3.3 ELECTRIC CURRENT

Electric current is defined as the net amount of charge that passes through the given cross-section per unit time

$$I = \frac{dQ}{dt} \quad [A]$$

The direction of current is generally taken as being that of positive charges.
The current density j is a vector the magnitude of which is defined as a current passing through a given cross-sectional area. Thus the current dI passing through an elementary oriented area $d\mathbf{S}$ equals

$$dI = \mathbf{j}\,d\mathbf{S}$$

The total current passing through the whole cross-sectional area is

$$I = \iint_S \mathbf{j}\,d\mathbf{S}$$

From a microscopic point of view, current density is related to the number of charge carriers per unit volume n, their charge e, and their drift velocity v by

$$\mathbf{j} = e\,n\,\mathbf{v} = \rho\mathbf{v}$$

where $\rho = e\,n$ is the volume charge density.
The conservation of an electric charge law (equation of continuity) expresses the fact that no current can flow away from a closed surface without diminishing the amount of charge that is there

$$\iint_{(S)} \mathbf{j}\,d\mathbf{S} = -\frac{d}{dt}\iiint_V \rho\,dV$$

Ohm's law states that the current in a conductor is proportional to the potential difference applied to its two ends. The proportionality constant is called the resistance R of the material so

$$U = I\,R$$

where the unit of resistance is called the ohm (Ω).
The differential form of Ohm's law expresses the relation between the current density **j** and the electric field **E**

$$\mathbf{j} = \gamma\,\mathbf{E}$$

where the proportionality constant γ is called the conductivity of the conductor. Conductivity can also be expressed using microscopic quantities, e.g.

$$\gamma = \frac{n\,e^2\,\bar{t}}{2\,m}$$

where n is the number of free electrons per unit of volume, $\bar{t}$ is the average time between two succeeding collisions of electrons and m is the mass of an electron.
The reciprocal of conductivity is called the resistivity or

$$\rho = \frac{1}{\gamma}$$

The electromotive force U_e over a closed path C is defined as the work done on a unit positive charge by the total electrical force acting on the charge as it traverses the path or

$$U_e = \oint_C \mathbf{E}\,d\mathbf{l}\,.$$

Ohm's law for a closed electric circuit can be written in the form

$$U_e = I(R + R_i)$$

where R_i is the internal resistance of the source, R is the resistance of the wire and I denotes the current in the conductor.

Kirchhoff's first law states that the algebraic sum of the currents into the junction point equals zero or

$$\sum_{k=1}^{n} I_k = 0.$$

Kirchhoff's second law states that the algebraic sum of the changes in potential around any closed path of the circuit equals zero or

$$\sum_{k=1}^{n} U_{ek} = \sum_{k=1}^{n} R_k I_k .$$

Joule's law expresses the rate at which energy is transformed in a resistance R from an electric to another form of energy (such as heat)

$$P = U.I = R.I^2 = \frac{U^2}{R} \quad [W]$$

Joule's law in the differential form states that power dissipated per unit volume by the flow of an electric current is

$$p = \mathbf{E}.\mathbf{j} \quad \left[W/m^3\right]$$

If an external electric field is applied to an assemblage of free electrons the current that arises is called **the conduction current.** The current density of this current is

$$\mathbf{j} = \rho \mathbf{v} \qquad \text{or} \qquad \mathbf{j} = \gamma \mathbf{E}.$$

Convection current is the current due to the motion of free charges which move together with macroscopic bodies.

Displacement current is the current which is due to the relative displacement of the induced charges and the time variations of the electric field. The current density of the displacement current is

$$\mathbf{j} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \mathbf{P}}{\partial t} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where **P** is the polarisation vector.

Problem 3-56. The current in a wire is a function of time $I = I_0 + k\,t$ where $I_0 = 4\,A$ and $k = 2\,A/s$. Determine:

a) How much charge passes a cross section of the wire during the time interval between $t_1 = 2\,s$ and $t_2 = 6\,s$.

b) What constant current I would transport the same charge in the same time interval?

Solution: From the general definition of an electric current $I=\dfrac{dQ}{dt}$ we can obtain

$$Q=\int_{t_1}^{t_2} I\,dt.$$

Thus we have:

a) $$Q=\int_{t_1}^{t_2}(I_0+kt)dt=\int_{2}^{6}(4+2t)dt=48C$$

b) $$I=\frac{Q}{(t_2-t_1)}=\frac{48}{(6-2)}=12\ A$$

Problem 3-57. A copper wire of diameter d = 3.2 mm carries a steady current I = 5 A. If each copper atom contributes one free electron to the conduction of the current, find:

a) the current density,

b) the drift speed of the current carrying electrons,

c) the root-mean square velocity of these electrons assuming that they behave like an ideal gas at a temperature of 20° C.

