1 OSCILLATIONS

1.1 Oscillations of Spring

Let us assume an object oscillating on the end of a spring. We assume that the mass of the spring can be ignored and that the spring is mounted horizontally so that the object of mass m slides without friction on the horizontal surface (see Fig.1-1).

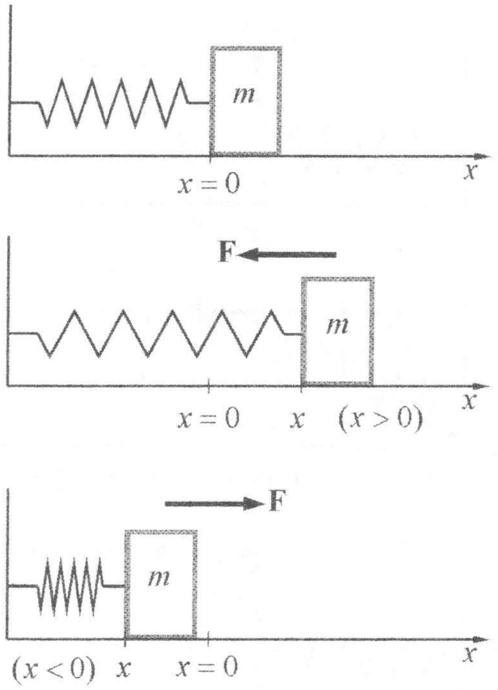

Figure 1-1

Any spring has a natural length at which it exerts no force on the mass m. It is called the **equilibrium position.** If the mass m is moved either to the left, which compresses the spring, or to the right, which stretches it, the spring exerts a force on the mass m which acts in the direction of returning it to the equilibrium position. This force is called the **restoring force.** The magnitude of the restoring force F is found to be directly proportional to the distance x the spring has been stretched or compressed

$$F = -k\,x\,. \tag{1-1}$$

This equation is accurate as long as the spring is not compressed or stretched beyond the elastic region. The minus sign indicates that the restoring force is always in the direction opposite to the displacement x. The proportionality constant k in Eq.1-1 is called the spring constant. It represents the force needed to displace the system from equilibrium of the unit length. Its dimension is $[k] = \mathrm{N\,m^{-1}}$.
Let us examine what happens when the spring is stretched a distance $x = A$ and released. Its initial speed equals zero. The spring exerts a force on the mass m that pulls it toward the equilibrium position (see Fig.1-2).
The mass is accelerated and it passes the equilibrium position with maximum speed v_0. As the mass reaches the equilibrium position, the force on it equals zero. As it moves father to

the left, the force on it acts to slow the mass and it stops at $x = -A$ and begins moving back in the opposite direction until it reaches the original starting point $x = A$ and it repeats the motion.

