# 2 WAVE MOTION

The study of wave phenomena is very important since it occurs in many areas of physics. In this chapter we will concentrate on mechanical waves – waves that travel in a material medium.

## 2.1 Description of Wave Motion

First, let us consider a single pulse which can be formed on a rope by a quick up and down motion of its end. If the hand pulls up on one end of the rope and because the end piece is attached to adjacent pieces, these also feel an upward force and they also begin to move upward. We say that the wave moves along the rope.

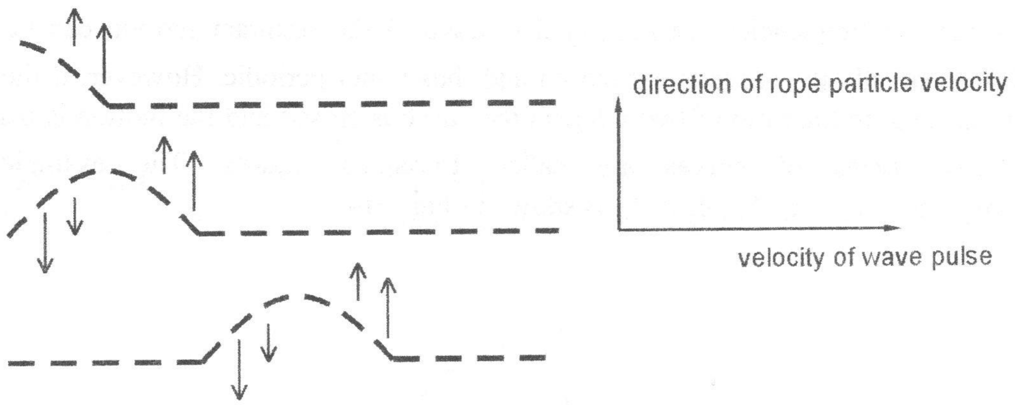


Figure 2-1

The source of a travelling wave pulse is a disturbance and cohesive forces between adjacent pieces of rope which cause the pulse to travel. If the source vibrates sinusoidally in the simple harmonic motion, the wave itself, if the medium is perfectly elastic, will have a sinusoidal shape both in space and in time.
To describe a periodic wave we introduce some of the important quantities (see Fig.2-2).

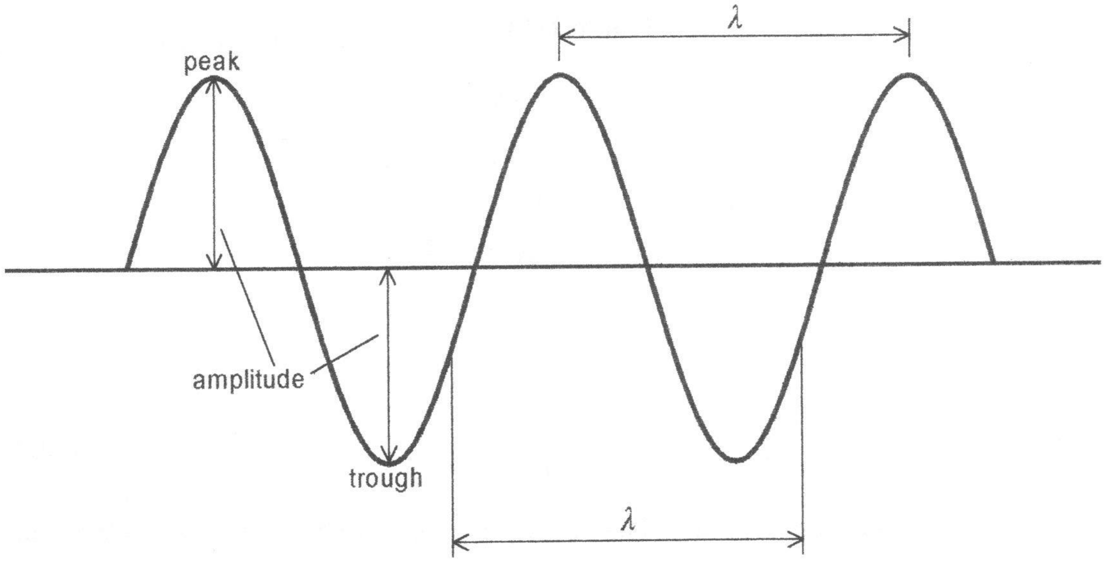


Figure 2-2

The high points on a wave are called peaks., the low points troughs. The maximum height of a peak or depth of a trough relative to the equilibrium level is called the **amplitude** of the wave. The distance between two successive peaks is called the **wavelength** $\lambda$. It is also equal to the distance between any two successive identical points on the wave. The number of peaks that pass a given point per unit time is called the **frequency** $f$. It is also equal to the complete cycles of a given point. The **period** $T$ is equal to the reciprocal value of the frequency $T=1/f$.

The **velocity** of a wave $v$ is the velocity at which, for example, peaks appear to move. This velocity is often called as the **phase velocity**. We must distinguish the velocity of a wave from that of a particle motion.
A wave peak travels a distance of wavelength $\lambda$ in period $T$. Thus the velocity of a wave $v$ is equal to

$$v=\frac{\lambda}{T}=\lambda f \; . \qquad (2\text{-}1)$$

We will assume, of course, that $v$ does not depend on $\lambda$ or $f$ unless otherwise stated. From the last expression we have $\lambda = vT$ and we can also say, that the wavelength $\lambda$ is equal to the distance travelled by the wave in a time of the one oscillation, that is, in a time equal to one period $T$.
The velocity of a wave depends on the properties of the medium in which it travels. For example, the velocity of a wave on a stretched string depends on the tension in the string $\tau$ and on the mass per unit length $\mu$ of the string by the relationship

$$v=\sqrt{\frac{\tau}{\mu}} \; . \qquad (2\text{-}2)$$

If the particles of the medium (such as a rope) vibrate up and down in a direction perpendicular (transverse) to the motion of the wave itself we call such a wave a **transverse wave**. But if the vibration of the particles of the medium is along the same direction as the motion of the wave, such a wave is called a **longitudinal wave**. The longitudinal wave is characterised by compressions and expansions that propagate along the spring that correspond to the peaks and troughs of a transverse wave. An important example of a longitudinal wave is a sound wave in air. The wavelength of a longitudinal wave is the distance between successive compressions (or expansions). Its frequency is the number of compressions that pass a given point per second. And its velocity is the velocity with which each compression appears to move and is equal to the product $v=\lambda f$.
The velocity of a longitudinal wave has similar form as Eq.2-2. For a longitudinal wave travelling along solid rod

$$v=\sqrt{\frac{E}{\rho}} \; , \qquad (2\text{-}3)$$

where $E$ is the elastic modulus of the material and $\rho$ is its density.
For a longitudinal wave travelling in a liquid or gas

$$v=\sqrt{\frac{K}{\rho}} \; , \qquad (2\text{-}4)$$

where $K$ is the bulk modulus and $\rho$ again the density.