The molar weight of copper $M_{Cu}=63.5\ kg.kmol^{-1}$, density $\rho=8.89\times 10^3\ kg.m^{-3}$; Avogadro's number $N_A=6.02\times 10^{26}\ kmol^{-1}$.

Solution:

a) $$j=\frac{I}{\pi r^2}=\frac{5}{\pi\left(1.6\times10^{-3}\right)^2}=6.2\times10^5\,A.m^{-2}$$

b) Since we assume that there is one free electron per atom, the density of free electrons, n, is the same as the density of Cu atoms. The number of Cu atoms per one kilogram is N_A/M_{Cu} and therefore the number of Cu atoms per m^3 is

$$n=\frac{N_A}{M_{Cu}}\rho=\frac{6.02\times10^{26}}{63.5}\,8.89\times10^3=8.4\times10^{28}\ m^{-3}$$

Then, the drift speed is

$$v=\frac{j}{e\,n}=\frac{6.2\times10^5}{1.6\times10^{-19}\times8.4\times10^{28}}=4.6\times10^{-5}\ m.s^{-1}$$

c) If we treat the free electrons as an ideal gas (a rough approximation) then their root-mean square velocity at temperature 20° C is

$$v_{rms}=\sqrt{\frac{3kT}{m}}=\sqrt{\frac{3\times1.38\times10^{-23}\times293}{9.1\times10^{-31}}}=1.15\times10^5\ m.s^{-1}$$

Thus we see that the drift speed is very much less than the root-mean square velocity of electrons.

Problem 3-58. For the conditions in problem 3-57 determine the mean time $\bar{t}$ between collisions and the mean free path λ for free electrons in copper. (Assume that free electrons behave like an ideal gas). Resistivity of copper is $\rho_{Cu}=1.7\times10^{-8}\ \Omega.m$.

Solution: The conductivity of a material is expressed as

$$\gamma = \frac{ne^2 \bar{t}}{2m}$$

Resistivity is defined as $\rho = \frac{1}{\gamma} = \frac{2m}{ne^2 \bar{t}}$. Thus the mean free time between collisions of electrons with copper ions is

$$\bar{t} = \frac{2m}{ne^2 \rho_{Cu}}$$

The density of free electrons was calculated in problem 3-57. We can therefore substitute the numerical values

$$\bar{t} = \frac{2 \times 9.1 \times 10^{-31}}{8.4 \times 10^{28} \times \left(1.6 \times 10^{-19}\right)^2 \times 1.7 \times 10^{-8}} = 4.9 \times 10^{-14} s$$

The mean free path of an electron is

$$\lambda = \bar{t} v_{rms} = 4.9 \times 10^{-14} \times 1.15 \times 10^5 = 5.6 \times 10^{-9} m.$$

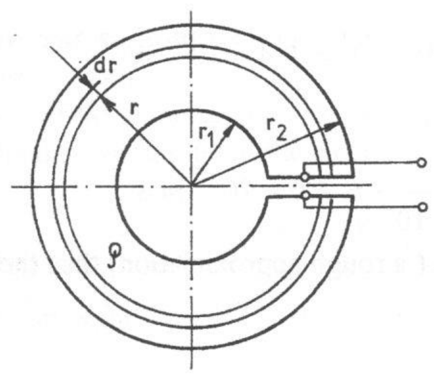
Problem 3-59. A hollow cylindrical resistor with inner radius r_1, outer radius r_2 and small length *h* is made of a material whose resistivity is ρ. Calculate the resistance of the resistor if:

a) the resistor is radially cut and the current enters the resistor through the cut sides,

b) current flows radially outward.

