5. SPECIAL THEORY OF RELATIVITY

Albert Einstein recognized the contradiction between the predictions of classical physics and those of electromagnetism as regards the propagation of light. To explain these contradictions he proposed the two following postulates:

1. Relativity principle: **The laws of physics have the same form in all inertial reference frames.**
2. Principle of the constancy of the speed of light: **Light propagates through empty space at a definite speed independent of the speed of the source or observer.**

The principle of the constancy of the speed of light is in contradiction with with our everyday experience and therefore with Galilean transformation. Consequently a new transformation for objects moving with velocites close to the speed of light was needed. This transformation is called **Lorentz's transformation**:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The special theory of relativity deals with phenomena which take place in inertial reference frames which move at a speed close to the speed of light. Lorentz's transformation reduces to the Galilean transformation when the relative velocity of the two reference frames v is small compared to the speed of light.

The first consequence of Lorentz's transformation is **length contraction**:

the length of an object is measured to be shorter when it is moving than when it is at rest, or

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

where l is the length of the object in the reference frame with respect to which an object is moving and l_0 is the length of an object as measured by an observer at rest with respect to the object. The length l_0 is called the proper length. The coordinates of the end points are measured simultaneously.

The second consequence of Lorentz's transformation is **time dilatation**:

the time interval between two events which take place at the same place is greater for an observer on the earth than for a travelling observer

$$\Delta t' = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$

where $\Delta t'$ is the proper time which is measured in the reference frame with respect to which the object is at rest and Δt is the time that is measured in the reference frame with respect to which the object moves.

The third consequence of Lorentz's transformation is that there are new formulae for the relativistic **addition of velocities**

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \qquad u_y = \frac{\sqrt{1 - \frac{v^2}{c^2}}\, u'_y}{1 + \frac{u'_x v}{c^2}} \qquad u_z = \frac{\sqrt{1 - \frac{v^2}{c^2}}\, u'_z}{1 + \frac{u'_x v}{c^2}}$$

where primed velocities denote velocities of an object with respect to a moving reference frame, v is the velocity of the primed reference frame with respect to the unprimed reference frame and unprimed velocities are velocities of an object with respect to the unprimed reference frame.

The mass of an object depends on the velocity as

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the rest mass of an object, i.e. the mass measured in a reference frame in which the object is at rest and m is the mass as it will be measured when the object moves at speed v.

The relativistic momentum is

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The relativistic kinetic energy is

$$E_K = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

The rest energy of an object is

$$E_0 = m_0 c^2$$

The **total energy** of an object is

$$E = mc^2$$

The interval in four-dimensional space is defined as

$$\Delta s_{AB} = \left[c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \right]^{\frac{1}{2}}$$

For the components of a four-vector in four-dimensional Minkowski space-time we can write

$$x_1 = x \qquad x_2 = y \qquad x_3 = z \qquad x_4 = ict$$

Thus Lorentz's transformation is

$$x_1 = \frac{x_1' - \frac{iv}{c} x_4'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad x_2 = x_2' \qquad x_3 = x_3' \qquad x_4 = \frac{x_4' + \frac{iv}{c} x_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where i is an imaginary unit and x_1 x_2 x_3 and x_4 are components of any four-vector.

Problem 5-1. The spherical light wavefront emitted from the origin of reference frame S at time $t=0$ is described by the equation $x^2 + y^2 + z^2 = (ct)^2$.

Find the form of this light wavefront in the reference frame S' which is moving with respect to the reference frame S at constant speed v close to the speed of light.

Solution: To find the shape of the wavefront in the reference frame S' we have to substitute in the equation for a spherical wavefront for x, y, z and t from the inverse Lorentz transformation, or

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad y = y' \qquad z = z' \qquad t = \frac{t' + \frac{x'v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

After a little rearrangement we obtain

$$(x')^2 + (y')^2 + (z')^2 = (ct')^2$$

We therefore see that the light wavefront in the reference frame S' will also have the form of a sphere.

Problem 5-2. A stick of 1 m length moves about an observer on the earth in the direction parallel to its length at such a speed that the observer measures its length to be 0.5 m. Determine the speed of the stick.

