3. PHYSICAL FIELDS

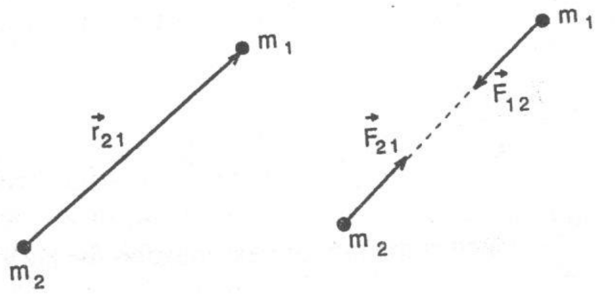
3.1 GRAVITATION

Newton's law of universal gravitation states that every particle in the Universe attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of this force is along a line joining the two particles.

$$F = G\frac{m_1 m_2}{r^2}$$

where m_1 *and* m_2 are the masses of the two particles, r is the distance between them, and G is a universal constant that has the same numerical value for all objects and that has been measured experimentally. The accepted value is $G \approx 6.67\times10^{-11}\, N\, m^2\, kg^{-2}$.

Newton's law of universal gravitation can be expressed in vector form as



$$\mathbf{F}_{12} = -G\frac{m_1 m_2}{r_{21}^2}\mathbf{r}_{21}^0$$

where $\mathbf{F}_{12}$ is the vector force on mass m_1 exerted by mass m_2, which is a distance r_{21} away; $\mathbf{r}_{21}^0$ is a unit vector that points from particle 2 toward particle 1 along the line joining them so that $\mathbf{r}_{21}^0 = \mathbf{r}_{21}/r_{21}$, where $\mathbf{r}_{21}$ is the displacement vector as shown in Fig. The minus sign is necessary because the force on particle 1 due to particle 2 points in the direction opposite to $\mathbf{r}_{21}^0$. The displacement vector $\mathbf{r}_{12}$ is a vector of the same magnitude as $\mathbf{r}_{21}$, but it points in the opposite direction so that

$$\mathbf{r}_{12} = -\mathbf{r}_{21}$$

By the third law of motion, the force $\mathbf{F}_{21}$ acting on m_2 exerted by m_1 must have the same magnitude as $\mathbf{F}_{12}$ but acts in the opposite direction, so that

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = G\frac{m_1 m_2}{r_{21}^2}\mathbf{r}_{21}^0 = -G\frac{m_2 m_1}{r_{12}^2}\mathbf{r}_{12}^0$$

When the law of universal gravitation is applied to the gravitational force between the Earth and an object of mass m at its surface, this force of gravity due to the Earth is the weight of the object, which we have been writing as mg. Thus,

$$mg = G\frac{m m_e}{r_e^2}$$

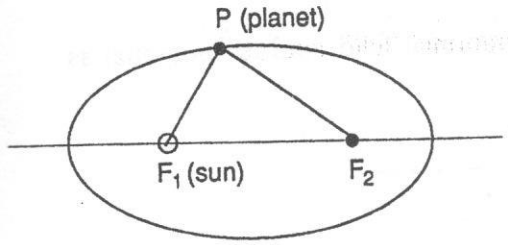
where m_e is the mass of the Earth and r_e is the radius of the Earth.

Hence,

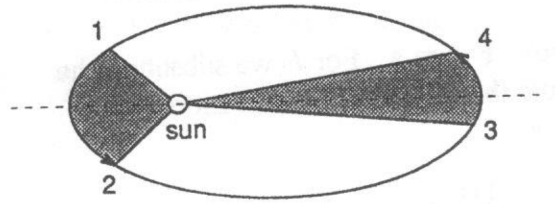
$$g = G\frac{m_e}{r_e^2}$$

Thus the acceleration of gravity at the surface of the Earth, g, is determined by m_e *and* r_e.. However, this expression does not give precise values for g at various locations because the earth is not a perfect sphere and its rotation also has an effect on the measurement of g.