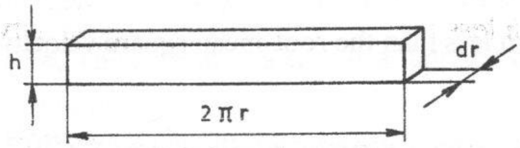
Solution: A resistor has different resistance, depending on how the potential difference is applied to it.

a) If the cylindrical resistor is radially cut (see the figure) and the current enters it through the cut sides, then the elementary conductor, which is shown in the figure, has a length ($2\pi r$) and a cross-section ($h.dr$). The conductivity of this elementary conductor is therefore



$$d\gamma = \frac{1}{dR} = \frac{1}{\rho \frac{2\pi r}{h\,dr}} = \frac{h}{\rho\, 2\pi} \frac{dr}{r}$$

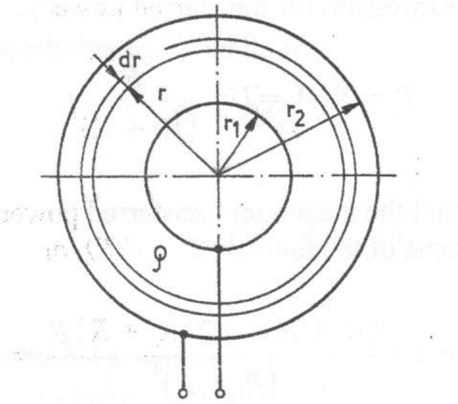
After integration for the total conductivity of the ring we have

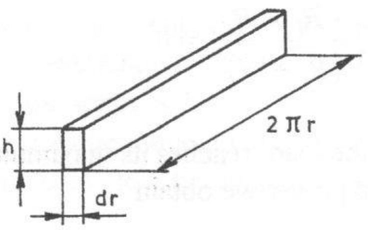


$$\gamma = \frac{h}{2\pi\rho} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{h}{2\pi\rho} \ln \frac{r_2}{r_1}$$

The resistance of the ring is the reciprocal of the conductivity or

$$R = \frac{1}{\gamma} = \frac{2\pi\rho}{h} \frac{1}{\ln \frac{r_2}{r_1}}.$$





b) To determine the resistance of this resistor in which the current flows radially outward we divide the resistor into concentric cylindrical shells (see the figure) and then we integrate it from radius r_1 to radius r_2. The length of each elementary conductor is *dr* and its cross-section is ($2\pi rh$). Thus we have

$$dR = \rho \frac{dr}{2\pi r h}$$

The total resistance obtained by integration is

$$R = \frac{\rho}{2\pi h}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\rho}{2\pi h}\ln\frac{r_2}{r_1}$$

Problem 3-60. A rectangular carbon block has the length $a = 10$ cm and a cross-section $b.c = 3\ cm\ .2\ cm$. Determine the electric field E inside the block and the current density j if the potential difference 10 V is applied on its ends. The conductivity of carbon is $\gamma = 1.6\times10^4\ S.m^{-1}$.

Solution: The electric field inside the carbon block is

$$E = \frac{U}{a} = \frac{10}{0,1} = 100\ V.m^{-1}$$

The current density is

$$j = \frac{I}{b.c}$$

From Ohm's law we have

$$I = \frac{U}{R} = \frac{U}{\frac{1}{\gamma}\frac{a}{b.c}} = \frac{\gamma\ b.c.U}{a}$$

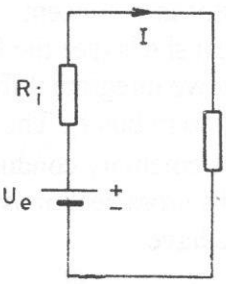
Thus, for the current density we have

$$j = \frac{\gamma\ b.c.U}{a.b.c} = \frac{\gamma\ U}{a} = \frac{1.6\times10^4\times10}{0.1} = 1.6\times10^6\ A/m^2$$

Problem 3-61. The external resistance R is connected across a source of an electromotive force $U_e = 1,5\ V$ with the internal resistance $R_i = 1\ \Omega$. Determine the optimal external resistance R so as to maximise power transfer to the load. What will be maximal power P_{max} transferred to the load?

Solution: The electric circuit is shown in the figure. From Ohm's law for a closed electric circuit we have:

$$I = \frac{U_e}{R_i + R}$$



The expression for transferred power is

$$P = R.I^2 = U_e^2 \frac{R}{(R_i + R)^2}$$

We find the maximum transferred power as an extreme of the function $P = P(R)$ or

