delta t-1\right)=3.8\times10^{-5}\,s.$$

Problem 2-2. The bimetallic expansion strip in a thermostat consists of two metal strips of copper and iron riveted together to form a straight piece at temperature *t=0°C*. The thickness of both strips is the same *d=2 mm*. When the strip is heated it bends into an arc. Determine the radius of the arc when the temperature is *400°C*.
$(\alpha_{Cu}=17\times10^{-6}K^{-1};\alpha_{Fe}=12\times10^{-6}K^{-1})$.

Solution: Since $\alpha_{Cu}\rangle\alpha_{Fe}$ the centre of the arc will be on the side of the iron strip.



If the length of the strip at temperature $t=0°C$ is l_0 then at temperature $t=400°C$ the copper strip will have the length

$$l_{Cu}=l_0(1+\alpha_{Cu}\Delta t).$$

Similarly, for an iron strip we obtain

$$l_{Fe}=l_0(1+\alpha_{Fe}\Delta t).$$

The ratio of the lengths of the strips is therefore

$$\frac{l_{Cu}}{l_{Fe}}=\frac{1+\alpha_{Cu}\Delta t}{1+\alpha_{Fe}\Delta t}=(1+\alpha_{Cu}\Delta t)(1+\alpha_{Fe}\Delta t)^{-1}$$

To simplify the expression $(1+\alpha_{Fe}\Delta t)^{-1}$ we use the binomial expansion (see problem 2-1):

$$(1+\alpha_{Fe}\Delta t)^{-1}=1-\alpha_{Fe}\Delta t.$$

Thus we obtain

$$\frac{l_{Cu}}{l_{Fe}}=(1+\alpha_{Cu}\Delta t)(1+\alpha_{Fe}\Delta t)^{-1}=1+\alpha_{Cu}\Delta t-\alpha_{Fe}\Delta t-\alpha_{Fe}\alpha_{Cu}\Delta t^2=$$
$$=1+(\alpha_{Cu}-\alpha_{Fe})\Delta t$$

Since the term $(\alpha_{Fe}\alpha_{Cu}\Delta t^2)$ is much more smaller than the other terms, we can disregard it. As is seen from the figure, l_{Cu} and l_{Fe} are the lengths of the arcs of circles with radii $\left(r+\frac{d}{2}\right)$ and $\left(r-\frac{d}{2}\right)$ respectively. Expressing the angle φ in radians we have

$$l_{Cu}=\left(r+\frac{d}{2}\right)\varphi \qquad \text{and} \qquad l_{Fe}=\left(r-\frac{d}{2}\right)\varphi$$

Thus for the ratio $\frac{l_{Cu}}{l_{Fe}}$ we obtain

$$\frac{l_{Cu}}{l_{Fe}} = \frac{r+\frac{d}{2}}{r-\frac{d}{2}} = 1+(\alpha_{Cu}-\alpha_{Fe})\Delta t.$$

After a little rearrangement we finally obtain for r:

$$r = \frac{d}{(\alpha_{Cu}-\alpha_{Fe})\Delta t} + \frac{d}{2} = 1.001m$$

Problem 2-3. If the density of copper at temperature $t_1 = 20°C$ is $\rho_1 = 8.91\times 10^3\ kg.m^3$ what it will be its density at $t_2 = 70°C$. ($\alpha = 1.7\times 10^{-5} K^{-1}$).

Solution: The mass of the material does not change with temperature. Thus we can write

$$m = \rho_1 V_1 = \rho_2 V_2.$$

We can express the dependence of the volume on temperature as

$$V = V_0[1+\beta(t-t_0)]$$

where V_0 is the volume corresponding to the temperature $0°C$ and the coefficient of volume expansion β is approximately equal to 3α.
Substituting for volume into the first equation we have

$$\rho_1 V_0\,[1+\beta(t_1-t_0)] = \rho_2 V_0\,[1+\beta(t_2-t_0)]$$

Since $t_0 = 0°C$ we can write

$$\rho_2 = \rho_1\frac{1+\beta\, t_1}{1+\beta\, t_2}.$$

Taking into account that $\beta\, t_2 \langle\langle 1$ we have, using first term of the binomial expansion