Kepler's laws



Kepler's first law: the path of each planet about the sun is an ellipse with the Sun at one focus.
We know that an ellipse is a closed curve such that the sum of the distances from any point P on the curve to two fixed points, called the foci F_1 *and* F_2, remains constant. That is, the sum of the distances $F_1P + F_2P$ is the same for all points on the curve.



Kepler's second law: each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal periods of time. Planets move fastest in that part of their orbit where they are closest to the Sun.

Kepler's third law: the ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of their semimajor axis.
That is

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$$

where T_1 *and* T_2 represent the periods of two planets, and r_1 *and* r_2 are their semimajor axes.

Gravitational field intensity g is defined as the gravitational force per unit mass at any point in space:

$$\mathbf{g} = \frac{\mathbf{F}}{m}$$

The unit of **g** is N/kg.
It is clear that gravitational field intensity has a magnitude equal to the acceleration due to gravity at that point.

The potential energy of mass m in the gravitational field due to the mass M placed at the origin of the chosen coordinate system equals

$$PE = -G\frac{mM}{r}$$

if we choose zero PE at infinity.

The potential of the gravitational field gives us the potential energy of a unit mass:

$$V = -G\frac{M}{r}$$

There is a relation between the intensity of the gravitational field and the potential as follows:

$$\mathbf{g} = -\ \mathrm{grad}\ V$$

Problem 3-1. Try to derive Kepler's third law for the special case of a circular orbit.

Solution: We start from the second law of motion F = m a . For F we substitute the law of universal gravitation and for a we substitute the centripetal acceleration v^2/r:

$$G\frac{m_1 M}{r_1^2} = m_1\frac{v_1^2}{r_1} \qquad (1)$$

Here m_1 is the mass of a planet, r_1 is its orbit radius, v_1 is the speed of the planet in orbit, M is the mass of the Sun, since it is the gravitational attraction of the Sun that keeps each planet in its orbit.
The period of its motion is

$$T_1 = \frac{2\pi r_1}{v_1}$$

We substitute for v_1 into Eq.(1) and obtain

$$\frac{T_1^2}{r_1^3} = \frac{4\pi^2}{GM}$$

The same result can be derived for the second planet:

$$\frac{T_2^2}{r_2^3} = \frac{4\pi^2}{GM}$$

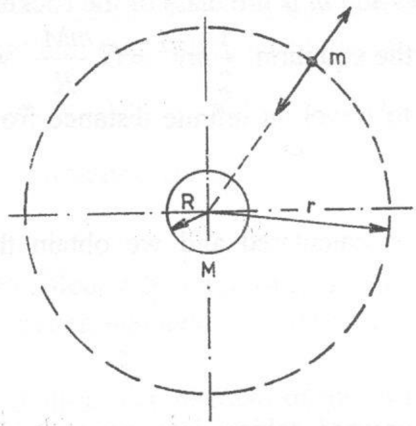
where T_2 *and* r_2 are the period and orbit radius, respectively, for the second planet.
From last two relations we obtain

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$$

which is Kepler's third law.

Problem 3-2. Calculate the orbital velocity and the distance from the Earth of a stationary satellite.

Solution: For a satellite that moves (approximately) with uniform circular motion the acceleration is v^2/r. The force that gives a satellite this acceleration is the force of gravity. When we apply the second law of motion, we obtain



$$G\frac{mM}{r^2} = m\frac{v^2}{r}$$

where r is the distance of the satellite from the earth's center and v is its velocity; m is the mass of the satellite and M is the mass of the Earth. From this relation we obtain the orbital velocity

$$v = \sqrt{G\frac{M}{r}}$$

When we put r = R=the radius of the Earth we obtain $v \approx 7912$ m/s.
The period of the satellite is $T = \frac{2\pi r}{v} = 5065 s$.

