3. ELECTROMAGNETIC WAVES

3.1 MAXWELL'S EQUATIONS

Maxwell's equations are a set of four equations containing the most important laws of electricity and magnetism. The properties of electromagnetic waves are also described by these equations. Maxwell's equations are usually written in two equivalent forms, integral and differential.

	Integral form	Differential form
1.	$\oint_L \vec{B}\, d\vec{l} = \iint_S \left(\mu_0 \vec{j} + \mu_0 \frac{\partial \vec{D}}{\partial t} \right) d\vec{S}$	$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$
2.	$\oint_L \vec{E}\, d\vec{l} = -\frac{d}{dt} \iint_S \vec{B}\, d\vec{S}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
3.	$\oiint_S \vec{D}\, d\vec{S} = \iiint_V \rho\, dV$	$\nabla . \vec{E} = \frac{\rho}{\varepsilon_0}$
4.	$\oiint_S \vec{B}\, d\vec{S} = 0$	$\nabla . \vec{B} = 0$

These four fundamental equations are usually written together with the following four equations, which express the influence of the medium on the electric and magnetic fields:

1.	$\vec{j} = \sigma \vec{E}$	Ohm's law
2.	$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$ or $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$	————
3.	$\vec{B} = \mu_0 \mu_r \vec{H}$ or $\vec{B} = \mu_0 \vec{H} + \vec{J}$	————
4.	$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$	Lorentz's force

3.2. PROPERTIES OF ELECTROMAGNETIC WAVES

From the standpoint of technical applications, the study of electromagnetic waves in a nonconducting homogeneous and isotropic medium without electric charges and permanent magnets is very important.
The **wave equations** for electric and magnetic components of an electromagnetic wave are respectively

$$\Delta \vec{E} = \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \Delta \vec{H} = \varepsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2}.$$

The velocity of electromagnetic wave propagation is

$$c = \frac{1}{\sqrt{\varepsilon \mu}}$$

Electromagnetic waves far away from the source are **plane transverse** waves. For a wave travelling along the x axis, the relation between the electric and the magnetic component is

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} E_y \left(t - \frac{x}{c} \right) \qquad H_y = - \sqrt{\frac{\varepsilon}{\mu}} E_z \left(t - \frac{x}{c} \right)$$

The angle between the electric $\vec{E}$ and magnetic $\vec{H}$ component of an electromagnetic wave is $\frac{\pi}{2}$.

Poynting's vector is defined as the rate of energy transport per unit area and per unit of time, or

$$\vec{S} = \vec{E} \times \vec{H} \quad [W/m^2].$$

The conservation of energy law for an electromagnetic field is

$$div\, \vec{S} = - \frac{\partial w_{em}}{\partial t}$$

where w_{em} is the **density of energy** of the electromagnetic field:

$$w_{em} = \frac{1}{2}\left[\vec{E} . \vec{D} + \vec{H} . \vec{B} \right] \qquad [J/m^3].$$

The intensity of an electromagnetic wave is defined as the power transmitted in one period per unit area perpendicular to the energy flow:

$$\vec{I} = \frac{1}{T} \int_0^T \vec{S}\, dt \qquad [W/m^2].$$

3.3. WAVE NATURE OF LIGHT

Visible light constitutes only a small fraction of the spectrum of electromagnetic waves ranging from $3.9x10^{-7}$ to $7.6x10^{-7}$ m. The wave theory of light is based on **Huyghens' principle**:

Every point on a wavefront may be considered as a source of secondary spherical wavelets which spread out in the forward direction at the speed of light in the medium. The new wavefront is then the surface tangent to all these secondary wavelets.

A wavefront is a surface joining all adjacent points on a wave which have the same phase.

Huyghens' principle immediately leads to a satisfactory explanation of the reflection and refraction of light.

Diffraction is the bending of waves into the shadow region when they pass through holes or slits comparable in size to the wavelength. It comes from the superposition of an infinite number of wavelets arising from different places on the same wavefront (e.g. single slit diffraction).

Interference is the superposition of two or more beams of light so that they reinforce each other (constructive interference - bright fringes) or cancel each other out (destructive interference - dark fringes). It arises from the superposition of a finite number of waves coming from different coherent sources (double slit experiment).

Dispersion of light is the dependence of the index of refraction of a material on wavelength, leading to a difference in the angle of refraction.

Single-slit diffraction:

We find that dark fringes occur at distances: $$y_n = \frac{\lambda\, L}{w}\, n$$

where L is the distance between the screen and the slit, w is the width of the slit, λ is the wavelength of the light and n is an integral number.

Double slit experiment:

We find bright fringes at distances $$y_{\max} = \frac{x}{d}\, k\, \lambda,$$

and we find dark fringes at distances $$y'_{\min} = \frac{x}{d}\left(2k + 1\right)\frac{\lambda}{2}$$

where d is the distance between the centres of the slits, λ is the wavelength of the light and $k=0,1,2, \dots$

Interference by thin film:

Destructive interference occurs in reflected light and constructive interference occurs in transmitted light if

$$2d\sqrt{n_2^2 - n_1^2 \sin^2\alpha} = k\lambda .$$

Constructive interference occurs in reflected light and the destructive interference occurs in transmitted light if

$$2d\sqrt{n_2^2 - n_1^2 \sin^2\alpha} = \left(2k + 1\right)\frac{\lambda}{2}$$

where d is the thickness of the film, n_2 is the index of refraction of the film, α is the angle of incidence, λ is the wavelength of the light and $k=0,1,2, \dots$.