$$(1+\beta t_2)^{-1} = 1-\beta\, t_2.$$

Substituting for the denominator into the expression for ρ_2 we obtain

$$\rho_2 = \rho_1(1+\beta t_1)(1-\beta t_2) \approx \rho_1[1-\beta(t_2-t_1)]$$

Since the term ($\beta^2 t_1 t_2$) is much smaller than the other terms we can disregard it.
Finally, for the density of copper at $70°C$ we obtain

$$\rho_2 = 8.91\times 10^3[1-3\times 1.7\times 10^{-5}(70-20)] = 8.89\times 10^3 kg.m^{-3}.$$

Problem 2-4. A horizontal iron rod is rigidly connected to two vertical walls. Calculate what stress is developed in the rod when the temperature increases by $5\,°C$. ($E = 2\times 10^{11}\ Pa$; $\alpha = 12\times10^{-6}\ K^{-1}$).

Solution: If the free rod is heated it increases its length as

$$\Delta l = l_0\, \alpha\, \Delta t.$$

We can express the strain as

$$\frac{\Delta l}{l_0} = \alpha\, \Delta t$$

If the rigid walls do not allow the expansion of the bar, a stress appears in the bar. This stress is equal to the stress that is required to compress the bar to its original length. Thus, following Hooke's law, we can write

$$\frac{\Delta l}{l_0} = \frac{\sigma}{E}$$

From a comparison of the foregoing two equations we obtain for the required stress

$$\sigma = \alpha\, E\, \Delta t = 12\times 10^{-6}\times 2\times 10^{11}\times 5 = 1.2\times 10^{7}\ Pa$$

Problem 2-5. At what temperature does the specific volume of water have the minimum value? For water within the temperature range 0 - 33°C coefficients A, B and C have the following values: $A = -\ 6.427\times 10^{-5}\ K^{-1}$; $B = 8.5053\times 10^{-6}\ K^{-2}$; $C = -\ 6.79\times 10^{-8}\ K^{-3}$.

Solution: We have to find the extreme of the function

$$V = V_0\left(1 + At + Bt^2 + Ct^3\right) \qquad (1)$$

Equalling the first derivative to zero

$$\frac{dV}{dt} = V_0\left(A + 2Bt + 3Ct^2\right) = 0$$

and taking into account that $V_0 \neq 0$, we obtain a quadratic equation. The roots of this equation are

$$t_{1,2} = \frac{-\ B \pm \sqrt{B^2 - 3AC}}{3C}$$

From this equation we find

$$t_1 = 3.98\,°C \quad \text{and} \quad t_2 = 79.5°C.$$

Since the temperature t_2 is outside the definition range of coefficients A, B and C, the temperature corresponding to the minimum specific volume of water is $3.98\,°C$. To be sure that t_1 is the minimum, we take the second derivative of equation 1 with respect to the temperature or

$$\frac{d^2V}{dt^2} = V_0\left(2B + 6Ct\right).$$

Substituting the values of coefficients B and C and temperature t_1 into the previous equation, we obtain

$$\frac{d^2V}{dt^2} \rangle 0.$$

We see that temperature t_1 corresponds to the minimum of specific volume of water within this temperature range.

Problem 2-6. Derive the expression for the density of a gas as a function of its temperature and pressure.

Solution: Density is defined as $\rho = \frac{m}{V}$. We know that with increasing temperature the mass of a gas does not change. The volume of the gas can be expressed by the ideal gas law:

$$\frac{P.V}{T} = \frac{P_0.V_0}{T_0} \qquad \Rightarrow \qquad V = \frac{P_0.V_0.T}{T_0.P}.$$

Substituting for volume into the expression for density we have

$$\rho = \frac{m}{V} = m\frac{T_0.P}{P_0.V_0.T} = \frac{m}{V_0}\frac{P.T_0}{P_0.T} = \rho_0\frac{P.T_0}{P_0.T}$$

The thermodynamic temperature T can be converted to Celsius temperature t, using the formula $T = T_0 + t$. Thus we have

$$\rho = \rho_0\frac{P.T_0}{P_0.(T_0+t)} = \rho_0\frac{P}{P_0}\frac{1}{\left(1+\frac{t}{T_0}\right)} = \rho_0\frac{P}{P_0}\frac{1}{(1+\gamma t)}$$

where $\gamma = 1/273.15$ is the expansion coefficient.