The waves can be also classified according to their **polarisation**. If all particles vibrate in the same plane we say that a wave is polarised. In Fig.2-3 we have examples of vertically and horizontally polarised waves.

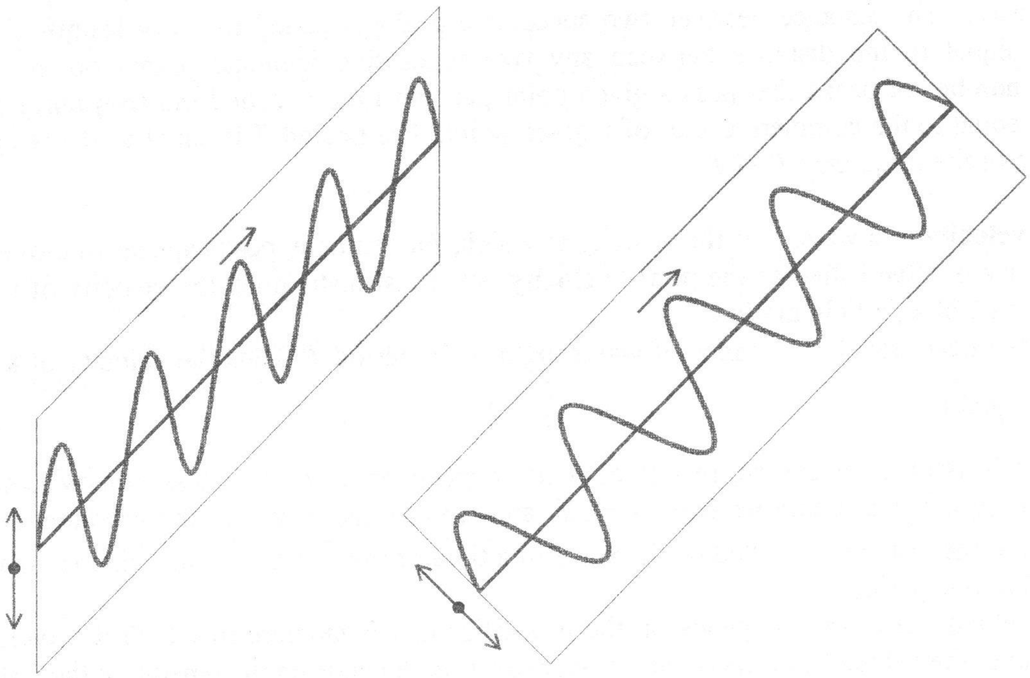

Figure 2-3

The **polarisation plane** is defined as the plane perpendicular to the plane in which particles vibrate. Polarisation is only possible with transverse waves. In the longitudinal wave the particles vibrate in the direction in which the wave travels and that is why there is no sense to talk about polarisation plane.

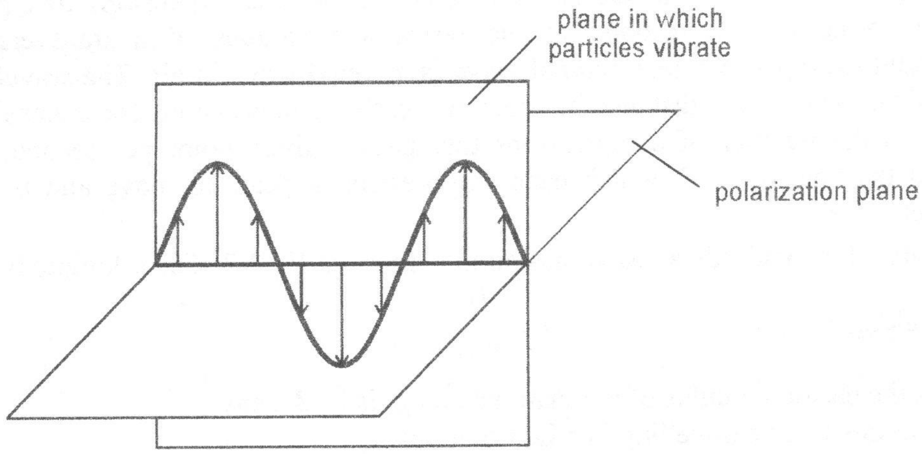


Figure 2-4

## 2.2 Energy Transmitted by Waves

As waves travel through a medium the energy is transmitted from particle to particle and thus, waves transmit energy from one place to another. For a sinusoidal wave of frequency $f$ the particles move in the simple harmonic motion. An energy of each particle motion equals

$$E = \frac{1}{2} k A^2 \ ,$$

where $A$ is the amplitude of its motion (see Eq.1-19).
Using Eq.1-12 gives us the formula

$$E = 2\pi^2 m f^2 A^2 \qquad [\mathrm{J}] \, .$$

For the wave travelling in an elastic medium, we take $m = \rho V$, where $\rho$ is the density of the medium and $V$ is its volume. If the wave travels the distance $l$ in a time $t$, then $l = vt$, where $v$ is the velocity of the wave. The volume $V$ can be now expressed as $V = S.l$, where $S$ is the cross-sectional area through which the wave travels.
Thus

$$m = \rho V = \rho S l = \rho S v t$$

and

$$E = 2\pi^2 \rho S v t f^2 A^2 \qquad [\mathrm{J}] \, . \tag{2-5}$$

Eq.2-5 represents the energy passing through the cross-sectional area $S$ perpendicular to the direction of energy flow in a time $t$. We see that the **energy transmitted by a wave is proportional to the square of the amplitude**.
The average rate of energy transferred is the **average power** $\overline{P}$:

$$\overline{P} = \frac{E}{t} = 2\pi^2 \rho S v f^2 A^2 \qquad [\mathrm{W}] \, . \tag{2-6}$$

The average power transferred across unit area perpendicular to the direction of energy flow is referred to as **intensity** $I$ of a wave

$$I = \frac{\overline{P}}{S} = 2\pi^2 \rho v f^2 A^2 \qquad [\mathrm{W\ m^{-2}}] \, . \tag{2-7}$$