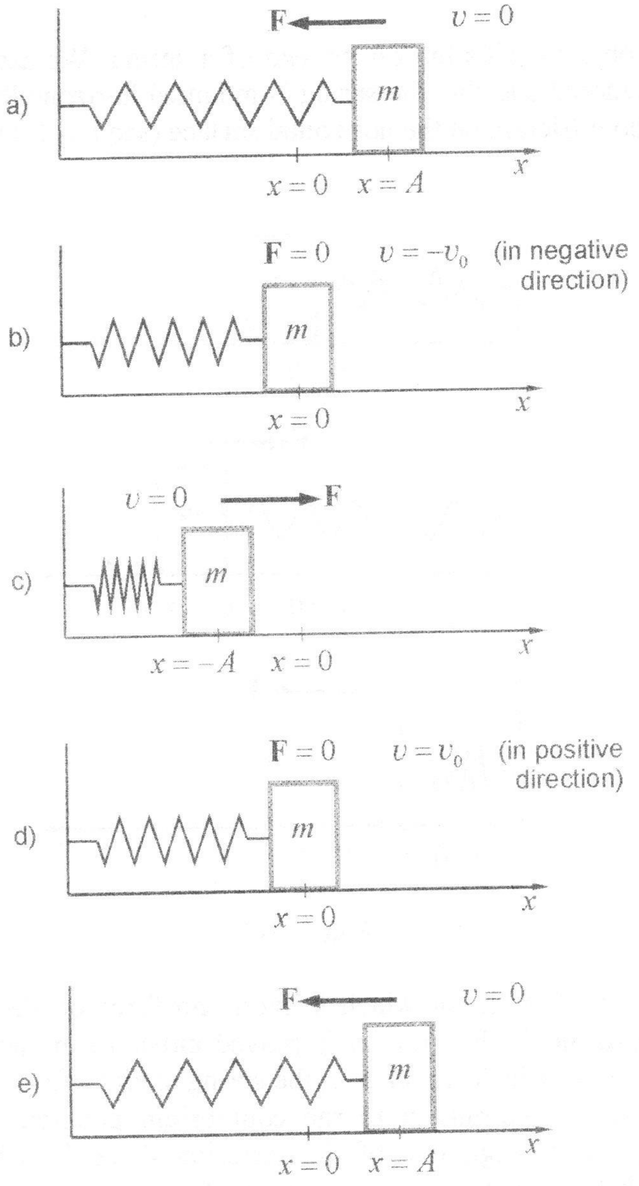

Figure 1-2

The distance x of the mass from the equilibrium point at any moment is called the **displacement**. The maximum displacement, the greatest distance from the equilibrium point, is called the **amplitude** A. The motion from some point back to that same point, say from $x = A$ to $x = -A$ back to $x = A$, is called one **cycle**. The **period** T is defined as the time required for one complete cycle. The **frequency** f is the number of complete cycles per second. Frequency is usually specified in **hertz** (Hz) where 1Hz = 1 cycle per second. It is evident that $f = \frac{1}{T}$ and $T = \frac{1}{f}$.

1.2 Simple Harmonic Motion

Any vibrating system for which the restoring force is directly proportional to the negative of the displacement (as in Eq.1-1) is said to exhibit **simple harmonic motion** and such a system is often called a **simple harmonic oscillator**.
Let us now determine the position x as a function of time. We make use of Newton's second law $F = m\,a$ and we have

$$m\frac{d^2x}{dt^2} = -k\,x\ , \tag{1-2}$$

where m is the mass which is oscillating.

After rearranging we obtain

$$\frac{d^2x}{dt^2} + \frac{k}{m}\,x = 0\ , \tag{1-3}$$

which is known as the **equation of motion** for the simple harmonic oscillator. General solution of this differential equation has form (as known from mathematics)

$$x(t) = a\cos\omega t + b\sin\omega t\ , \tag{1-4}$$

where a and b are arbitrary constants and the constant ω is called the **angular frequency.**

If we differentiate this function twice and substitute for x and its second derivative into Eq.1-3 we get for ω

$$\omega^2 = \frac{k}{m}\ ; \qquad [\omega] = s^{-1}\ . \tag{1-5}$$

So, the function in Eq.1-4 is the solution of Eq.1-3 if and only if Eq.1-5 holds.

The **speed** of the harmonic motion equals

$$v(t) = \frac{dx}{dt} = -a\,\omega\sin\omega t + b\,\omega\cos\omega t\ . \tag{1-6}$$

In real physical situations, the constants a and b are determined by initial conditions. Suppose, for example, the mass starts to move at its maximum displacement $x = A$ without pushing it ($v = 0$ at $t = 0$). Applying the initial conditions $x(0) = A$ and $v(0) = 0$ at $t = 0$, we have from Eqs.1-4 and 1-6

$$x(0) = a\cos 0 + b\sin 0 = A\ ,$$

$$v(0) = -a\,\omega\sin 0 + b\,\omega\cos 0 = 0\ .$$

Thus, $a = A$ and $b\omega = 0$ and so $b = 0$, and the motion is a cosine curve

$$x(t) = A\cos\omega t\ .$$

The equation 1-4 for $x(t)$ can be written in the following more convenient form

$$x(t) = A\cos(\omega t + \varphi)\ . \tag{1-7}$$

The physical interpretation of this equation is simpler than for Eq.1-4. As shown in Fig.1-3, A is simply the amplitude (which occurs when the cosine in Eq.1-7 has its maximum value of 1) and φ, called the **phase angle**, tells how long after or before $t = 0$ the peak at $x = A$ is reached. For $\varphi = 0$ we have