Diffraction grating:

The grating equation for principal maxima:

$$\sin\varphi = \frac{m\lambda}{d} \qquad m=0,1,2,\dots .$$

Polarisation of light:

A linearly polarised light wave is a wave in which all vectors of intensity of the electric field are oriented in the same direction.

A circularly polarised light wave is a wave in which the endpoints of the vector of intensity of the electric field rotate in a circle.

Complete **polarisation by reflection** is described by Brewster's law:

$$tg\alpha_B = \frac{n_2}{n_1}$$

where α_B is the angle of incident light (called Brewster's angle) for which complete polarisation occurs.

Problem 3-1. For a nonconducting homogeneous and isotropic medium, prove the existence of electromagnetic waves and find the velocity of their propagation.

Solution: To prove the existence of electromagnetic waves we will derive the wave equation for the electric and magnetic component of the wave, and from comparison with the general wave equation we will find the velocity of propagation of each component of the wave.

For a nonconducting homogeneous and isotropic medium we can write:

conductivity $\sigma = 0$,

volume density of electric charges $\rho = 0$,

$\vec{D} = \varepsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$.

Substituting these assumptions into the differential form of Maxwell's equations we obtain:

$$\nabla\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} \quad (1) \qquad \nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \quad (2)$$

$$\nabla.\vec{E} = 0 \quad (3) \qquad \nabla.\vec{H} = 0 \quad (4)$$

Taking the first derivative of Eq.1 and reversing the order of derivation we have

$$\nabla\times\frac{\partial\vec{H}}{\partial t} = \varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad (5)$$

Substituting Eq. 2 into Eq.5 we obtain

$$\nabla\times\left(\nabla\times\vec{E}\right) = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad (6)$$

From vector analysis we can use the following identity

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla . \vec{E}\right) - \Delta \vec{E}$$

Taking into account this identity and Eq. 6 we can write

$$\nabla\left(\nabla . \vec{E}\right) - \Delta \vec{E} = - \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}.$$

With respect to Eq.3 we therefore obtain

$$\Delta \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{7}$$

We can now repeat the same procedure starting from Eq.2. If we take the first derivative of this equation, and substitute for $\frac{\partial \vec{E}}{\partial t}$ from Eq. 1 we have

$$\Delta \vec{H} = \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}. \tag{8}$$

Eqs. 7 and 8 have the form of wave equations. Hence each of the components of the electromagnetic wave is described by the wave equation, therefore the existence of electromagnetic waves has been proved.

To find the velocity of propagation of these waves it is sufficient to compare Eq.7 and 8 with the general wave equation. From this comparison we find that the electric and magnetic components of electromagnetic wave propagate with the same velocity, or

$$v = \frac{1}{\sqrt{\mu \varepsilon}} \tag{9}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r$ and $\mu = \mu_0 \mu_r$. For a vacuum we have $\varepsilon_r = \mu_r = 1$, consequently the velocity of electromagnetic wave propagation is

$$v = c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^8 \ m/s. \tag{10}$$

The velocity of electromagnetic waves in a vacuum is a certain ultimate speed. We can rewrite Eq.9 into the following form

$$v = \frac{c_0}{\sqrt{\mu_r \varepsilon_r}} \tag{11}$$

Taking into account that for nonconducting media $\mu_r \cong 1$ and $\varepsilon_r > 1$ we see that the velocity of electromagnetic waves in these types of media is

$$v = \frac{c_0}{\sqrt{\varepsilon_r}} \tag{12}$$

Problem 3-2. Prove that an electromagnetic wave is a transverse wave.

Solution: The wave equations which we derived in problem 3-1 describe all types of electromagnetic waves in a homogeneous isotropic medium without charges and without permanent magnets. The most important and at the same time the simplest type of electromagnetic waves are so called plane waves, the wavefronts of which are planes. In fact most waves that are sufficiently far from the source can be considered as plane waves.

Let us assume that we have a plane wave travelling along the x axis, hence the wavefront is perpendicular to this axis. Thus the vectors $\vec{E}$ and $\vec{H}$ are the functions of x and t only, or

$$\vec{E} = \vec{E}(x,t) \qquad \text{and} \qquad \vec{H} = \vec{H}(x,t) \tag{13}$$

Wave equations can therefore be rewritten into a simpler form

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \mu\,\varepsilon\,\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \frac{\partial^2 \vec{H}}{\partial x^2} = \mu\,\varepsilon\,\frac{\partial^2 \vec{H}}{\partial t^2} \tag{14}$$

For further calculation it is useful to write Maxwell's equations 1-4 into components. The wave is a plane wave. Therefore following Eq. 13 partial derivatives with respect to y and z equal zero. Maxwell's equations therefore have the following form:

$$0 = \frac{\partial E_x}{\partial t} \tag{15}$$

$$-\frac{\partial H_z}{\partial x} = \varepsilon\,\frac{\partial E_y}{\partial t} \tag{16}$$

$$\frac{\partial H_y}{\partial x} = \varepsilon\,\frac{\partial E_z}{\partial t} \tag{17}$$

$$0 = \frac{\partial H_x}{\partial t} \tag{18}$$

$$\frac{\partial E_z}{\partial x} = \mu\,\frac{\partial H_y}{\partial t} \tag{19}$$

$$\frac{\partial E_y}{\partial x} = -\mu\,\frac{\partial H_z}{\partial t} \tag{20}$$

$$\frac{\partial E_x}{\partial x} = 0 \tag{21}$$

$$0 = \frac{\partial H_x}{\partial x} \tag{22}$$

From Eqs. 15 and 21 it is obvious that the first derivative of E_x both with respect to time t and with respect to x equals zero. Since for plane waves travelling along the direction of the x axis no component of $\vec{E}$ and $\vec{H}$ can depend either on y or on z, obviously E_x must be constant. We can arrive at the same conclusion for H_x from Eqs. 18 and 22. A constant electric and magnetic field can be produced by electric charges or permanent magnets, respectively. Since following our assumptions there are no charges or permanent magnets, both E_x and H_x equal zero.