Thus, the intensity is defined as the energy transmitted by a wave per unit time across unit area and, as we see, is proportional to the square of the wave amplitude.
If the wave flows out from the source in all directions it is a three-dimensional wave. As an example we have sound travelling in the open air. Such the wave is spherical in shape and is said to be a spherical wave. Thus wave is spread over the area of a sphere the value $S = 4\pi r^2$ of which increases. Because energy must be conserved, we can see from Eq.2-5 that as the area $S$ increases the amplitude $A$ must decreases.
Thus, for two different distances $r_1$ and $r_2$ from the source we can write equality

$$4\pi r_1^2 A_1^2 = 4\pi r_2^2 A_2^2$$

or

$$\frac{r_1}{r_2} = \frac{A_2}{A_1} \ , \tag{2-8}$$

where $A_1$ and $A_2$ are the amplitudes of the wave at $r_1$ and $r_2$, respectively. Thus the amplitude decreases inversely as the distance – at twice distance the amplitude is half.
Since the intensity $I$ is proportional to $A^2$ (see Eq.2-7), it must decreases as the square of the distance:

$$\frac{I_2}{I_1} = \frac{r_1^2}{r_2^2} \ , \tag{2-9}$$

where $I_1$ and $I_2$ are the intensities of the wave at $r_1$ and $r_2$, respectively.
In practice, frictional damping is present and some of the energy is transformed into thermal energy and the decrease of the amplitude and intensity will be more greater.
Note: for a one-dimensional wave (as a transverse wave on a string or a longitudinal wave travelling along a metal rod) the area $S$ remains constant and so the amplitude as well as the intensity also remain constant.

## 2.3 Representation of Travelling Wave

We consider a one-dimensional wave travelling along the $x$ axis and we assume the wave shape to be a sine curve. Let wavelength of the wave be $\lambda$ and its frequency be $f$.
We suppose the oscillations of the point $x = 0$ to be given by

$$u(t) = U \sin \omega t \ , \qquad (2\text{-}10)$$

where $\omega = 2\pi f$ is the **angular frequency** of the wave.
Let us now suppose the wave moves to the right with velocity $\upsilon$. After a time $t$ it has moved a distance $x = \upsilon t$. To describe the oscillations of this point whose distance from the $x = 0$ is $\upsilon t$ we must replace $t$ in Eq.2-10 by $\left(t - \frac{x}{\upsilon}\right)$:

$$u(x,t) = U \sin \omega \left(t - \frac{x}{\upsilon}\right) = U \sin \left(\omega t - \frac{\omega x}{\upsilon}\right) = U \sin(\omega t - kx) \ , \qquad (2\text{-}11)$$

where $k = \frac{\omega}{\upsilon} = \frac{2\pi}{T\upsilon} = \frac{2\pi}{\lambda}$ is called the **wave number**. The quantity $(\omega t - kx)$ is called the **phase** of the wave. The velocity $\upsilon$ of the wave, which is often called the **phase velocity** since it describes the velocity of the phase of the wave, can be also written in terms of $\omega$ and $k$:

$$\upsilon = \lambda f = \frac{2\pi}{k} \frac{\omega}{2\pi} = \frac{\omega}{k}$$

Eq.2-11 is the mathematical representation of a sinusoidal wave travelling along the $x$ axis to the right. It describes the displacement $u$ of the wave at any point $x$ at any time $t$.
<u>Note</u>: For a wave travelling along the $x$ axis to the left Eq.2-11 has form

$$u = U \sin(\omega t + kx) \ .$$

Let us rewrite now the argument $\left(t - \frac{x}{\upsilon}\right)$ in Eq.2-11 for a three-dimensional wave into more useful form. The position of the vibrating particle (in one-dimensional case $x$) will be given now by the position vector $\mathbf{r}$:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \ .$$

The direction of the wave propagation will be given by the **unit vector s** in the direction of the wave propagation:

$$\mathbf{s} = \mathbf{i}s_x + \mathbf{j}s_y + \mathbf{k}s_z \ .$$

Thus we can write for the **wave vector k**:

$$\mathbf{k} = k.\mathbf{s} \ , \qquad \text{where } k = \frac{2\pi}{\lambda} \text{ is the wave number.}$$

For the argument of the function describing three dimensional wave we can therefore write:

$$\omega t - \mathbf{k}\cdot\mathbf{r} = \omega t - k\,\mathbf{s}\cdot\mathbf{r} = \omega\left(t - \frac{k}{\omega}\mathbf{s}\cdot\mathbf{r}\right) = \omega\left(t - \frac{2\pi}{\lambda}\frac{T}{2\pi}\mathbf{s}\cdot\mathbf{r}\right)\omega\left(t - \frac{T}{\lambda}\mathbf{s}\cdot\mathbf{r}\right) = \omega\left(t - \frac{\mathbf{s}\cdot\mathbf{r}}{v}\right).$$

Thus, we can write for three dimensional wave

$$u = U\sin\omega\left(t - \frac{\mathbf{s}\cdot\mathbf{r}}{v}\right). \qquad (2\text{-}12)$$

This is valid for sinusoidal wave – for wave of any shape we will not specify the shape of the function and we write:

$$u = f\left(t - \frac{\mathbf{s}\cdot\mathbf{r}}{v}\right). \qquad (2\text{-}13)$$

## 2.4 Wave Equation and Superposition Principle

Let us consider the general class of functions $u(x, t) = f(x-vt)$ and $u(x, t) = f(x+vt)$, where $f$ is any differentiable function of $x$ and $t$. Let the quantity $(x-vt)$ be represented by $z$ (so $z = x - vt$). If $u(x, t) = f(x-vt)$, we use chain rule for derivatives and find:

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial f}{\partial z}(-v)\ ,$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{\partial f}{\partial z}\right) = -v\frac{\partial^2 f}{\partial z^2}\frac{\partial z}{\partial t} = v^2\frac{\partial^2 f}{\partial z^2}\ . \qquad \text{(a)}$$

Similarly,

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z}\ ,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial z^2}\ . \qquad \text{(b)}$$

Comparing Eqs.(a) and (b) gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\ , \qquad (2\text{-}14)$$

where $v$ is the wave velocity. (For $u(x, t) = f(x+vt)$, the result is the same.)
Equation (2-14) is the **one-dimensional wave equation** that applies to waves in one dimension only. For waves spreading out in three dimensions the wave equation will be

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\ .$$

The wave equation is a linear one. If $u_1(x,t)$ and $u_2(x,t)$ will be two different solution of the wave equation, then the linear combination

$$u_3(x,t) = au_1 + bu_2\ ,$$

where $a$ and $b$ are constants, will be also a solution. This is the essence of the **superposition principle** which says that if two waves pass through the same region of space at the same time, the actual displacement is the sum of the separate displacements. The principle is valid for mechanical waves as long as the displacements are not too large, that is, as long as there is a linear relationship between the displacement and the restoring force of the oscillating medium. If the amplitude of a mechanical wave, for example, is so large that it goes beyond the elastic region of the medium and Hook's law is no longer held, the superposition principle is no longer held.