At height h above the earth the velocity of the satellite equals

$$v = \sqrt{G\frac{M}{(R+h)}}$$

For instance, if h = 5000 km the velocity of the satellite is $v = 5.92 \times 10^3 ms^{-1}$.

Note that the orbital velocity decreases with increasing height h and the period increases with increasing height:

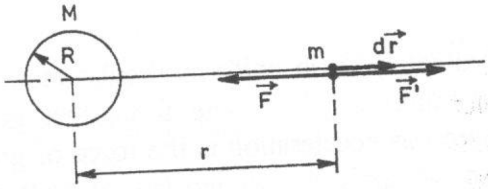
$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

The period of the stationary satellite is just 24 hours. Thus, the corresponding height $h \approx 35810$ km and the velocity $v \approx 3076$ m/s.

Problem 3-3. Calculate the work done by the engine of a rocket to free itself from the influence of the earth's gravity.

Solution: At every point of the rocket's path the force **F**′ of the engine is the same in magnitude as the force of gravity **F** but it has an opposite direction from that of the force of gravity . Thus, **F**′ = - **F** and the work done by this force equals

$$W = \int_R^\infty \mathbf{F}'d\mathbf{r} = -\int_R^\infty \mathbf{F}d\mathbf{r} = G\frac{mM}{R}$$



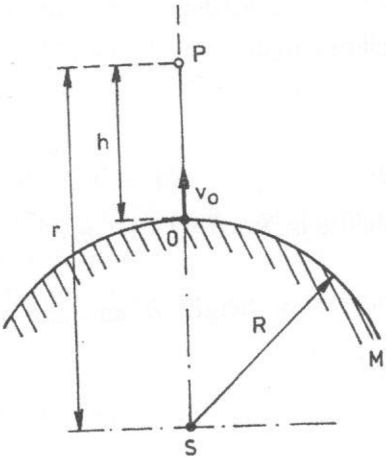
where R is the radius of the Earth, M is its mass and m is the mass of the rocket.

From the equation $\frac{1}{2}mv^2 = G\frac{mM}{R}$ we obtain the initial velocity of the rocket to enable it to travel an infinite distance from the Earth $v = \sqrt{2G\frac{M}{R}}$.

If we substitute the earth's constants the velocity is calculated and we obtain the magnitude v ≈ 11189 m/s.

Problem 3-4. Examine the projection vertically upward taking into account the variability of gravitational acceleration.

Solution: We use the principle of conservation of energy: the sum of KE and PE must be constant at any point of this motion path.



The potential energy of a body of mass m at point P at distance r from the earth's center equals (M is the mass of the Earth)

$$PE = -G\frac{mM}{r}$$

and on the earth's surface it is

$$PE = -G\frac{mM}{R}$$

Now we can write the equation

$$\frac{1}{2}mv_0^2 - G\frac{mM}{R} = \frac{1}{2}mv^2 - G\frac{mM}{r}$$

when v_0 is the initial velocity of the body.

From here the velocity at point P is calculated:

$$v = \sqrt{v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right)}$$

The top of the path is at distance $r_t = R + h_t$. This is given from the condition v = 0; So,

$$r_t = \frac{2GMR}{2GM - Rv_0^2}$$

Analysing this relation we can conclude:

if $2GM > Rv_0^2$, the body will return back to the earth after reaching the top of its path;

if $2GM \leq Rv_0^2$ the body will continue travelling to infinity, and the velocity

$$v_0 = \sqrt{\frac{2GM}{R}} \approx 11.2 \times 10^3\ ms^{-2}$$

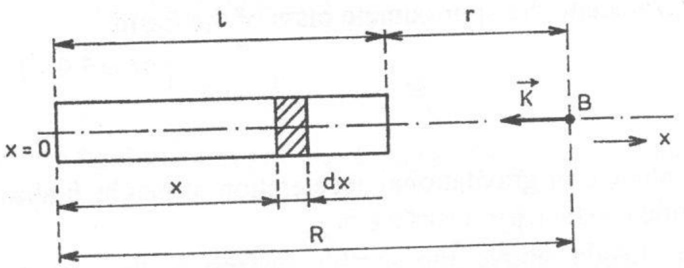
is the so-called the escape velocity. Its magnitude decreases as the height above the earth of the starting place increases:

$$v_h = \sqrt{\frac{2GM}{R+h}} = v_0\sqrt{\frac{R}{R+h}}$$

For instance, for h = 5000 km the escape velocity would be 8.4 km/s.