For the plane electromagnetic wave travelling along the x axis we arrived at the conclusion that

$$E_x = 0 \tag{23}$$

$$H_x = 0 \tag{24}$$

We can therefore conclude that the nonzero components of the electromagnetic wave are only E_y, E_z, H_y and H_z which are perpendicular to the direction of wave propagation. Hence an electromagnetic wave is a transverse wave.

Problem 3-3. Find the relation between the components of vectors $\vec{E}$ and $\vec{H}$ for a plane electromagnetic wave.

Solution: To find the components of vectors $\vec{E}$ and $\vec{H}$ for a plane electromagnetic wave it is necessary to rewrite into components the wave equations expressed by Eq. 14:

$$\frac{\partial^2 E_x}{\partial x^2} = \mu\,\varepsilon \frac{\partial^2 E_x}{\partial t^2} \qquad (25) \qquad\qquad \frac{\partial^2 H_x}{\partial x^2} = \mu\,\varepsilon \frac{\partial^2 H_x}{\partial t^2} \qquad (28)$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu\,\varepsilon \frac{\partial^2 E_y}{\partial t^2} \qquad (26) \qquad\qquad \frac{\partial^2 H_y}{\partial x^2} = \mu\,\varepsilon \frac{\partial^2 H_y}{\partial t^2} \qquad (29)$$

$$\frac{\partial^2 E_z}{\partial x^2} = \mu\,\varepsilon \frac{\partial^2 E_z}{\partial t^2} \qquad (27) \qquad\qquad \frac{\partial^2 H_z}{\partial x^2} = \mu\,\varepsilon \frac{\partial^2 H_z}{\partial t^2} \qquad (30)$$

If we restrict our considerations to the case of a wave travelling along the positive direction of the x axis only, the solution of these equations is the following:

A. Equations 25 and 28 (taking into account that an electromagnetic wave is a transverse wave):

$$E_x = 0 \qquad\qquad H_x = 0$$

B. Remaining equations:

$$E_y = E_y\left(t - \frac{x}{c}\right) \qquad\qquad H_y = H_y\left(t - \frac{x}{c}\right)$$

$$(31)$$

$$E_z = E_z\left(t - \frac{x}{c}\right) \qquad\qquad H_z = H_z\left(t - \frac{x}{c}\right)$$

where E_y, E_z, H_y and H_z are arbitrary functions of the argument $\left(t - \frac{x}{c}\right)$. These functions must at the same time obey Maxwell's equations for a plane wave, i.e. Eqs. 15 - 22. Hence we can arbitrary choose only two of them and the remaining two have to be found from Maxwell's equations.

Let us therefore assume that we know $E_y\left(t - \frac{x}{c}\right)$ and $E_z\left(t - \frac{x}{c}\right)$. The component of magnetic field strength H_z can be found from Eq. 16 (or Eq. 20):

$$-\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial}{\partial t}\left[E_y\left(t - \frac{x}{c}\right)\right] = \varepsilon\, E_y'\left(t - \frac{x}{c}\right)$$

Integrating this equation we obtain

$$H_z = -\varepsilon \int E_y'\left(t - \frac{x}{c}\right)dx + C$$

To solve this integral we use the substitution $q = \left(t - \frac{x}{c}\right)$. Thus we have $dx = -c\,dq$. The integration constant C constitutes the static component of the magnetic field, which equals zero because there are no charges and no permanent magnets.

$$H_z = \varepsilon c \int \frac{dE_y(q)}{dq} dq = \varepsilon\, cE_y\left(t - \frac{x}{c}\right) = \sqrt{\frac{\varepsilon}{\mu}} E_y\left(t - \frac{x}{c}\right)$$

Similarly from a known component of electric field $E_z\left(t - \frac{x}{c}\right)$ we can find from Eq. 17 (or from Eq. 19) the component of magnetic field strength H_y:

$$H_y = -\sqrt{\frac{\varepsilon}{\mu}} E_z\left(t - \frac{x}{c}\right)$$

Finally we can conclude:

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} E_y \qquad\qquad H_y = -\sqrt{\frac{\varepsilon}{\mu}} E_z \qquad (32)$$

For a wave travelling in the opposite direction we can obtain the following results:

$$H_z = -\sqrt{\frac{\varepsilon}{\mu}} E_y \qquad\qquad H_y = \sqrt{\frac{\varepsilon}{\mu}} E_z \qquad (33)$$

Problem 3-4. Calculate the characteristic impedance (wave resistance) of the plane electromagnetic wave in a vacuum.