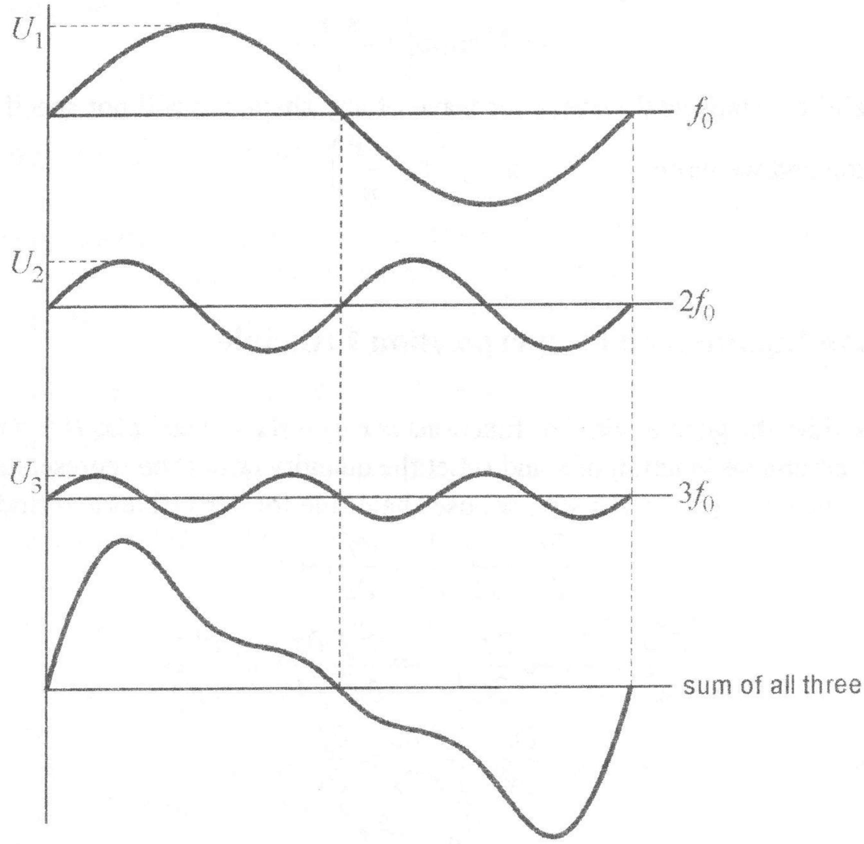


Figure 2-5

Fig.2-5 shows the application of the superposition principle to three waves of different amplitudes and frequencies. At the instant shown, the actual amplitude at any position is the algebraic sum of the amplitudes of the three waves at that position. The actual wave, as we see, is not a sinusoidal wave and is called a complex wave.

<u>Notes</u>: 1) Any complex wave can be considered as composed of many simple sinusoidal waves of different amplitudes, wavelengths and frequencies. This is known as Fourier's theorem. A complex periodic wave of period $T$ can be represented as a sum of pure sinusoidal terms whose frequencies are integral multiples of $f = 1/T$. If the wave is not periodic, the sum becomes an integral – called Fourier integral.

2) When the restoring force is not proportional to the displacement for mechanical waves in some medium, the velocity of waves depends on the frequency. This is called **dispersion**. The different sinusoidal waves that compose a complex wave will travel with different velocities in such a case and a complex wave will change shape if the medium is dispersive.

## 2.5 Reflection and Refraction of Waves

### Reflection

When a wave strikes a boundary some of its energy is reflected and some is transmitted or absorbed.

First, we will consider a one-dimensional wave travelling down a rope. If the end of the rope will be fixed the reflected wave is inverted and it will be 180° out of phase with the incident wave. If the end is free the reflected wave is not inverted and it will have no phase change.

Secondly, for a two or three dimensional wave we consider so-called **wavefront** by which we mean the line or surface of all points having the same phase. A line which has the direction of wave motion and is perpendicular to the wavefront is called a **ray**.

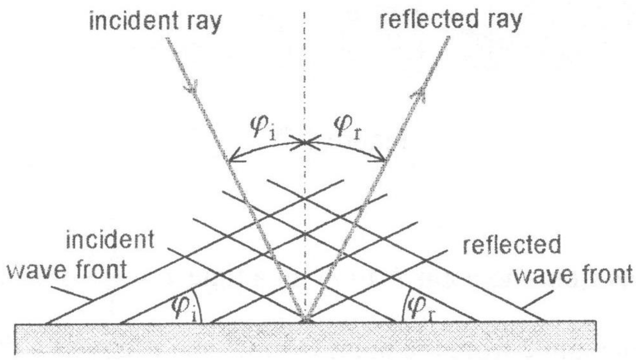


Figure 2-6

In Fig.2-6 $\varphi_i$ represents the **angle of incidence** which is defined as the angle the incident ray makes with the perpendicular to the reflecting surface (we see that it is also equal to the angle the wavefront makes with a tangent to the surface). The **angle of reflection** $\varphi_r$ is the corresponding angle for the reflected wave.

The **law of reflection** states that the angle of reflection $\varphi_r$ equals the angle of incidence $\varphi_i$.

Important note: when the wave strikes the boundary between the two media, part of its energy is reflected and part of energy is transmitted. If the second medium has a greater density than the first, the less energy is transmitted and the reflected wave will be 180° out of phase with the incident wave.

### Refraction

When a wave passing a boundary into a medium where its velocity is different the transmitted wave will move in a different direction than the incident wave. Such a wave is called the **refracted wave**.

Let us now consider wavefront $\overline{OA}$ of the incident wave in Fig.2-7. Let $\upsilon_1$ and $\upsilon_2$ be wave velocities in the medium 1 and the medium 2, respectively, and let $\varphi_1$ be the angle of the incident wave.