$$x(t) = A\cos\omega t \ ,$$

for $\varphi = -\pi/2$ we have

$$x(t) = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin\omega t \ .$$

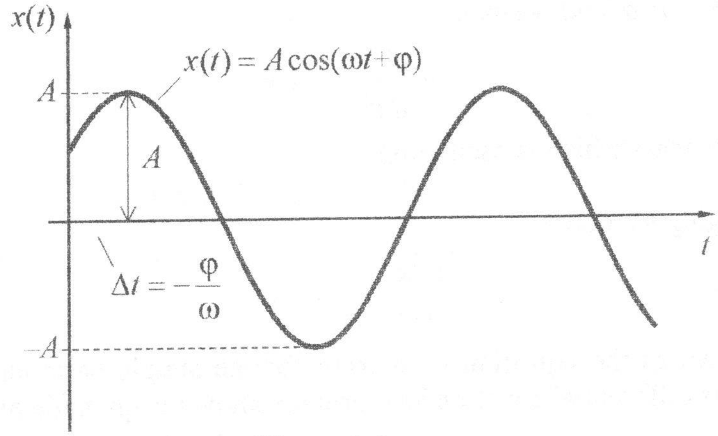

Figure 1-3

Since the simple harmonic oscillator repeats its motion after the time equal to its period T and since sine or cosine function repeats itself after every 2π radians, we have from Eq.1-7 that $\omega T = 2\pi$.

Hence
$$\omega = \frac{2\pi}{T} = 2\pi f \ , \tag{1-8}$$
where f is the frequency of the motion and ω is the angular frequency.

Now we can also write Eq.1-7 as

$$x(t) = A\cos\left(\frac{2\pi}{T}t + \varphi\right) \ , \tag{1-9}$$

or
$$x(t) = A\cos(2\pi f t + \varphi) \ , \tag{1-10}$$

where, because of Eq.1-5

$$T = 2\pi\sqrt{\frac{m}{k}} \ , \quad [T] = \mathrm{s} \ , \tag{1-11}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ , \quad [f] = \mathrm{s}^{-1} \ . \tag{1-12}$$

It is clear that the frequency and period do not depend on the amplitude. It is also clear that the greater the mass m the lower the frequency and the stiffer the spring the higher the frequency. The frequency given by Eq.1-12 at which a simple harmonic oscillator oscillates is called its **natural frequency**.

The **velocity** and **acceleration** of the simple harmonic oscillator can be obtained by differentiation of Eq.1-7

$$v(t)=\frac{dx}{dt}=-\omega A\sin(\omega t+\varphi)\ , \quad (1\text{-}13)$$

$$a(t)=\frac{d^2x}{dt^2}=\frac{dv}{dt}=-\omega^2 A\cos(\omega t+\varphi)=-\omega^2 x(t)\ . \quad (1\text{-}14)$$

In Fig. 1-4 we plot displacement, velocity and acceleration of a simple oscillating system as a function of time for the case when $\varphi = 0$.

We can see that the speed reaches its maximum of

$$v_{\max}=\omega A=\sqrt{\frac{k}{m}}A \quad (1\text{-}15)$$

when the oscillator is passing through its equilibrium point $x = 0$ and the speed is zero at points of maximum displacement $x \pm A$.

The acceleration of the oscillation motion has its maximum value

$$a_{\max}=\omega^2 A=\frac{k}{m}A \quad (1\text{-}16)$$

at $x = \pm A$ and is zero at $x = 0$ since at $x = 0$ the restoring force $F = -kx$ equals zero.