Solution: The electric component of an electromagnetic wave travelling along the positive direction of the x axis can be expressed as

$$\vec{E} = \vec{j}E_y + \vec{k}E_z$$

The magnitude of intensity of the electric field is $E = \sqrt{E_y^2 + E_z^2}$

Similarly for the magnetic component we have $\vec{H} = \vec{j}H_y + \vec{k}H_z$

$$H = \sqrt{H_y^2 + H_z^2} = \sqrt{\frac{\varepsilon}{\mu}}\sqrt{E_z^2 + E_y^2} = \sqrt{\frac{\varepsilon}{\mu}} E$$

The ratio $\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$ for a given medium is constant, and it is called the characteristic impedance. For a vacuum it has the following value:

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 377\,\Omega$$

Problem 3-5. Prove that the electric and magnetic field vectors in a plane wave are perpendicular to each other.. Assume a wave traveling along the positive direction of the x axis.

Solution: We can find the angle between vectors $\vec{E}$ and $\vec{H}$ for a plane electromagnetic wave using the properties of the dot product of two vectors:

$$\vec{E}\vec{H} = E.H\cos\alpha$$

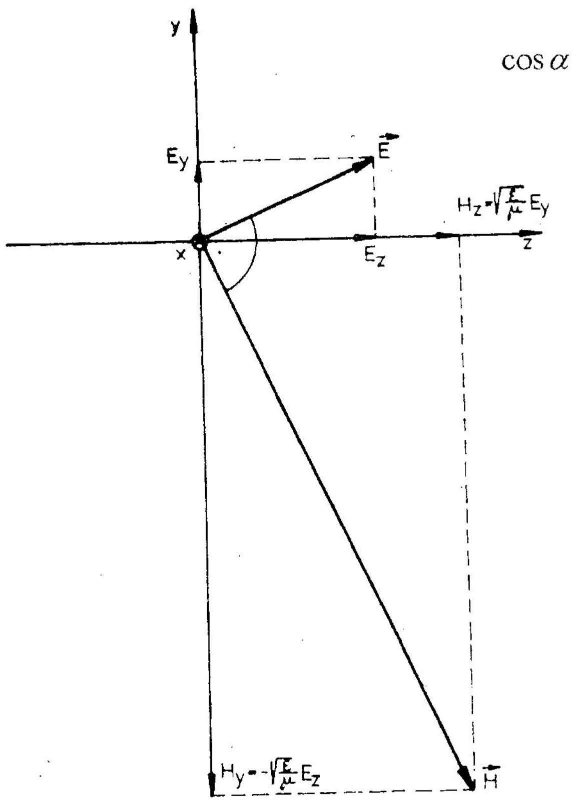
Thus we have

$$\cos\alpha = \frac{\vec{E}\vec{H}}{EH} = \frac{\left(\vec{j}E_y + \vec{k}E_z\right)\left(\vec{j}H_y + \vec{k}H_z\right)}{EH}$$

For components H_y and H_z we can substitute from Eqs. 32. Thus we obtain:

$$\cos\alpha = \frac{\left(\vec{j}E_y + \vec{k}E_z\right)\left(-\vec{j}E_z + \vec{k}E_y\right)\sqrt{\frac{\varepsilon}{\mu}}}{EH} =$$

$$= \frac{\left(-E_yE_z + E_yE_z\right)\sqrt{\frac{\varepsilon}{\mu}}}{EH} = 0$$



Hence we arrive at the conclusion that, for a plane electromagnetic wave, vectors $\vec{E}$ and $\vec{H}$ are perpendicular to each other. These two vectors are oriented in such a way that the resultant vector given by the cross product of these two vectors points in the direction of the wave propagation. The position of vectors $\vec{E}$ and $\vec{H}$ can be seen in the figure. The positive direction of the x axis is inward.

Problem 3-6. Calculate the intensity of a linearly polarised plane sinusoidal electromagnetic wave traveling along the positive direction of the x axis.

Solution: We can express Poynting's vector for a linearly polarised plane sinusoidal electromagnetic wave traveling along the positive direction of the x axis as

$$\vec{S} = \vec{E} \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & E_y & E_z \\ 0 & H_y & H_z \end{vmatrix} = \vec{i}\left(E_yH_z - E_zH_y\right) = \vec{i}\sqrt{\frac{\varepsilon}{\mu}}\left(E_y^2 + E_z^2\right) = \vec{i}\sqrt{\frac{\varepsilon}{\mu}}E^2$$

We see that Poynting's vector has the same orientation as the positive direction of the x axis. For a linearly polarised plane sinusoidal wave, both components of the intensity of the electric field are in phase, but they can have different magnitudes. Thus we can write

$$E_y = A\sin\omega\left(t - \frac{x}{c}\right) \qquad\qquad E_z = B\sin\omega\left(t - \frac{x}{c}\right)$$

For Poynting's vector we therefore obtain

$$\vec{S} = \vec{i}\sqrt{\frac{\varepsilon}{\mu}}\left(A^2 + B^2\right)\sin^2\omega\left(t - \frac{x}{c}\right)$$

The intensity of the wave is given by the following expression

$$\vec{I} = \frac{1}{T}\int_0^T \vec{S}dt$$

Substituting for Poynting's vector we obtain

$$I = \frac{1}{T}\int_0^T Sdt = \frac{1}{T}\sqrt{\frac{\varepsilon}{\mu}}\left(A^2 + B^2\right)\int_0^T \sin^2\omega\left(t - \frac{x}{c}\right)dt = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(A^2 + B^2\right)$$

Notice:

$$\int_0^T \sin^2\omega\left(t - \frac{x}{c}\right)dt = \int_0^T \frac{1 - \cos 2\omega\left(t - \frac{x}{c}\right)}{2}dt = \frac{T}{2} - \frac{1}{2}\int_0^T \cos 2\omega\left(t - \frac{x}{c}\right)dt =$$

$$= \frac{T}{2}$$

Problem 3-7. Calculate the intensity of the nonpolarised sinusoidal electromagnetic wave travelling along the positive direction of the x axis.