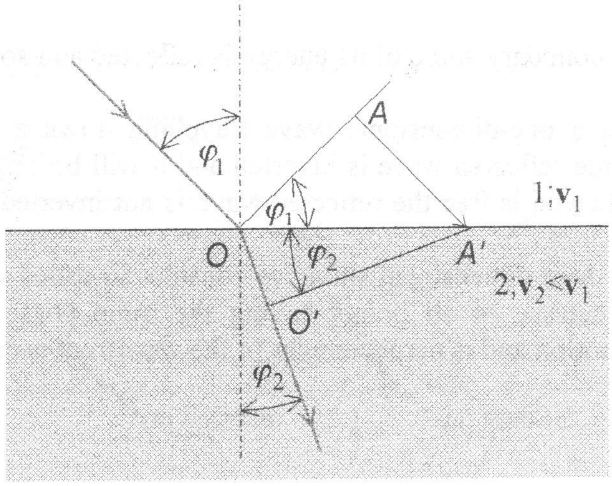


Figure 2-7

The wavefront $\overline{OA}$ gets from the point A to A' in a time $t = \frac{\overline{AA'}}{v_1}$. In the same time, this wavefront gets in the medium 2 from the point O to O' and it travels a distance OO'=$v_2 t$. For triangles OAA' and OA'O' we may write:

$$\sin\varphi_1 = \frac{\overline{AA'}}{\overline{OA'}} = \frac{v_1 t}{\overline{OA'}} ,$$

and
$$\sin\varphi_2 = \frac{\overline{OO'}}{\overline{OA'}} = \frac{v_2 t}{\overline{OA'}} .$$

Dividing these equations we find the **law of refraction**

$$\frac{\sin\varphi_2}{\sin\varphi_1} = \frac{v_2}{v_1} ,$$

where $\varphi_2$ represents the **angle of refraction**.

## 2.6 Interference of Waves

Interference refers to the physical effects that will happen when two or more waves pass through the same region of space at the same time.
Let us consider two one-dimensional waves of equal frequencies $\omega$, different amplitudes $U_1 \neq U_2$ and different phase constant $\varphi_1 \neq \varphi_2$. We shall suppose that the displacement directions of the first and the second wave are the same and the waves travel down a positive direction of $x$-axis. So, these waves will be described as follows:

$$u_1 = U_1 \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi_1\right],$$

$$u_2 = U_2 \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi_2\right],$$

where $v$ is the velocity of the waves.
Using the principle of superposition gives the resultant wave

$$u = \left(U_1 \cos\varphi_1 + U_2 \cos\varphi_2\right)\sin\omega\left(t - \frac{x}{v}\right) + \left(U_1 \sin\varphi_1 + U_2 \sin\varphi_2\right)\cos\omega\left(t - \frac{x}{v}\right) . \quad \text{(a)}$$

The terms in parentheses are constants and we see that the resultant wave is also harmonic with the same frequency. To find the amplitude and the phase angle of this resultant wave we express it in standard form

$$u = U \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi\right] ,$$

or after easy arrangement

$$u = U \cos\varphi \sin\omega\left(t - \frac{x}{v}\right) + U \sin\varphi \cos\left(t - \frac{x}{v}\right) ,$$

where $U$ and $\varphi$ represent the amplitude and the phase angle of the resultant wave. (b)
As Eqs.(a) and (b) must be identically equal for any $x$ and any $t$, we obtain the following expressions:

$$U \cos\varphi = U_1 \cos\varphi_1 + U_2 \cos\varphi_2 ,$$
$$U \sin\varphi = U_1 \sin\varphi_1 + U_2 \sin\varphi_2 .$$

Dividing of these two equations we find the phase angle $\varphi$:

$$\mathrm{tg}\varphi = \frac{U_1 \sin\varphi_1 + U_2 \sin\varphi_2}{U_1 \cos\varphi_1 + U_2 \cos\varphi_2} . \quad \text{(2-15)}$$

To find the amplitude $U$ we square the equations and then we add them:

$$U^2 = U_1^2 + 2U_1U_2 \cos\left(\varphi_2 - \varphi_1\right) + U_2^2 . \quad \text{(2-16)}$$

From this result it is clear that the amplitude $U$ depends on phase difference between the waves.
Instead of the phase difference we use more frequently so-called **path difference**. It is defined as the difference between the paths travelled by the waves from the source to the point of interference. We now find the relationship between the phase difference

and the path difference. We shall consider the one-dimensional wave of the sine course with the angular frequency $\omega$.
The displacement at any point $x$ and at any time $t$ is described by the function $u(x,t)$ of two variables $x$ and $t$

$$u(x,t) = U \sin \omega \left( t - \frac{x}{\upsilon} \right) , \qquad (2\text{-}17)$$

where $U$ and $\upsilon$ are the amplitude and velocity of the wave, respectively.
If we choose the fixed point $x_0$, the function $u(x_0,t)$ describes the oscillating motion at this point $x_0$, thus for $x = x_0$ we can write the function in (2-17)

$$u(t) = U \sin \left( \omega t - \omega \frac{x_0}{\upsilon} \right) = U \sin (\omega t - \varphi) ,$$

where $\varphi = \omega \frac{x_0}{\upsilon}$ is constant and represents the **phase delay** of the point $x_0$ with respect to the point in origin. The greater distance of $x_0$ the greater delay $\varphi$.
Thus, for any point $x$ we have the relationship

$$\varphi = \omega \frac{x}{\upsilon} = \frac{2\pi}{T} \frac{x}{\upsilon} = 2\pi \frac{x}{\lambda} . \qquad (2\text{-}18)$$

The phase difference $\varphi_2 - \varphi_1$ of two waves of different phase angles (or phase delays) $\varphi_1$ and $\varphi_2$ is now equal to

$$\varphi_2 - \varphi_1 = \frac{2\pi}{\lambda}(x_2 - x_1) , \qquad (2\text{-}19)$$

or

$$x_2 - x_1 = \frac{\lambda}{2\pi}(\varphi_2 - \varphi_1) , \qquad (2\text{-}20)$$

where $(x_2 - x_1)$ represents the path difference $d = x_2 - x_1$.

Let us now return to the expression(2-16) in more detail.
If the phase difference

$$\varphi_2 - \varphi_1 = 2k\pi , \quad \text{where } k = = 0,1,2,\ldots$$

or the path difference is equal to the even multiple of the half-wavelength

$$d = 2k\frac{\lambda}{2} , \quad k = 0,1,2,\ldots$$

The amplitude of the resultant wave will have its **maximum** $U = U_1 + U_2$. This case is called **constructive interference**. In constructive interference the wave are **in phase**.

