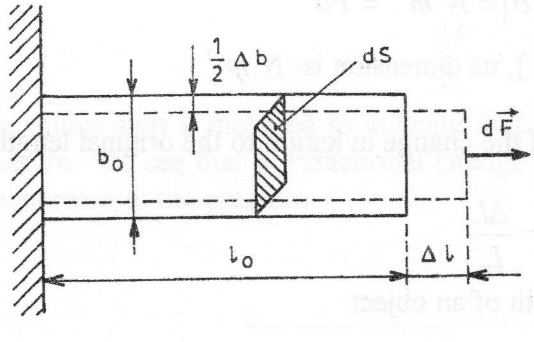
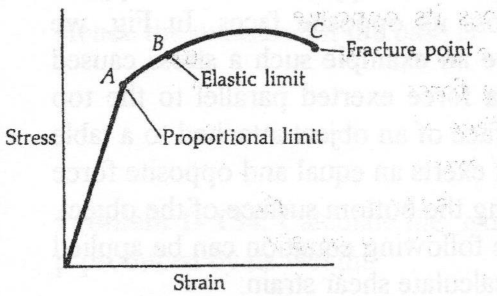
1.7 ELASTICITY



If a force is exerted on an object such as the metal bar in Fig., the length of the object changes. The bar is said to be under **tension or tensile stress.** If the amount of elongation ΔL, is small compared to the length of the object, experiments show that ΔL is proportional to the force exerted on the object. This proportionality can be written as an equation

$$dF = k.\Delta L$$

Here dF represents the force on the object, ΔL is the increase in length, and k is the proportionality constant. However, this equation is found to be valid only up to a point called the **proportional limit.**



The next figure shows a typical graph of elongation versus applied force. Beyond the proportional limit the graph deviates from a straight line and no simple relationship exists between the force and the elongation. Up to a point farther along the curve called the **elastic limit**, the object will return to its original length if the applied force is removed. The region from the origin to the elastic limit is called the **elastic region**. If the object is stretched beyond the elastic limit, it enters the **plastic region** and it does not return to the original length upon removal of the external force, but remains permanently deformed.

The amount of elongation of an object depends not only on the force applied to it, but also on the material from which it is made. This elongation is proportional to the original length and inversely proportional to the cross-sectional area. That is, the longer the object, the more it elongates for a given force and the fatter it is the less it elongates. These experimental findings can be combined into the relation

$$\Delta L = \frac{1}{E}\frac{dF}{dS}L_0 \qquad (1)$$

where L_0 is the original length of the object, dS is the cross-sectional area, and ΔL is the change in length due to the applied force. **E** is the constant of proportionality known as the **elastic modulus or Young's modulus**, [E] = $N.m^{-2}$ and its value depends only on the material and is independent of the object's size or shape. We see that the change in length of an object is directly proportional to the product of the object's length L_0 and the force per unit area dF/dS applied to it.

In practice we define the force per unit area, as the **stress σ** :

$$\sigma = \frac{dF}{dS}; \qquad [\sigma] = N.m^{-2} = Pa$$

The unit of stress is called the Pascal [Pa]; its dimension is $N.m^{-2}$.

Also, the **strain ε** is defined as the ratio of the change in length to the original length:

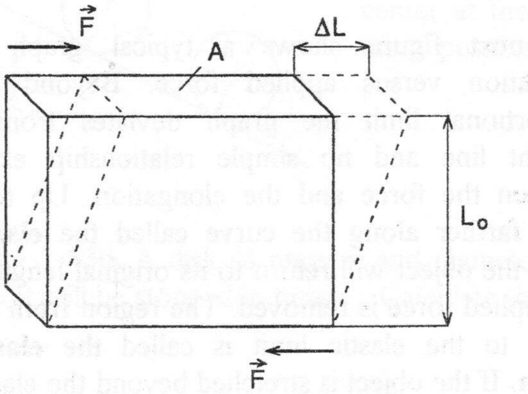
$$\varepsilon = \frac{\Delta L}{L_0}$$

Strain is thus the fractional change in length of an object.

Thus, Eq.(1) can be rewritten as

$$\sigma = E\,\varepsilon$$

This relationship can be interpreted as the general form of **Hook's law** for all longitudinal deformations:**the stress is directly proportional to the strain**.