Solution: For a nonpolarised plane sinusoidal wave the y and z components of the intensity of the electric field are not in phase, and they can have different magnitudes. Thus we can write:

$$E_x = 0$$

$$E_y = A\sin\omega\left(t - \frac{x}{c}\right) \qquad\qquad E_z = B\sin\left[\omega\left(t - \frac{x}{c}\right) + \varphi\right]$$

For Poynting's vector we have

$$\vec{S} = \sqrt{\frac{\varepsilon}{\mu}}E^2\vec{i} = \sqrt{\frac{\varepsilon}{\mu}}\left\{A^2\sin^2\omega\left(t - \frac{x}{c}\right) + B^2\sin^2\left[\omega\left(t - \frac{x}{c}\right) + \varphi\right]\right\}\vec{i}$$

Finally for the intensity of the nonpolarised wave we have

$$I = \frac{1}{T}\int_0^T S dt = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(A^2 + B^2\right)$$

Comparison with the expression for the intensity of the polarised wave shows that both expressions are the same. Hence we can conclude that the intensity of the wave does not depend on the phase shift between the components of the intensity of the electric field, which are perpendicular to each other.

Problem 3-8. Solar electromagnetic radiation intercepted by the earth has an average energy flux *1353 J/s.m²* (assuming that the surface is perpendicular to the energy flow). Calculate the effective values of the intensity of the electric field and the magnetic field strength.

Solution: The earth's radius is very small in comparison with the distance from the sun, so we may assume that the radiation is in the form of plane waves by the time it reaches us. The power transmitted by radiation per unit area is in fact equal to the average value of Poynting's vector, for which we have obtained (see problem 3-6):

$$S = \sqrt{\frac{\varepsilon}{\mu}} E^2$$

The average value of the transmitted power is equal to the intensity of the wave for which we have obtained

$$I = \frac{1}{T}\int_0^T S dt$$

Substituting for S we have

$$I = \frac{1}{T}\int_0^T S dt = \sqrt{\frac{\varepsilon}{\mu}} \frac{1}{T}\int_0^T E^2 dt$$

Taking into account that

$$E_{ef}^2 = \frac{1}{T}\int_0^T E^2 dt \qquad \text{and} \qquad \frac{E_{ef}}{H_{ef}} = \sqrt{\frac{\mu}{\varepsilon}}$$

we can write

$$I = \sqrt{\frac{\varepsilon}{\mu}} E_{ef}^2 = E_{ef} H_{ef} = \sqrt{\frac{\mu}{\varepsilon}} H_{ef}^2$$

Hence for effective values of the electric and magnetic component of the wave we obtain

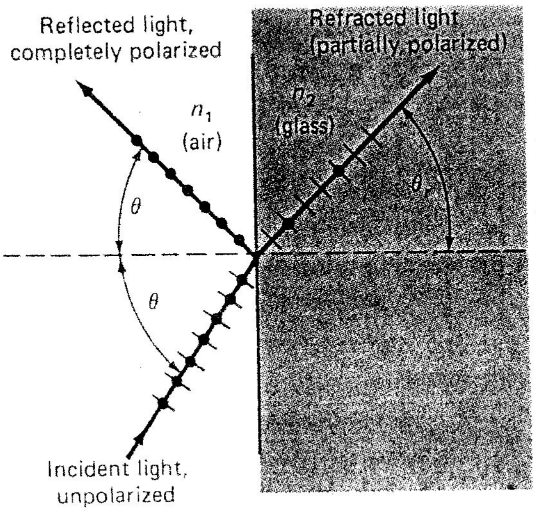
$$E_{ef} = \sqrt{I\sqrt{\frac{\mu_0}{\varepsilon_0}}} = \sqrt{1395 \times 377} = 724\, V.m^{-1}$$

$$H_{ef} = \sqrt{I\sqrt{\frac{\varepsilon_0}{\mu_0}}} = \sqrt{\frac{1395}{377}} = 1.92\, A.m^{-1}$$

Problem 3-9. Find the angle of incidence θ_B for which complete polarisation of reflected light occurs.

Solution. If unpolarised light strikes a non-metallic material such as glass or wate, the wave becomes partially polarised by reflection.

The Figure shows an unpolarised light beam, incident on a sheet of glass at angle of incidence θ. The components of the vector of intensity of an electric field in the plane of incidence are indicated by short lines, and points indicate the components perpendicular to the plane of incidence. It is found by experiment that the reflected light is partially linearly polarised, i.e. the vectors of an intensity of electric field point in all directions normal to the direction of propagation, but the amplitudes of these vibrations are greater parallel to the surface of the glass (i.e. normal to the plane of the paper) than they are at right angles to the plane.



However for one particular angle of incidence θ_B, called Brewster's angle, the linear polarisation produced at the reflected light is complete. This occurs when all the components of the vector of the intensity of an electric field not parallel to the surface of the glass are refracted through the glass, and only the parallel components are refracted. The angle for which complete polarisation occurs can be found from Fresnel's formulae, which describe the conditions of polarisation among incident, reflected and refracted waves.