If the phase difference

$$\varphi_2 - \varphi_1 = (2k+1)\pi , \quad k = 0,1,2,\ldots$$

or the path difference is equal to the odd multiple of the half-wavelength

$$d = (2k+1)\frac{\lambda}{2} , \quad k = 0,1,2,\ldots$$

the amplitude of the resultant wave will have its **minimum**

$$U = \left\langle \begin{matrix} U_1 - U_2 & (\text{if } U_1 > U_2) \\ U_2 - U_1 & (\text{if } U_2 > U_1) \end{matrix} \right. .$$

This case is called **destructive interference.** In destructive interference the waves are **out of phase** by one-half wavelength or 180°. For the special case of $U_1 = U_2$ the resultant amplitude $U = 0$ and this point will be at rest, that is, there are no oscillations at this point.

## 2.7 Standing Waves

If we vibrate one end of a rope and the other end is kept fixed, a wave will travel down to the fixed end and be reflected back. There will be waves travelling in both directions and the wave travelling down the rope will interfere with the reflected wave coming back. If vibrations of the rope will be at just the right frequency, these two waves will interfere and a standing wave will be produced. Thus, a **standing wave** is the result of the interference of two waves travelling in opposite directions. The points of destructive interference, called **nodes**, and of constructive interference, called **antinodes**, remain in fixed positions. Standing waves occur at more than one frequency. Fig.2-8 shows the standing waves produced at the lowest frequency of vibrations and at twice and three times the lowest frequency.

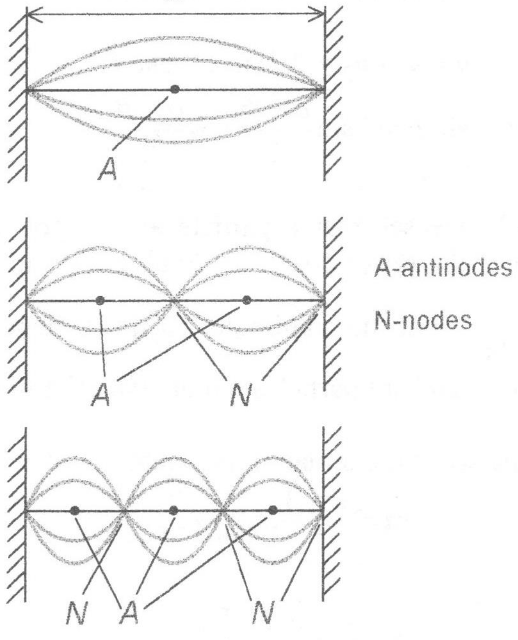


Figure 2-8

The frequencies at which standing waves are produced are called the **natural or resonant frequencies**, each of which is a integer multiple of the lowest resonant frequency. The lowest frequency, called the **fundamental frequency**, corresponds to one antinode. So, the whole length $L$ corresponds to one-half wavelength, $L = \frac{1}{2}\lambda_1$, where $\lambda_1$ is the wavelength of the **fundamental**. The other frequencies are called **overtones** which are integral multiples of the fundamental and are also called **harmonics**, with the fundamental are referred to as the first harmonic. The next mode after the fundamental is called the second harmonic or first overtone and the length $L$ corresponds to one complete wavelength, $L = \lambda_2$. For the higher harmonic, $L = \frac{3}{2}\lambda_3$, $L = 2\lambda_4$ and so on.

In general, we can write $\quad L = n\dfrac{\lambda_n}{2}$ , $\quad$ where $n = 1,2,3,\ldots$ $\quad$ (2-21)

The integer $n$ labels the number of the harmonic.

From Eq.2-21 we have $$\lambda_n = \frac{2L}{n}, \qquad n = 1,2,3,\ldots \tag{2-22}$$

To find the frequency $f$ of each harmonics we use $f = \dfrac{v}{\lambda}$.

We have known that a standing wave can be considered as a result of the interference of two waves travelling in opposite directions. Each of these waves can be described as a function of position $x$ and time $t$:

$$u_1 = U\sin(kx - \omega t),$$
$$u_2 = U\sin(kx + \omega t).$$

We assume the amplitude are equal as are the frequencies and wavelengths.
The sum of these two travelling waves produces a standing wave

$$u = u_1 + u_2 = 2U\sin kx \cos\omega t \quad . \tag{2-23}$$

(We used the identity $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$ ).

From the result in Eq.2-23 we see that a particle at any position $x$ vibrates in simple harmonic motion because of the factor $\cos\omega t$ . We also see that all particles vibrate with the same frequency $f = \dfrac{\omega}{2\pi}$ , but the amplitude $2U\sin kx$ depends on $x$. (Compare this fact to a travelling wave for which all particles vibrate with the same amplitude).

The amplitude of a standing wave has a maximum equal to $2U$, when

$$kx = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$$

that is, at points $$x = \frac{\lambda}{4}, \frac{3}{4}\lambda, \frac{5}{4}\lambda, \ldots \tag{2-24}$$

what are the **positions of the antinodes**.

The amplitude of a standing wave equals zero, when

$$kx = n\pi, \qquad n = 0,1,2,\ldots$$

that is at points $$x = 0, \frac{\lambda}{2}, \lambda, 3\frac{\lambda}{2}, \ldots, n\frac{\lambda}{2}\ , \tag{2-25}$$

what are the **positions of the nodes**.

<u>Note</u>: If two waves producing the standing wave have different amplitudes, we talk about partial standing wave. The important feature of it is that its nodes are not motionless. The amplitude of the vibrations of nodes is non-zero and depends on the difference between the amplitudes of original waves.

## 2.8 Sound Wave

The source of a sound wave is a vibrating object. The sound energy is transferred from the source in the form of **longitudinal** sound waves. Sound cannot travel in the absence of matter. That is why, for example, a bell ringing inside an evacuated space cannot be heard. The **speed** of sound is different in different materials. In air at 0 °C sound travels at a speed of 331.3 m/s. Generally, the sound speed depends on the elastic or bulk modulus and the density of the material. For a longitudinal wave travelling in solids is given by

$$v = \sqrt{\frac{E}{\rho}} .$$

For gases and fluids is given by

$$v = \sqrt{\frac{K}{\rho}} , \tag{2-26}$$

where $E$ and $K$ are elastic modulus and bulk modulus, respectively and $\rho$ the density.
The value of the sound speed depends on temperature, too, but this is significant mainly for gases. For example, in air the speed increases approximately 0,6 m/s for each Celsius degree increase

$$v \approx (331.3 + 0.6\,T) \text{ m/s} ,$$

where $T$ is the temperature in °C.