Solution: To solve this problem we have to use the expression for length contraction:

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}}$$

After a little rearrangement we obtain for velocity:

$$v = c\sqrt{1 - \left(\frac{l}{l_0}\right)^2} = 3 \times 10^8 \sqrt{1 - (0.5)^2} = 2.6 \times 10^8 \ m\, s^{-1}$$

Problem 5-3. A spaceship is moving at a speed of $0.998c$. How much time is required to reach a star which is *100* light-years away?

Solution: If the star is 100 light-years away it would take 100 years as measured on the earth (travelling at the speed of light), to reach the star. To determine the time required to travel this distance by an astronaut in a spaceship we have to use the time dilatation formula, or

$$\Delta t' = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$

where the time for the trip, as measured on earth, is Δt, and as measured by the astronauts in their rest frame, it is $\Delta t'$. After substitution we obtain

$$\Delta t' = 100 \sqrt{1 - 0.998^2} = 6.3 \text{ } years.$$

Problem 5-4. An astronaut on the moon observes two spaceships approaching him from opposite directions at speeds $0.8c$ and $0.9c$ respectively. Calculate the relative velocity of these spaceships as measured by the observer on board one of them.

Solution: One of the spaceships approaches the astronaut on the moon at velocity u'_x, and the second spaceship approaches this astronaut at velocity v. The speed of the moon with respect to the second spaceship is $-v$ (the reference frames are approaching each other). Consequently the speed of the first spaceship with respect to the astronaut in the second spaceship is

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{0.8c + 0.9c}{1 + \frac{0.8c \text{ x } 0.9c}{c^2}} = 0.988c$$

Problem 5-5. An iron parallelepiped moves with respect to a reference frame S connected to the ground at speed $0.98c$ in the direction of its x-axis in such a way that its sides are parallel with the axis of the reference frame S. Calculate the mass density of the iron with respect to the reference frame S.

Solution: The mass density of iron at speed $v=0$ m/s is

$$\rho_0 = \frac{m_0}{V_0} = 7.8 \text{ x } 10^3 \text{ } kg.m^{-3}$$

The mass density of a parallelepiped which is moving with respect to the reference frame S can be found from

$$\rho_0 = \frac{m}{V} = \frac{m}{abc} = \frac{\dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}}{a_0\sqrt{1-\dfrac{v^2}{c^2}}\, b_0\, c_0} = \frac{\rho_0}{1-\dfrac{v^2}{c^2}} = \frac{7.8 \times 10^3}{1-(0.98)^2} = 195 \times 10^3\ kg.m^{-3}$$

because only side a of the parallelepiped which is parallel to the direction of motion is contracted.

- **Problem 5-6.** Estimate the kinetic energy of a *π-meson* which allows it to pass through the atmosphere of the earth and to reach the earth's surface. Assume the height of the atmosphere $h=300\ km$, lifetime of a *π-meson* $\tau=2.5x10^{-8}\ s$ and the rest mass of a *π-meson* $m_0=2.5x10^{-25} kg$.

Solution: From the point of view of an observer on the earth the time required by a *π-meson* to travel through the atmosphere is

$$t = \frac{h}{c} = \frac{300 \times 10^3}{3 \times 10^8} = 10^{-3}\ s$$

(We assume that the speed of a *π-meson* is very close to the speed of light). From the comparison of this time with the lifetime of *π-meson* we can conclude that no *π-meson* can ever reach the surface of the earth, which is in contradiction with reality. We have to estimate the kinetic energy of a *π-meson*

$$E_K = m_0 c^2 \left(\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} - 1 \right)$$

From this expression we see that we have to determine the term $\sqrt{1-\frac{v^2}{c^2}}$. This term can be found from the time dilatation formula. Substituting into this formula we have

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{2.5 \times 10^{-8}}{10^{-3}} = 2.5 \times 10^{-5}$$

Finally we can substitute into the expression for kinetic energy

$$E_K = 2.5 \times 10^{-25} \left(3 \times 10^8\right)^2 \left(\frac{1}{2.5 \times 10^{-5}} - 1 \right) = 9 \times 10^{-4}\ J$$

Problem 5-7. Calculate the mass and velocity of an electron which is accelerated with voltage $10^5\ V$.