Thus for components of the intensities of the electric field of the incident (E_{1p}) and reflected (E_{2p}) waves polarised in the plane of the paper we have

$$E_{2p} = E_{1p}\tan(\theta - \theta_r)\cot(\theta + \theta_r)$$

where θ is angle of incidence and θ_r is angle of refraction. For complete polarisation E_{2p} must be equal to zero, hence

$$\theta_B + \theta_{B,r} = 0$$

We see that, for this case, the incident and refracted beams are perpendicular to each other. Substituting this result into Snell's law we obtain

$$\frac{\sin\theta_B}{\sin\theta_{B,r}} = \frac{\sin\theta_B}{\sin(90° - \theta_B)} = \frac{\sin\theta_B}{\cos\theta_B} = \tan\theta_B = \frac{n_2}{n_1} = n$$

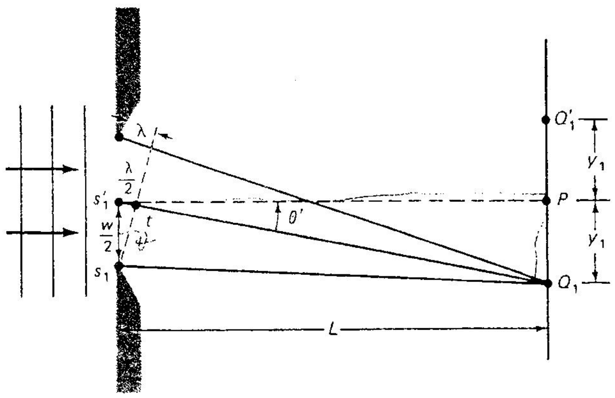
where n is the index of refraction of medium 2 with respect to medium 1. Equation

$$\tan\theta_B = n$$

is called Brewster's law.

Problem 3-10. A monochromatic light of wavelength λ falls on a narrow slit of width w and then on a screen a large distance L from the slit. Calculate the coordinates of the points at which we find dark fringes in the single slit experiment.

Solution: The Huyghens' wavelets that go straight through the slit will arrive at point P on the screen in phase, and will produce a bright central image of the slit. To find the positions of the dark fringes i.e. coordinates y_n of points Q_n we have to find the points at which the Huyghens' wavelets from different parts of the slit cancel out, because they travel different distances and arrive at the screen 180° out of phase. These are the positions at which each Huyghens' wavelet has superimposed on it another wavelet which is $\lambda/2$ out of phase with it.



Let us divide the slit into two zones, a top half and a bottom half, and consider the superposition of Huyghens' wavelets, one from the top half and one from the bottom half, arising at points separated by a distance $w/2$ across the slit. If point Q_l is chosen as indicated above, then a wavelet from s_l will cancel a wavelet from s_1' at Q_l. We can therefore write for the first dark fringe

$$\sin\theta' = \frac{\frac{\lambda}{2}}{\frac{w}{2}} = \frac{\lambda}{w}$$

Since angle θ' is very small we can also write $\sin\theta' = \tan\theta' = \frac{y_1}{L}$. Combining these two equations we obtain for the coordinate of the first dark fringe

$$y_1 = \frac{L\lambda}{w}$$

In a fashion similar to that used for the first dark fringe we find that dark fringes occur at distances

$$y_n = n\frac{L\lambda}{w}$$ where n is an integral number.

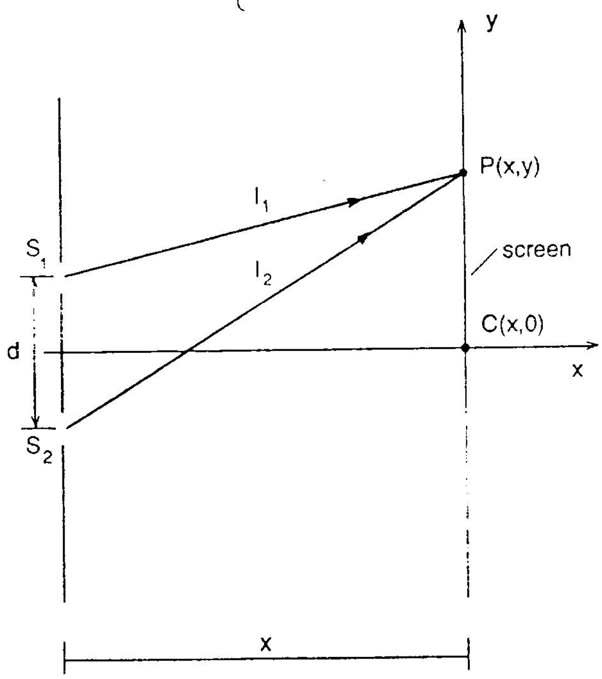
Problem 3-11. A monochromatic light of wavelength λ falls on two slits separated by distance d and then on a screen a large distance x from the slits. Calculate the co-ordinates of the points at which we find dark and bright fringes in the double slit experiment.

Solution: Huyghens' wavelets produced by diffraction at the slits combine to form cylindrical wavefronts (because of the long narrow slits) and these wavefronts are superimposed on the screen. This leads to constructive and destructive interference, depending on the path difference between the wavelets from the two slits, and hence to bright and dark vertical fringes on the screen.