The human ear responds to frequencies in the range about 20 Hz to about 20 kHz which is called the audible range. These limits can vary from one person to another. It is also known that the higher age the lower upper limit of the audible range. Sound waves whose frequency is outside the audible range cannot be heard. Frequencies above 20 kHz are called ultrasonic. It is proved that ultrasonic frequencies can be heard by many animals (for example, a dog can hear sounds as high as 50 kHz, a bat can detect frequencies as high as 100 kHz). Sound waves whose frequencies are below the audible range - less than 20 Hz – are called infrasonic (for example, the waves produced by earthquake are infrasonic ones).
We have already known that a one-dimensional sinusoidal wave travelling along the $x$-axis is represented by the relation describing the displacement at position $x$ at time $t$

$$u(x,t) = U \sin(kx - \omega t) , \tag{2-27}$$

where $U$ is the amplitude of the wave (or maximum value of its displacement), the wave number $k$ is related to the wavelength $\lambda$ by $k = 2\pi / \lambda$ and $\omega = 2\pi f$, where $f$ is the frequency. For a transverse wave the displacements are perpendicular to the direction of wave propagation. But for a **longitudinal (sound) wave the displacement is along the direction of wave propagation**. That is, it is parallel to $x$-axis and represents the displacement of a volume element from its equilibrium position. That is why longitudinal (sound) waves can also be considered from the point of view of variations in pressure rather than displacement. Therefore, longitudinal waves are often referred to as **pressure waves**.

We now describe the pressure variation in a travelling longitudinal wave. We know that a pressure change $p$ causes the fractional change in volume $\Delta V / V$ of the medium. This is given by the relation (see Physics I)

$$p = -K \frac{\Delta V}{V} ,$$

where $K$ is the bulk modulus and $p$ now represents the pressure difference from the normal pressure in the absence of a wave. The negative sign relates to the fact that the volume decreases $(\Delta V < 0)$ if the pressure is increased. Let us now consider a cylindrical layer of fluid of thickness $\Delta x$ and area $S$ through which the longitudinal wave travels (Fig.2-9). Its volume is $V = S\,\Delta x$.

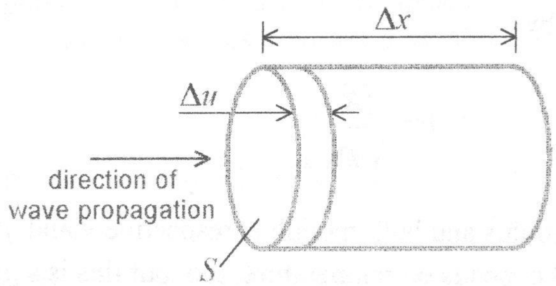


Figure 2-9

The pressure variation in the wave causes the change in the volume by an amount $\Delta V = S\,\Delta u$, where $\Delta u$ is the change in thickness of this layer as it is compressed or expanded. Hence, we have

$$p = -K \frac{S\,\Delta u}{S\,\Delta x} .$$

Taking the limit of $\Delta x \to 0$ yields

$$p = -K \frac{\partial u}{\partial x} ,$$

where it is used the partial derivative since $u$ is a function of both $x$ and $t$.

For a sinusoidal wave the displacement is given by Eq.2-27. Then, we have for the pressure variation

$$p = -(KU\,k)\cos(kx - \omega t) . \tag{2-28}$$

We see that the pressure varies sinusoidally as well, but is out of phase from the displacement by 90° or a quarter wavelength. Where the pressure variation is a maximum or minimum the displacement from equilibrium is zero and where the pressure variation is zero the displacement is a maximum or minimum (Fig.2-10).

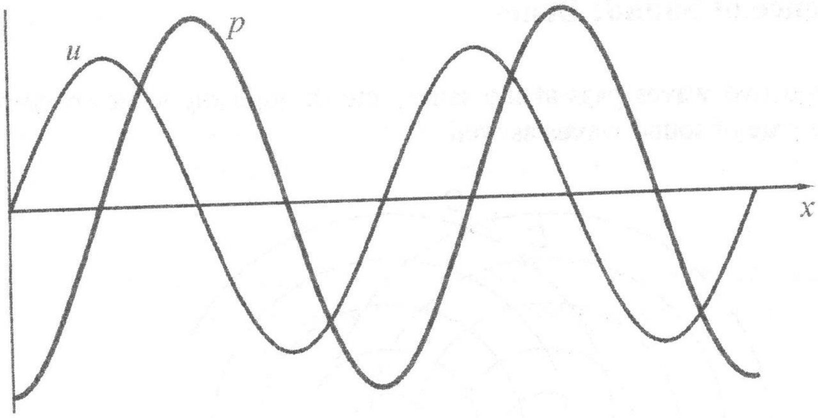


Figure 2-10

The quantity $KUk$ is called the pressure amplitude $p_m$. It represents the maximum and minimum amounts by which the pressure varies from the normal ambient pressure.
By using Eq.2-26 for the wave velocity we can express the pressure amplitude as

$$p_m = KUk = \rho \upsilon^2 Uk = 2\pi\rho\upsilon Uf \ ,$$

and Eq.2-28 can be expressed in the form

$$p = -p_m \cos(kx - \omega t) .$$

The intensity of sound is defined as the energy transmitted by a sound wave per unit time across unit area and, as we saw, is proportional to the square of the wave amplitude. The human ear can detect sounds with an intensity as low $10^{-12}$ W/m$^2$ and as high as 1W/m$^2$. In practice it is usual to specify sound intensity levels using a logarithmic scale. The unit on this scale is a bel or a **decibel** (1dB = 0,1 bel). The intensity level $\beta$ of any sound is defined in terms of its intensity I, as follows

$$\beta(\text{in dB}) = 10\log\frac{I}{I_0} \ ,$$

where $I_0$ is the intensity of some reference level. $I_0$ is usually taken as the minimum intensity audible to an average person (threshold of hearing) which is $I_0 = 10^{-12}\ \text{Wm}^{-2}$. Thus, for example, the intensity level of a sound whose intensity $I = 10^{-10}\ \text{Wm}^{-2}$ will be

$$\beta = 10\log\frac{10^{-10}}{10^{-12}} = 20\,\text{dB} .$$

Notice that an increase in intensity by a factor of 10 corresponds to a level increase of 10 dB. An increase in intensity by a factor of 100 corresponds to a level increase of 20 dB. Thus, for example, 50 dB sound is 100 times more in intensity than 30 dB sound.

## 2.9 Interference of Sound; Beats

As we saw, when two waves pass at the same time through the same space region, they interfere. This is true of sound waves as well.