Problem 2-7. One gram of oxygen is enclosed in a vessel at a pressure of *1 MPa* and at a temperature of *47°C*. Due to leakage in the enclosure of the vessel, the amount of oxygen in the vessel decreases in such a way that after a certain time the pressure in the vessel is *5/8* of the initial pressure and temperature is decreased to *27°C*. Determine the volume of the vessel and the amount of oxygen that has flowed out of the vessel. ($M_{O_2} = 32\,kg.kmol^{-1}$).

Solution: In order to apply the ideal gas law the temperature must be converted from Celsius to the thermodynamic temperature.

Initial state: $\left[m_1 = 10^{-3}\,kg;\ P_1 = 10^6\,Pa;\ T_1 = 320\,K\right]$

The volume remains constant!

Final state: $\left[m_2 = ?; P_2 = \frac{5}{8} P_1; T_2 = 300K\right]$

For the initial state we have

$$P_1.V = \frac{m_1}{M} R.T_1$$

The volume of the vessel is therefore

$$V = \frac{m_1}{M} \frac{R\ T_1}{P_1} = \frac{10^{-3} \times 8.3 \times 10^3 \times 320}{32 \times 1 \times 10^6} = 83 \times 10^{-6}\ m^3$$

For the final state we have

$$P_2.V = \frac{m_2}{M} R.T_2$$

From this equation we determine the mass of oxygen in the final state

$$m_2 = \frac{P_2.M}{R.T_2} V$$

Substituting for volume into this equation we have

$$m_2 = \frac{P_2.M}{R.T_2} \frac{m_1.R.T_1}{M.P_1} = \frac{P_2.m_1.T_1}{T_2.P_1} = \frac{\frac{5}{8} P_1 \times 10^{-3} \times 320}{300 \times P_1} = 0.666 \times 10^{-3}\ kg$$

Thus the amount of oxygen that flowed out of the vessel is

$$\Delta m = 1 - 0.666 = 0.334\ g.$$

Problem 2-8. A tank of the volume $V=20\ dm^3$ contains compressed air at a pressure $P=20$ *MPa* and at a temperature $t=20\,°C$. Calculate the volume of water that can be replaced by the air from the tank, if the tank is submerged *30 m* below the surface of the water and the temperature is $5\,°C$. The barometric pressure is $1.03 \times 10^5 Pa$; $\rho_{H_2O} = 10^3\ kg.m^{-3}$.

Solution: In order to apply the ideal gas law, the temperature must be converted from degrees Celsius to the thermodynamic temperature.

Initial state: $\left[V_1 = 20\ dm^3;\ P_1 = 2 \times 10^7\ Pa;\ T_1 = 293\ K\right]$

The amount of air is constant!

Final state: $\left[V_2 = ?;\ P_2 = P_{Bar} + \rho g h;\ T_2 = 278\ K\right]$

Pressure P_2 is equal to the sum of the barometric and hydrostatic pressures or

$$P_2 = P_{Bar} + \rho g h = 1.03 \times 10^5 + 10^3 \times 9.81 \times 30 = 4 \times 10^5\ Pa$$

We use the ideal gas law for each of the states

$$P_1.V_1 = nR.T_1 \qquad\qquad P_2.V_2 = nR.T_2$$

Since the amount of air remains constant we can write

$$n.R = \frac{P_1.V_1}{T_1} = \frac{P_2.V_2}{T_2}$$

From this equation we determine V_2:

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$$V_2 = \frac{P_1}{T_1}\frac{V_1}{}\frac{T_2}{P_2} = \frac{2\times 10^7 \times 20}{293}\ \frac{278}{4\times 10^5} = \cancel{940\,dm}^3$$