$$\frac{dP}{dR} = U_e^2 \frac{(R_i + R)^2 - 2(R_i + R)R}{(R_i + R)^4} =$$

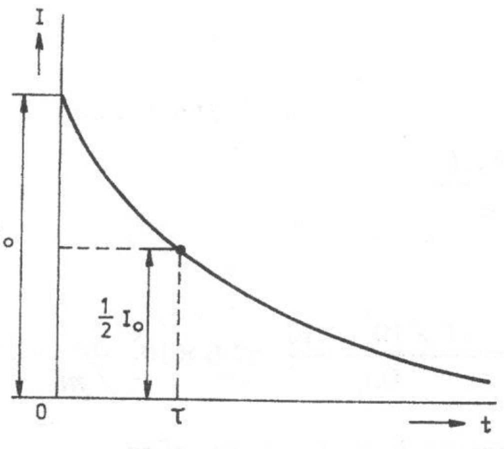
$$= U_e^2 \frac{R_i - R}{(R_i + R)^3} = 0$$

The power transferred from the source to the load reaches its maximum when $R = R_i$. Thus for the maximum transferred power we obtain

$$P_{max} = U_e^2 \frac{R_i}{(R_i + R_i)^2} = \frac{U_e^2}{4R_i} = 0.55\ W$$

Problem 3-62. An electric charge $Q = 40\ C$ passes through the conductor of a resistance $R = 5\ \Omega$. Determine the work that has been done if the current flowing through the conductor decreases to zero in such a way that during each $\tau = 16\ s$ it was decreased to one half.

Solution: The dependence of current on time is expressed by an exponential function:



$$I = I_0\, e^{-bt}$$

The charge passing through the conductor in the time interval from zero to infinity is

$$Q = \int_0^\infty I\, dt = \int_0^\infty I_0\, e^{-bt} dt = \frac{I_0}{b}$$

From this we have

$$I_0 = Q.b$$

The constant b can be found from the initial conditions:

for $t = 0$ $\quad I = I_0$ $\quad$ and for $\quad t = \tau \quad I = \frac{I_0}{2}$.

For the current we thus have

$$\frac{I_0}{2} = I_0 e^{-b\tau} \quad \text{and} \quad b = \frac{\ln 2}{\tau}$$

The work that has been done is

$$A = \int_0^\infty RI^2 dt = RI_0^2 \int_0^\infty e^{-2bt} dt = RI_0^2 \left[\frac{e^{-2bt}}{-2b} \right]_0^\infty = \frac{RI_0^2}{2b} =$$

$$= \frac{RQ^2 b^2}{2b} = \frac{RQ^2}{2} \frac{\ln 2}{\tau} = \frac{5 \times 40^2 \times 0.693}{2 \times 16} = 173\ J$$

Problem 3-63. The capacitor of capacitance $C = 10\ \mu F$ is connected to the source of saw tooth pulse voltage $U = kt\ (0 \leq t \leq 100\ s)$. Calculate:

a) the displacement current if $k = 0{,}5\ V/s$,

b) the conduction current if the insulation resistance of the dielectric material of the capacitor is $R = 10^6\ \Omega$,

c) the conduction and the displacement current if the capacitor is continually connected to a potential difference of $U = 500\ V$.

Solution:

a) For the displacement current we have $I_d = S \dfrac{\partial D}{\partial t}$

Substituting for $D = \dfrac{Q}{S} = \dfrac{C.U}{S}$ we obtain

$$I_d = C \frac{dU}{dt} = C \frac{d}{dt}(kt) = ck = 10 \times 10^{-6} \times 0.5 = 5 \times 10^{-6} A$$

We see that the displacement current remains constant.

b) The conduction current: $I_c = \dfrac{U}{R} = \dfrac{kt}{R} = 0.5 \times 10^{-6}\ t$

We see that the conduction current is directly proportional to time.

c) Since in this case is D = constant, the displacement current $I_d = 0$ and the conduction current I_c is the leakage current of the capacitor, or

$$I_c = \frac{U}{R} = \frac{500}{10^6} = 5 \times 10^{-4}\ A.$$

Problem 3-64. The conduction current of the current density $j_c = j_0 . \sin \omega t$ passes through the medium of conductivity γ and relative permittivity $\varepsilon_r = 1$. Calculate:

a) the current density of the displacement current,

b) the ratio of amplitudes of the displacement and the conduction current, if $\gamma = 10^7\ S.m^{-1}$.