Problem 3-5. A uniform rod has mass *m* and length L. Calculate the potential and the gravitational field intensity of the rod at the point B on its axis. (see Fig.)

Solution: An element of the rod has the mass $dm = \rho S\,dx$. (ρ is density, S is cross-sectional area) This mass causes the potential dV at point B



$$dV = -G\frac{dm}{R-x} = -GS\rho\frac{dx}{R-x}$$

and the potential due to the whole rod equals

$$V = \int_{x=0}^{L} dV = -G\frac{m}{L}\ln\frac{r+L}{r}$$

The intensity of the gravitational field equals

$$\mathbf{g} = -\mathbf{i}\frac{dV}{dr} = -\mathbf{i}G\frac{m}{r(r+L)}$$

Problem 3-6. At what altitude above the earth's surface would the acceleration of gravity be $4.9\,ms^{-2}$?
The mass of the Earth is $6 \times 10^{24}\ kg$ and its mean radius is $6.4 \times 10^6\ m$.

$\left[2.6 \times 10^6\ m\right]$

Problem 3-7. A projectile is fired vertically upward from the earth's surface with an initial velocity of 10 km/s. Disregarding atmospheric friction, how far above the surface of the Earth will it go? Take the earth's radius as 6400 km.

$\left[2.6 \times 10^4\ km\right]$

Problem 3-8. Calculate the orbital velocity required for a satellite to stay in a circular orbit 200 km above the earth's surface. What is its period?

$$\left[v = 7.79 \times 10^3\ ms^{-1}\right]$$

[88.4 min]

Problem 3-9. Consider the line joining the earth's center and that of the Moon. Find a place on this line at which the attractive force of the Earth has the same magnitude as the attractive force of the Moon.
The distance between the earth's center and that of the Moon is $384 \times 10^3\ km$, the earth's mass is 81-times greater than that of the Moon.

[$384 \times 10^2\ km$ from the moon's center]

Problem 3-10. Given the radius of the Earth $R = 6378 \times 10^3\ m$, the gravitational acceleration on the earth's surface $g_0 = 9.81 ms^{-2}$ and a universal constant $G = 6.67 \times 10^{-11}\ N m^2\ kg^{-2}$, calculate the approximate mass of the Earth.

$$\left[M \approx 5.9 \times 10^{24}\ kg\right]$$

Problem 3-11. a) Find the change in gravitational acceleration at height h above the earth's surface, if its magnitude on the earth's surface is g_0.
b) At what height above the earth's surface is the gravitational acceleration n-times smaller than its magnitude g_0 on the earth's surface?
(the radius R of the earth is given)

$$\left[g_h \approx g_0\left(1 - \frac{2h}{R}\right)\right]$$

$$\left[h = -R + \frac{R}{\sqrt{n}}\right]$$

Problem 3-12. Consider the earth's motion to be approximately in a circular path. The period of the earth's motion around the sun T ≈ 365 days and the radius of the earth's path $r \approx 150 \times 10^6\ km$. Find the mass of the Sun.

$$\left[M = \frac{1}{G}\frac{4\pi^2 r^3}{T^2} \approx 2 \times 10^{30}\ kg\right]$$

Problem 3-13. The mass of the Sun is 355×10^3 times greater than that of the Earth. The radius of the Sun is 112 times greater than that of the Earth. Find the gravitational acceleration on the sun's surface.

$$\left[\approx 277 ms^{-2}\right]$$