An object under **shear stress** has equal and opposite forces applied across its opposite faces. In Fig. we have an example such a stress caused by a force exerted parallel to the top surface of an object attached to a table that exerts an equal and opposite force along the bottom surface of the object. The following equation can be applied to calculate shear strain:

$$\Delta L = \frac{1}{G}\frac{F}{A} L_0$$

where A is the area of the surface **parallel** to the applied tangential force, and ΔL is **perpendicular** to L_0 . The constant of proportionality G is called the **shear modulus** $[G] = N.m^{-2}$ and is generally one-half to one-third the value of the elastic modulus E. We can define the shear stress τ:

$$\tau = \frac{F}{A} \qquad [\tau] = N.m^{-2}$$

and we can write **Hook's law for shear stress** as

$$\Delta L = \frac{\tau}{G} L_0$$

If an object is subjected to pressure on all sides, its volume will decrease. Pressure is defined as force per unit area and is thus the equivalent of "**stress**". The change in volume ΔV, is found to be proportional to the original volume V_0 and to the increase in pressure Δp. We thus obtain a relation in the same form as Eq.(1) but with a proportionality constant called the **bulk modulus**, B, $[B] = N.m^{-2}$:

$$\frac{\Delta V}{V_0} = -\frac{1}{B}\Delta p$$

The minus sign is included to indicate that the volume decreases with an increase in pressure. We see that the fractional change in volume of an object is proportional to the increase in the pressure.

Problem 1-133. What is the pressure exerted at its base by a column of material of density ρ, height *h* and cross-sectional area *A*?

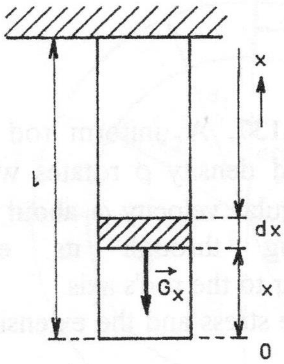
Solution: The force on the base is due to the weight of material in the column,

$$F = mg = \rho A h g$$

Hence the pressure on the base is

$$p = \frac{mg}{A} = \rho g h$$

Problem 1- 134. Calculate the extension of a suspended rope of length *L* and density ρ produced by its weight.



Solution: It is important to bear in mind that the stress along the suspended rope is not constant. At distance *x* from the free end the stress equals

$$\sigma_x = \frac{G_x}{A} = \rho g x$$

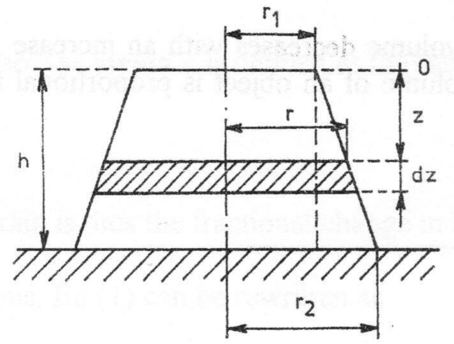
where $G_x = Axg\rho$, when *A* indicates the cross-sectional area of the rope.

The stress σ_x causes the extension of the rope element dx:

$$\Delta(dx) = \frac{\sigma_x}{E}dx = \frac{\rho g}{E}xdx$$

and the total extension of the rope equals $\Delta L = \int_0^L \Delta(dx) = \frac{1}{2}\frac{\rho g}{E}L^2$

Problem 1-135. A uniformly distributed force is exerted on the upper base of a frustum cone whose radii are r_1 and r_2. Calculate the contraction of its height.



Solution: From Hook's law

$$\Delta(dz) = \frac{\sigma}{E}dz = \frac{1}{E}\frac{F}{\pi r^2}dz = \frac{F}{\pi E}\frac{dz}{\left(r_1 + \frac{r_2 - r_1}{h}z\right)^2}$$

and the total contraction equals

$$\Delta h = \int_0^h \Delta(dz) = \frac{Fh}{\pi E r_1 r_2}$$

Problem 1-136. One end of a wire is fixed and the other end has a force F exerted on it in direction of its length L. An extension of the wire was measured to be ΔL. Calculate the original diameter of the wire.

Solution: We can write Hook's law

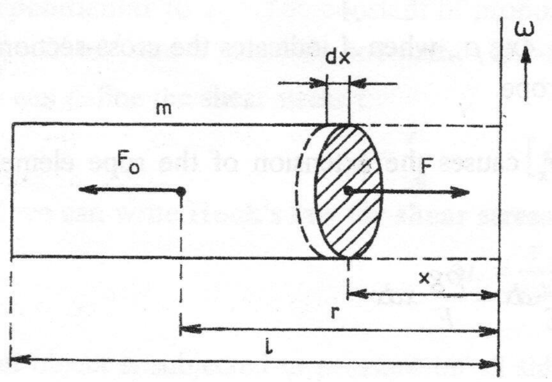
$$\frac{F}{A} = E\frac{\Delta L}{L}$$

and from here

$$A = \pi\frac{d^2}{4} = \frac{FL}{E\Delta L}$$

and thus

$$d = 2\sqrt{\frac{FL}{\pi E \Delta L}}$$



Problem 1-137. A uniform rod of length L and density ρ rotates with constant angular velocity ω about an axis passing through its end perpendicular to the rod's axis. Calculate the stress and the extension of the rod.