Solution:

a) The current density of the displacement current is

$$j_d = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t} = \varepsilon \frac{1}{\gamma} \frac{\partial j_c}{\partial t} = \frac{\varepsilon}{\gamma} j_0\, \omega \cos \omega t = \frac{\varepsilon_0 \varepsilon_r}{\gamma} j_0\, 2\pi f \cos \omega t$$

where we have used the Ohm's law in the differential form $j_c = \gamma E$.
From the expressions for conduction and displacement current it is obvious that there is a phase shift 90° between these two currents.

b) The ratio of amplitudes is

$$\frac{j_d}{j_c} = \frac{\varepsilon_0 \varepsilon_r j_0 2\pi f}{\gamma}\frac{1}{j_0} = \frac{\varepsilon_0 \varepsilon_r 2\pi}{\gamma} f \cong 10^{-18} f.$$

Problem 3-65. How much charge passes a cross section of the wire during a time interval $t = 10\ s$ if:

a) there is a steady current $I = 5\ A$,
b) the current linearly increases from 0 to 3 A.

$[a)\ Q = 50C]$ $[b)\ Q = 15C]$

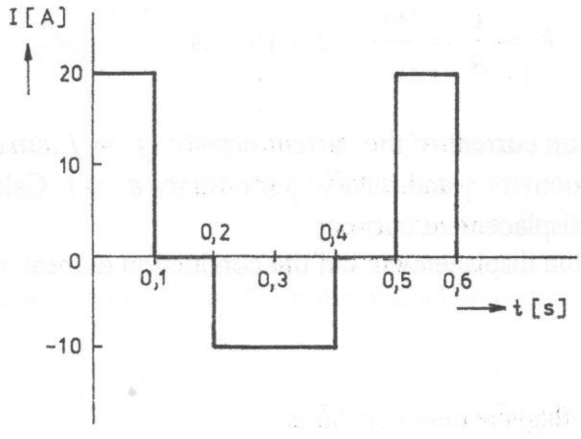
Problem 3-66. A current of 1 A flows in a wire. How many electrons per second flow past any point in the wire?

$[n = 6.25 \times 10^{18}]$

Problem 3-67. A copper wire of a cross section $S = 4\ mm^2$ carries a current of $I = 10\ A$.
a) What is the electric field in the wire if the resistivity of copper $\rho = 1.7 \times 10^{-8}\ \Omega.m$?
b) What is the voltage between two points at a distance $d=50\ cm$?

$[E = 4.25 \times 10^{-2} V/m]$ $[U=0{,}021\ V]$

Problem 3-68. The current in a wire is a function of time which is shown in the figure. Find the average value of current $\bar{I}$ in the time interval <0;0,5 s>.



$[\bar{I} = 0\ A]$

Problem 3-69. A battery of electromotive force $U_e = 1.06\,V$ and internal resistance $R_i = 1.8\ \Omega$ has a coil of resistance $R = 6.0\ \Omega$ connected across its terminals.
a) Find the potential difference U between the battery terminals.
b) Find the current in the circuit.
c) Find the dissipated power.

$[U = 0.815\ V]$ $[I = 0.136\ A]$ $[P = 33.2\ mW]$

Problem 3-70. The resistance of an immersion heater coil is 16 Ω. Calculate the time in which 400 g of water in the pot begins to boil if the initial temperature of water is 10° C and the efficiency of the heater is 60 %. The heater is designed to operate from 120 V. ($c_w = 4.18 \times 10^3\ J.kg^{-1}\ K^{-1}$).

$[\tau = 418\ s]$

Problem 3-71. The capacitor of a capacitance C = 100 μF and resistance of a dielectric material R = 10^6 Ω is connected to the source the voltage of which can be arbitrarily varied.
a) At voltage $U_1 = 100\ V$ calculate the displacement and the conduction current.
b) Assume that the voltage linearly increases from initial value $U_1 = 100\ V$ with a slope k = 10 V/s. Calculate the conduction and the displacement current after 20 s.

$[a)\ I_d = 0\ A \quad I_c = 10^{-4}\ A]$

$[b)\ I_c = 3 \times 10^{-4}\ A \quad I_d = 10^{-3}\ A]$

3.4 MAGNETIC FIELD

The magnetic field is associated with moving electric charges.
The basic magnetic field vector **B**, which is called **magnetic induction,** is operationally defined by the force acting on a moving electric charge q in a magnetic field as

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

where **v** denotes the velocity of the moving charge.
The magnetic field exerts a **force on a current carrying conductor** placed in this magnetic field

$$\mathbf{F} = \int_l (I.d\mathbf{l} \times \mathbf{B})$$

where **dl** is a vector in the direction of the current with a magnitude equal to the length of the current carrying element.