To find the co-ordinates of the dark and bright fringes we have first of all to determine the path difference between the wavelets from the two slits. The first wavelet coming from the slit S_1 has to travel the path l_1 to reach point P. The second wavelet coming from slit S_2 has to travel path l_2 to reach point P. The path difference δ is therefore

$$\delta = l_2 - l_1 = \sqrt{x^2 + \left(y + \frac{d}{2}\right)^2} - \sqrt{x^2 + \left(y - \frac{d}{2}\right)^2} =$$

$$= x\left\{\left[1 + \left(\frac{y + \frac{d}{2}}{x}\right)^2\right]^{1/2} - \left[1 + \left(\frac{y - \frac{d}{2}}{x}\right)^2\right]^{1/2}\right\}$$



To simplify the previous expression we may apply a binomial expansion

$$(1 \pm a)^n = 1 \pm na \pm$$
$$\pm \frac{n(n-1)}{2 \times 1} a^2 \pm \ldots$$

In reality the distance d between the slits and the positions y of a point P are very small compared to the distance x between the slits and the screen, consequently

$$a = \left(\frac{y \pm \frac{d}{2}}{x}\right)^2 \ll 1$$

We can therefore restrict the polynomial expansion to the first two terms only. Thus for the path difference we have:

$$\delta \approx x\left[1 + \frac{1}{2}\left(\frac{y}{x} + \frac{d}{2x}\right)^2 - 1 - \frac{1}{2}\left(\frac{y}{x} - \frac{d}{2x}\right)^2\right] \approx \frac{d\,y}{x}$$

The y co-ordinate of point P at which we observe interference is

$$y = \frac{x}{d}\delta$$

The bright fringes will occur at points where the path difference δ is equal to the even multiple of half of the wavelength, or

$$\delta = 2k\frac{\lambda}{2} \qquad \text{where } k = 0,1,2,\ldots$$

Thus for the co-ordinates of the bright fringes we have

$$y_{bright} = k\lambda\,\frac{x}{d}$$

The dark fringes will occur at points where the path difference δ is equal to the odd multiple of half of the wavelength, or

$$\delta = (2k+1)\frac{\lambda}{2} \qquad \text{where } k = 0,1,2,\ldots$$

Thus for the co-ordinates of the dark fringes we have

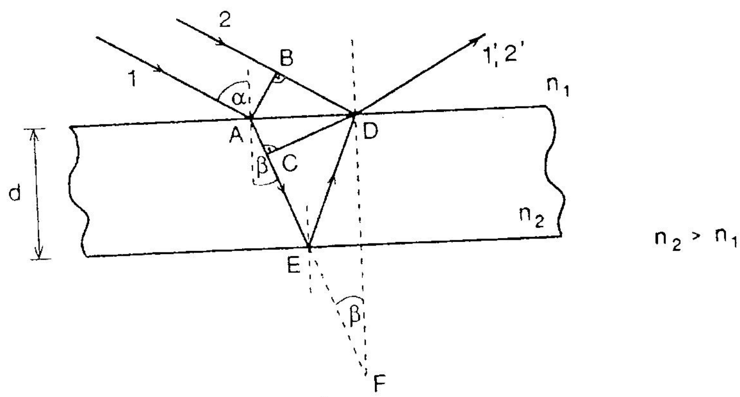
$$y_{dark} = (2k+1)\,\frac{x}{d}\,\frac{\lambda}{2}$$

For the distance between two neighbouring dark (bright) fringes we obtain

$$\Delta y = \frac{x}{d}\lambda$$

The pattern which will be seen on the screen will consist of alternating dark and bright fringes, symmetric about a central bright fringe (the central maximum).

Problem 3-12. A beam of parallel rays from a medium of refractive index n_1 is incident at an angle α upon one surface of a glass plate of refractive index n_2. The thickness of the glass plate is d. Investigate the interference in reflected and transmitted light.



Solution:

I. *Interference in reflected light.*
The incident rays (beam 1 and 2) are in part reflected (beam 1 into 1') and in part refracted (beam 2) at the upper surface of the plate. The refracted ray, after reflection from the lower surface of the plate, emerges after refraction at the upper surface in a direction parallel to that of the reflected rays from the upper surface (beam 2').

These two rays (beams 1' and 2') will produce destructive or constructive interference depending on the path difference. Let us calculate geometrical path difference δ. The Figure shows that

$$\delta = CE + ED = CE + EF = CF = 2\delta\cos\beta$$

Since we shall compare the geometrical path difference with the wavelength λ which is given for a vacuum, we must express the path difference for a vacuum as well. When a light wave travels from one medium to another, its frequency does not change but its wavelength does, as follows:

$$\frac{\lambda_2}{\lambda_1} = \frac{v_2 t}{v_1 t} = \frac{v_2}{v_1} = \frac{n_1}{n_2}$$

If medium 1 is a vacuum or air we put $n_1 = 1$ and $\lambda_1 = \lambda$. Then the wavelength in another medium of index of refraction n_2 will be

$$\lambda_{n_2} = \frac{\lambda}{n_2}$$

Hence to obtain the path difference δ' corresponding to a vacuum we have to multiply the geometrical path difference δ by index of refraction n_2. Thus we have

$$\delta' = \delta\, n_2 = 2d\, n_2 \cos\beta$$

In the calculation of the path difference δ' we have also to take into account the fact that the wave reflected on the upper surface is reflected by a material whose index of refraction is greater than that in which the wave travels, and thus the wave phase changes by $180°$. This change of phase corresponds to the path difference $\lambda/2$. Thus for the path difference δ' we have

$$\delta' = 2d\, n_2 \cos\beta + \frac{\lambda}{2}$$

Since the angle of incidence α is easier to measure than the angle of refraction β it is useful to introduce the angle α into the expression for the path difference. To do this we use Snell's law

$$n_1 \sin\alpha = n_2 \sin\beta$$

Taking the square of this equation we obtain

$$n_1^2 \sin^2\alpha = n_2^2 \sin^2\beta = n_2^2\left(1 - \cos^2\beta\right)$$

and after a little rearrangement

$$n_2 \cos\beta = \sqrt{n_2^2 - n_1^2 \sin^2\alpha}$$

Substituting this into the expression for the path difference we have

$$\delta' = 2d\sqrt{n_2^2 - n_1^2 \sin^2\alpha} + \frac{\lambda}{2}$$

If this path difference is equal to the even multiple of halves of the wavelength, that is