Solution: At any cut-area of the rod at distance x from the rotational axis there acts the force F, which must be in equilibrium to centrifugal force F_0.

Thus, we write

$$F = mr\omega^2$$

where

$$m = \rho A(L-x), \quad r = \frac{L+x}{2}$$

The stress at distance x from the axis of rotation equals

$$\sigma_x = \frac{F}{A} = \frac{mr\omega^2}{A} = \frac{\rho}{2}\omega^2\left(L^2 - x^2\right)$$

as seen, the stress is not constant along the rod.

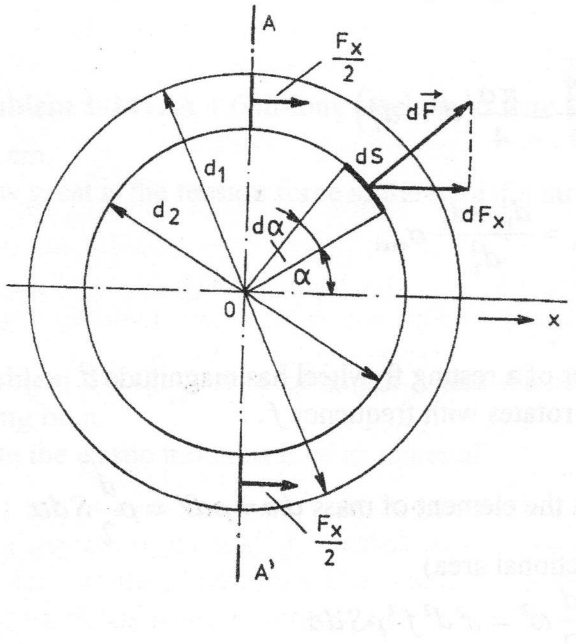
Therefore, we shall consider a section of the rod of length dx; its extension will be equal to

$$\Delta(dx) = \frac{\sigma_x}{E}dx$$

and the total extension of the rod is now equal to

$$\Delta L = \int_{x=0}^{L} \Delta(dx) = \frac{1}{E}\int_{x=0}^{L} \sigma_x dx = \frac{\rho\omega^2 L^3}{3E}$$

Problem 1-138. Calculate the maximum admissible pressure inside a glass cylindrical tube whose outer diameter is d_1 and whose inner diameter is d_2 if you know the magnitude of σ_{max}.



Solution: The force dF = p dS is exerted perpendicularly on the surface element $dS = \frac{d_2}{2}L\,d\alpha$; when p is the pressure inside the tube and L is its length.

Let us choose $A - A'$ cut section of the tube. The stress in this cut section is produced by the x-component of the force dF:(see Fig.)

$$dF_x = p\,dS\cos\alpha = p\frac{d_2}{2}L\cos\alpha\,d\alpha$$

The total force acting in the $A - A'$ cut section is

$$F_x = p\frac{d_2}{2}L\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\alpha d\alpha = pd_2 L$$

This force F_x acts on the surface of the $A - A'$ cut section $(d_1 - d_2)L$. Thus, this force produces the stress σ on this surface. Thus, we can write the relationship

$$pd_2L = (d_1 - d_2)L\sigma$$

and from here we get

$$p_{max} = \frac{d_1 - d_2}{d_2}\sigma_{max}$$

Problem 1-139. Solve the previous problem for the case of a glass spherical flask.

Solution: Force F_x can be now expressed as the product of the pressure and the surface of a circle of diameter d_2; thus,