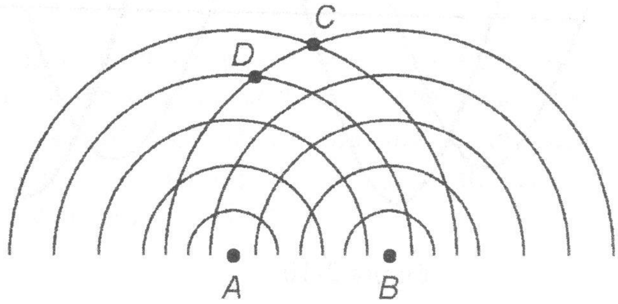


Figure 2-11

Let us consider a simple configuration of two sound sources A and B as in Fig.2-11. Let us assume the two sources are emitting sound waves of the same frequency and let them be in phase. Let the curves in figure represent the peaks of waves from each source. We see that at point C, which is the same distance from each source, constructive interference occurs since both waves at the same time have compression or rarefaction. Generally, we can say that **constructive interference** occurs at any point for which the difference of its distances from both sources is equal to a whole wavelength. As for point D, the wave from source B must travel a greater distance than the wave from A. Generally, we can say that **destructive interference** occurs at any point whose distance from one source is greater than its distance from the other by $\frac{1}{2}\lambda, \frac{3}{2}\lambda, \frac{5}{2}\lambda, \ldots$

An interesting example of interference occurs if two waves of the same amplitude and close in frequency but not the same meet at any point. We shall examine the interference of these waves.

Let two waves of frequencies $f_1$ and $f_2$ be represented at a fixed point by

$$u_1 = U \sin 2\pi f_1 t \,,$$
$$u_2 = U \sin 2\pi f_2 t \,.$$

By the principle of superposition, the displacement of the resultant wave, is

$$u = \left[ 2U \cos 2\pi \left( \frac{f_1 - f_2}{2} \right) t \right] \sin 2\pi \left( \frac{f_1 + f_2}{2} \right) t \,. \qquad (2\text{-}29)$$

As a result we obtained a wave whose frequency is equal to the average frequency $(f_1 + f_2)/2$ of the two components and whose amplitude is given by the expression in brackets. This amplitude varies in time from zero to a maximum of $2U$ with a frequency of $(f_1 - f_2)/2$. A **beat** occurs when $\cos 2\pi\left[(f_1 - f_2)/2\right]t$ equals $\pm 1$. So, we have two beats per one cycle and thus the **frequency of beats** must be

$$2\left( \frac{f_1 - f_2}{2} \right) = f_1 - f_2 \,. \qquad (2\text{-}30)$$

Hence, the beat frequency is equal to the difference in frequency of the component waves.

## 2.10 Doppler Effect

This effect, which occurs for all types of waves, relates to the change in wavelength and frequency when a source of sound is moving toward or away from an observer.
We shall have a look at this effect in detail. We shall assume the air is at rest in our reference frame.

In Fig.2-12a the **sound source is at rest**. The distance between two successive wake picks is $\lambda$. If the frequency of the source is $f$ then the time between emissions of successive wave peaks is equal to the period

$$T=\frac{1}{f}=\frac{\lambda}{v} ,$$

where $v$ is the velocity of sound wave in air.

In Fig.2-12b the **source is moving** with a velocity $v_s$ **toward stationary observer**. In a time $T$ the first wave peak has moved a distance $d=\lambda=vT$. In this same time the source has moved a distance $d_s=v_sT$.
The distance between successive wave peaks determines the new wavelength

$$\lambda'=d-d_s=\lambda-v_sT=\lambda-v_s\frac{\lambda}{v}=\lambda\left(1-\frac{v_s}{v}\right).$$

Thus, the change in wavelength

$$\Delta\lambda=\lambda'-\lambda=-v_s\frac{\lambda}{v} ,$$

$\Delta\lambda$ is proportional to the speed $v_s$ of the source.
And the **new frequency** is given by

$$f'=\frac{v}{\lambda'}=\frac{v}{\lambda\left(1-\frac{v_s}{v}\right)} ,$$

and since $\frac{v}{\lambda}=f$,

$$f'=\frac{f}{1-\frac{v_s}{v}}. \qquad \begin{bmatrix}\text{sorce moving toward}\\ \text{stationary observer}\end{bmatrix}$$

$f'>f$, because of the denominator is less of 1.

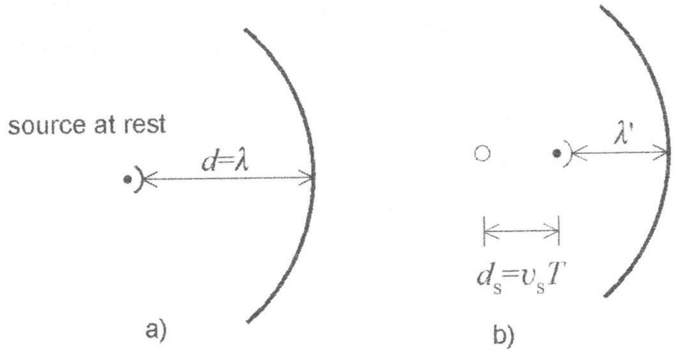


Figure 2-12

If the source will move **away from the observer** the new wavelength will be

$$\lambda' = d + d_s$$

and the change in wavelength will be

$$\Delta\lambda = \lambda' - \lambda = v_s \frac{\lambda}{v} ,$$

and **new frequency** will be

$$f' = \frac{f}{1 + \frac{v_s}{v}} . \qquad \begin{bmatrix} \text{source moving away from} \\ \text{stationary observer} \end{bmatrix}$$

$f' < f$, because of the denominator is greater of 1.

Doppler effect also occurs when the source is at rest and the **observer is in motion**. In this case the wavelength $\lambda$ is not changed but the wave velocity with respect to the observer is changed. Let the **observer move toward the source** with a velocity $v_0$. The wave velocity relative to the observer is $v' = v + v_0$, where $v$ is the sound velocity in the still air.

Hence, the **new frequency** is

$$f' = \frac{v'}{\lambda} = \frac{v + v_0}{\lambda} ,$$

since $\lambda = \frac{v}{f}$,

$$f' = \left(1 + \frac{v_0}{v}\right) f. \qquad \begin{bmatrix} \text{observer moving toward} \\ \text{stationary source} \end{bmatrix}$$

If the **observer is moving away from the source**, the wave velocity relative to the observer is now $v' = v - v_0$ and

$$f' = \left(1 - \frac{v_0}{v}\right) f. \qquad \begin{bmatrix} \text{observer moving away} \\ \text{from stationary source} \end{bmatrix}$$