Solution: The kinetic energy of an electron can be expressed as a function of its accelerating voltage, or

$$E_K = eU$$

and its total relativistic energy is equal to the sum of its kinetic energy plus rest energy, or

$$mc^2 = E_K + m_0 c^2$$

From the combination of these two expressions we can find the mass of an accelerated electron

$$m = \frac{E_K + m_0 c^2}{c^2} = \frac{1.6 \times 10^{-19} \times 10^5 + 9.1 \times 10^{-31} \times \left(3 \times 10^8\right)^2}{\left(3 \times 10^8\right)^2} = 1.09 \times 10^{-30}\ kg$$

From the ratio of $\frac{m}{m_0} = \frac{1.09 \times 10^{-30}}{9.1 \times 10^{-31}} = 1.2$ we can see that the mass of the electron has been increased by 20%.

The velocity of an electron can be found from the expression for its total energy

$$\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} c^2 = E_K + m_0 c^2$$

For the square of the velocity we obtain after a little rearrangement

$$v^2 = c^2 \left[1 - \left(\frac{m_0 c^2}{E_K + m_0 c^2} \right)^2 \right] =$$

$$= \left(3 \times 10^8\right)^2 \left[1 - \left(\frac{9.1 \times 10^{-31} \left(3 \times 10^8\right)^2}{10^5 \times 1.6 \times 10^{-19} + 9.1 \times 10^{-31} \left(3 \times 10^8\right)^2} \right)^2 \right] = 2.7 \times 10^{16}$$

The velocity of an electron is therefore $v = 1.63 \times 10^8\ m/s$.

Problem 5-8. How much energy will be released if the π^0 meson is transformed by decay completely into electromagnetic radiation? The rest mass of a π^0 meson is $m_0 = 2.4 \times 10^{-28}\ kg$.

Solution: The rest energy of a π^0 meson is

$$E_0 = m_0 c^2 = 2.4 \times 10^{-28} \times (3 \times 10^8)^2 = 2.2 \times 10^{-11}\ J$$

This is how much energy would be released if the π^0 decayed at rest.

Problem 5-9. An ellipsoid whose ratio of semimajor axis is 3:1 moves in the direction of its longer axis. Determine the speed of the ellipsoid with respect to the observer at rest at which this observer will see a sphere instead of an ellipsoid.

$$\left[2\sqrt{2}\ x\, 10^{8}\ ms^{-1}\right]$$

Problem 5-10. What will be the mean lifetime of a muon as measured in the laboratory if it is travelling at speed $0.6c$ with respect to the laboratory? Its mean lifetime at rest is $2.2x10^{-6}$ s.

$$\left[2.8\ x\, 10^{-6}\ s\right]$$

Problem 5-11. Calculate the speed of an electron so that its mass equals the rest mass of a proton. The rest mass of an electron is $m_e = 9.1x10^{-31} kg$, and the rest mass of a proton is $m_p = 1.67x10^{-27} kg$.

$$\left[v \approx c\right]$$

Problem 5-12. Calculate the speed of an electron with kinetic energy $0{,}1\ MeV$,

a) following classical mechanics, and

b) following the special theory of relativity.

$$\left[a)\ v = 1.87\ x\, 10^{8}\ m/s,\ b)\ v = 1.64\ x\, 10^{8}\ m/s\right]$$

Problem 5-13. How much energy could be obtained if the rest mass of 1 mol of atomic hydrogen could be completely converted into energy. Avogadro's number $N_A = 6.02x10^{23}\ mol^{-1}$, $m_0 = 1.67x10^{-27}\ kg$.

$$\left[E_0 = 9\ x\, 10^{13}\ J\right]$$

Problem 5-14. Calculate the mass of an electron with a kinetic energy $1\ GeV$ expressed in multiples of the rest mass of the electron.

$$\left[\frac{m}{m_0} = 1995\right]$$

Problem 5-15. If the relativistic kinetic energy of a particle is 9 times its rest energy, at what fraction of the speed of light must the particle be travelling?

$$\left[v = 0.995c\right]$$

Problem 5-16. The initial velocity of an electron is $1.4x10^{8}\ m/s$. What amount of energy must be supplied to this electron to double its velocity.

$$\left[\Delta E = 1.3\ x\, 10^{-13}\ J\right]$$