$$\delta' = 2k\frac{\lambda}{2} \qquad \text{where } k = 0, 1, 2 \ldots$$

constructive interference occurs in the reflected light. Hence the condition for the constructive interference in reflected light is

$$2d\sqrt{n_2^2 - n_1^2 \sin^2\alpha} = \left(2k+1\right)\frac{\lambda}{2}$$

If the path difference δ' is equal to the odd multiple of halves of the wavelength, that is

$$\delta' = (2k + 1)\frac{\lambda}{2} \qquad \text{where } k = 0, 1, 2 \ldots$$

destructive interference occurs in the reflected light. Hence the condition for destructive interference in reflected light is

$$2d\sqrt{n_2^2 - n_1^2 \sin^2 \alpha} = k\lambda$$

II. *Interference in transmitted light.*
Since for transmitted light there is no reflection on the optically denser medium, there is no change in the phase of the wave. Thus for the path difference in transmitted light we have

$$\delta'' = 2d\sqrt{n_2^2 - n_1^2 \sin^2 \alpha}$$

If this path difference is equal to the even multiple of halves of the wavelength, that is

$$\delta'' = 2k\frac{\lambda}{2} \qquad \text{where } k = 0, 1, 2 \ldots$$

constructive interference occurs in the transmitted light. Hence the condition for the constructive interference in transmitted light is

$$2d\sqrt{n_2^2 - n_1^2 \sin^2 \alpha} = 2k\frac{\lambda}{2}$$

If the path difference δ' is equal to the odd multiple of halves of the wavelengths, that is

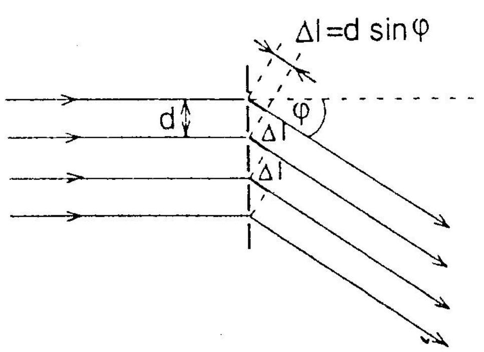
$$\delta'' = (2k + 1)\frac{\lambda}{2} \qquad \text{where } k = 0, 1, 2 \ldots$$

destructive interference occurs in the transmitted light. Hence the condition for the destructive interference in transmitted light is

$$2d\sqrt{n_2^2 - n_1^2 \sin^2 \alpha} = (2k + 1)\frac{\lambda}{2}$$

Problem 3-13. Monochromatic light falls on a diffraction grating as shown in the figure. Find the criterion to produce the brightest maximum.

Solution: We assume that parallel rays of light are incident on the grating, as shown in the figure. We also assume that the slits are narrow enough for the diffraction by each of them to spread light over a very large angle on to a distant screen behind the grating, and for interference to occur with light from all the other slits. Light rays that pass through each slit without deviation $(\varphi = 0)$ interfere constructively to produce a bright spot at the centre.



At an angle φ such that the rays from adjacent slits travel an extra distance $\Delta l = m\lambda$, where m is an integer, again constructive interference occurs. The integer m is called an order of the pattern. Thus the criterion for producing the brightest maximum is

$$\sin\varphi = \frac{m\lambda}{d},$$

where d is the distance between slits. These brightness maxima are much sharper and narrower than those for a double slit.

Problem 3-14. A diffraction grating with 10 000 lines/cm is illuminated by light of *400 nm* and *700 nm* wavelengths. Calculate the first and second order maxima for this light.

Solution: Since the grating contains 10^6 *lines/m,* the separation between slits is $1\times 10^{-6}\,m$. Hence for the first order maxima the angles for given wavelengths are

$$\sin\varphi_{400} = \frac{1\times 4\times 10^{-7}}{1\times 10^{-6}} = 0.400 \qquad \sin\varphi_{700} = \frac{1\times 7\times 10^{-7}}{1\times 10^{-6}} = 0.700$$

and thus $\varphi_{400} = 23.6^{o}$ and $\varphi_{700} = 44.0\ ^{o}$. For the second order maxima we have

$$\sin\varphi_{400} = \frac{2\times 4\times 10^{-7}}{1\times 10^{-6}} = 0.800 \qquad \sin\varphi_{700} = \frac{2\times 7\times 10^{-7}}{1\times 10^{-6}} = 1.400$$

For the *400 nm* wavelength for the angle of the second order maximum we therefore have $\varphi_{400} = 53.0^{o}$. For the *700 nm* wavelength the second order maximum does not exist, since $\sin\varphi$ cannot exceed 1. No higher order maxima will appear.

Problem 3-15. At what angle must light be incident on a piece of borosilicate crown glass, with an index of refraction *1.52* to produce completely polarised light by reflection?

$$\left[\ \theta_B = 57°\right]$$

Problem 3-16. Coherent light of wavelength *589 nm* from a small region of a sodium arc light falls on a double slit with a slit separation of *0.10 nm*. The interference pattern is produced on a screen *2.5 m* from the slit. Calculate the separation on the screen of the two fourth-order bright fringes on either side of the central image.

$$\left[2y_{bright} = 11.8\,cm\right]$$