Problem 2-9. Calculate the buoyant lift of a balloon filled with hydrogen at a height of *6000 m*, temperature $t = 0\,^{\circ}C$ and external pressure $P = 51\,750\ Pa$. The volume of the balloon is $V = 3\,000\ m^3$. The density of air under STP (standard temperature and pressure) is $\rho_0 = 1.293\ kg.m^{-3}$; the molar weight of hydrogen is *2 kg/kmol.*

Solution: The buoyant force of the balloon is equal to the weight of the air displaced by the balloon. If the buoyant force is greater than the weight of the balloon, then the balloon will fly. The weight of the air displaced by the balloon is $G_1 = \rho_1 V.g$ where ρ_1 is the density of air at corresponding temperature. For the weight of the balloon we take the weight of the hydrogen $G_2 = \rho_2 V.g$, where ρ_2 is the density of the hydrogen. The buoyant lift is equal to the difference between the buoyant force and the weight of the balloon or

$$F = G_1 - G_2 = V.g(\rho_1 - \rho_2)$$

We can use the ideal gas law to determine the density at a given temperature and pressure

$$\rho = \frac{m}{V} = \frac{P.M}{R.T}$$

Substituting corresponding values for hydrogen, we have

$$\rho_2 = \frac{51\,750 \times 2}{8.314 \times 273} = 0.0456\ kg.m^{-3}$$

The density of air at STP is

$$\rho_0 = \frac{P_0 M}{R\,T_0}.$$

The density of air at temperature T_0 and pressure P is

$$\rho_1 = \frac{P\,M}{R\,T_0}$$

We divide the previous two equations and obtain

$$\rho_1 = \rho_0 \frac{P}{P_0} = 1.293 \frac{51\,750}{1.035 \times 10^5} = 0.6465\, kg.m^{-3}.$$

Finally, for buoyant lift we obtain

$$F = 3\,000 \times 9.81\,(0.6465 - 0.0456) = 17\,684\ N.$$

Problem 2-10. A wheel of a coach wagon has a radius of $r_0=0.5\ m$ at temperature $t=0\,°C$. Calculate the difference between the number of revolutions of the wheel in summer at temperature $t_1=25\,°C$ and in winter at temperature $t_2=-25°C$ if the distance travelled by the wagon is $L=100\ km$. ($\alpha = 1.2\times 10^{-5}\ K^{-1}$).

$$\left[n_2 - n_1 = \frac{L}{2\pi r_0}\alpha(t_1 - t_2) = 19.1\right]$$

Problem 2-11. The $70\ l$ steel gas tank of a car is filled to the top with gasoline at temperature $20\,°C$. The car is then left in the sun and the tank reaches a temperature of $50\,°C$. How much gasoline do you expect to overflow from the tank?
($\beta_{gasoline} = 950\times 10^{-6}\ K^{-1}$; $\alpha_{Fe} = 12\times 10^{-6}\ K^{-1}$).

$$\left|\Delta V = V.\Delta t(\beta_{gasoline} - 3\alpha_{Fe}) = 1.925\,l\right|$$

Problem 2-12. An iron bar of cross section $S=2\ cm^2$ is heated from the temperature $t=0\,°C$ to the temperature $t=50\,°C$ and then it is suddenly cooled to the initial temperature. Calculate the minimum force acting in the direction of the axis of the bar preventing its contraction from the state when it was heated. ($E=2\times 10^{11}\ Pa$; $\alpha_{Fe} = 12\times 10^{-6}\ K^{-1}$).

$$[F = E.S.\Delta t.\alpha = 24\,000\ N]$$

Problem 2-13. How many molecules do you breathe in $1\ l$ of air?

$$[n = 2.7\times 10^{22}\ molecules]$$

Problem 2-14. The density of argon at temperature $27\,°C$ and pressure $82\,460\ Pa$ is $1.6\ kg.m^{-3}$. Calculate the mass of argon in a light bulb if the temperature inside the bulb is $127\,°C$, the pressure is $99\,750\ Pa$ and the volume of the light bulb is $100\ cm^3$.

$$[m = 0.145\ g]$$

Problem 2-15. An automobile tire is filled to a pressure of $200\ kPa$ at $10\,°C$. After driving $100\ km$ the temperature within the tire rises to $40\,°C$. What is the pressure within the tire now?

$$[p = 0.212\ Mpa]$$