$$F_x = p\frac{\pi d_2^2}{4}$$

But this force acts on the surface of the circular ring

$$S = \frac{\pi}{4}\left(d_1^2 - d_2^2\right)$$

and therefore

$$F_{x=}S\,\sigma = \frac{\pi\sigma}{4}\left(d_1^2 - d_2^2\right)$$

Thus, we can now write the equality

$$p\pi\frac{d_2^2}{4} = \frac{\pi\sigma}{4}\left(d_1^2 - d_2^2\right)$$

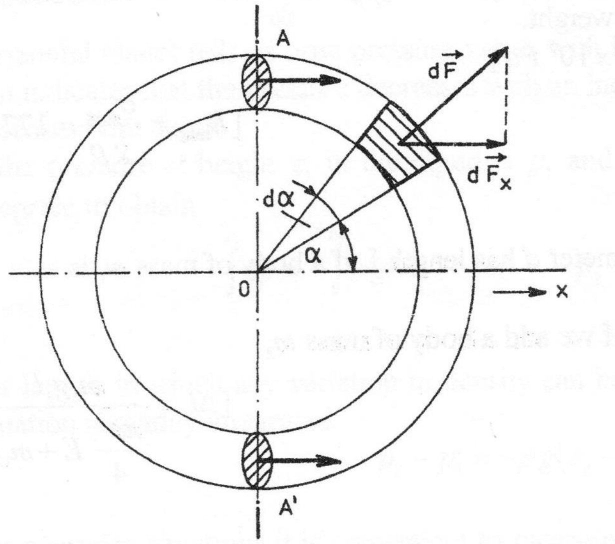
and from here

$$p_{max} = \frac{d_1^2 - d_2^2}{d_2^2}\sigma_{max}$$

Problem 1-140. The average diameter of a resting flywheel has magnitude d. Calculate its extension if the flywheel rotates with frequency f.

Solution: A centrifugal force on the element of mass $dm = \rho dV = \rho\frac{d}{2}S\,d\alpha$ equals (when S is the cross-sectional area)

$$dF = dm\frac{d}{2}\omega^2 = \pi^2 d^2 f^2 \rho S\,d\alpha$$



The force component which stresses the flywheel at the chosen cut section $A-A'$ (see Fig.) is equal to

$$dF_x = dF\cos\alpha = \pi^2 d^2 f^2 \rho S \cos\alpha d\alpha$$

Then, the total force acting in cut section $A-A'$ equals

$$F = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dF_x = 2\pi^2 d^2 f^2 \rho S$$

and the stress caused by this force is (note that the force acts on the surface of 2S)

$$\sigma = \frac{F}{2S} = \pi^2 d^2 f^2 \rho$$

The mean value of the circumference is $L = \pi d$.
The extension caused by the stress σ equals

$$\Delta L = L\frac{\sigma}{E} = \frac{\pi d}{E}\sigma$$

Thus, the extension of the diameter now equals

$$\Delta d = \frac{\Delta L}{\pi} = \frac{d}{E}\sigma = \frac{\pi^2 f^2 d^3 \rho}{E}$$

Problem 1-141. A 1.6 m-long steel piano wire $\left(E = 2\times 10^{11}\, N.m^{-2}\right)$ has a diameter of 0.2 cm.
How great is the tension force in the wire if it stretches 0.3 cm when tightened?

[F = 1200 N]

Problem 1-142. A wire of length L and diameter d stretches ΔL when a force F is acting on it.
State the elastic modulus E of its material.

$$\left[E = \frac{4FL}{\pi d^2 \Delta L} \right]$$

Problem 1-143. Calculate the maximum length of a suspended lead wire for which the wire does not break due to its own weight.
[$\rho = 11.3\times10^3\ kg.m^{-3}$, $\sigma_{max} = 19.6\times10^6\ Pa$]

$$[L_{max} = \frac{\sigma_{max}}{g\rho} = 177\,m]$$

Problem 1-144. A steel wire of diameter d has length L if a body of mass m_1 is suspended from it.
Calculate the extension of the wire if we add a body of mass m_2.

$$[\Delta L = \frac{m_2 g L}{\frac{\pi d^2}{4}E + m_1 g}]$$

Problem 1-145. A body of mass m is suspended from a steel wire of length L and cross-sectional area S. The body moves in a vertical circle with frequency f.
Determine the instant at which the wire is stretched to its maximum and calculate its maximum extension.

[Answer: the maximum stress comes at the lowest point of the path.]

$$[\Delta L_{max} = \frac{mL}{ES}(g + 4\pi^2 f^2 L)]$$

1.8 LIQUID MECHANICS

The density, ρ, of a substance is defined as its mass per unit volume $\rho = \frac{m}{V}$,
where m is the mass of an amount of the substance whose volume is V.
The SI unit for density is $kg.m^{-3}$.

Pressure is defined as the force per unit area, where force F is understood to be acting perpendicular to surface area A :

$$p = \frac{F}{A}$$

The unit of pressure is $N\,m^{-2}$. This unit is the pascal (Pa): 1 Pa = 1 $N\,m^{-2}$.

A liquid exerts a pressure in all directions. The force due to liquid pressure always acts perpendicularly to any surface it is in contact with.
The pressure, or force per unit area, in a liquid at rest is the **hydrostatic force**: at a given point in a liquid at rest, the pressure has the same magnitude regardless of the orientation of the surface on which it acts.