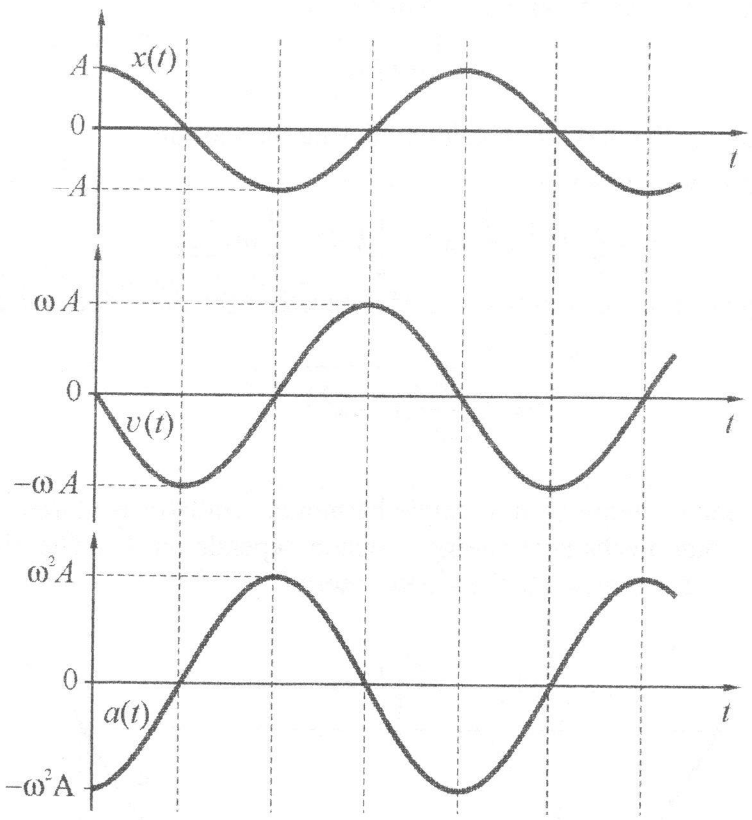

Figure 1-4

1.3 Energy in Simple Harmonic Oscillator

Let us assume a simple harmonic oscillator such as a mass m oscillating on the end of a massless spring. The motion of this oscillator is caused by the restoring force $F = -kx$. The **potential energy** of the simple harmonic oscillator will be given by

$$U = -\int F\,dx = \frac{1}{2}kx^2 \ , \tag{1-17}$$

where we set the constant of integration equal zero, so $U = 0$ at $x = 0$.
The **total mechanical energy** which remains constant is equal

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \ , \tag{1-18}$$

where v is the velocity at a distance x from the equilibrium.
At the extreme points, $x = A$ and $x = -A$, the velocity $v = 0$ and all the energy is the potential energy only

$$E = \frac{1}{2}kA^2 \tag{1-19}$$

and we can say, that the **total mechanical energy of a simple harmonic motion is proportional to the square of its amplitude**.
At the equilibrium, $x = 0$, all the energy is kinetic

$$E = \frac{1}{2}mv_{max}^2 \ , \tag{1-20}$$

where v_{max} represents the maximum velocity during the motion.
Hence, for any point we can write

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2 \ . \tag{1-21}$$

From this last equation we can obtain a useful equation for the velocity v as a function of x:

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \ . \tag{1-22}$$

In Fig. 1-5 the potential energy U of a simple harmonic oscillator is plotted. The horizontal line represents its total mechanical energy E which depends on A^2. The distance between this line and the U-curve represents the kinetic energy.

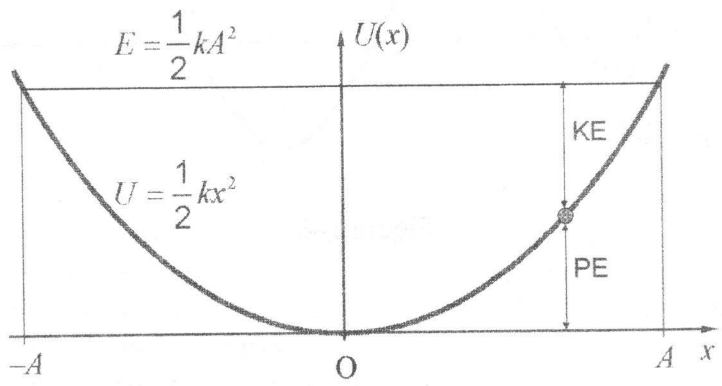

Figure 1-5

1.4 Simple Pendulum

A simple pendulum consists of a small object (called pendulum bob) suspended on the end of a light cord. We assume that the cord mass can be ignored relative to the mass of the bob.

The simple pendulum oscillates along the arc of a circle. Its equilibrium point is at the point O. As it passes through the equilibrium point it has its maximum speed. The displacement of the pendulum along the arc, x, is given $x = l\ \varphi$, where φ is the angle the cord makes with the vertical and l is its length.

The restoring force is equal to the component of the bob weight tangent to the arc

$$F = -mg \sin \varphi \ .$$

Since F is proportional to the $\sin\varphi$ and not to φ itself, the motion is not simple harmonic motion. The motion will be simple harmonic only in the case if the restoring force is proportional to x or to φ.

But if φ is small, then $\sin\varphi$ is very nearly equal to φ if specified in radians (this can be seen from the series expansion of $\sin\varphi$).

Thus, for small angles we can use approximation

$$F \doteq -mg\,\varphi = -\frac{mg}{l}x \ ,$$

where the term $\frac{mg}{l}$ represents the force constant k.

Hence, the period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{l}{g}} \ . \qquad (1\text{-}23)$$

Note, that the period does not depend on the mass of the pendulum bob.

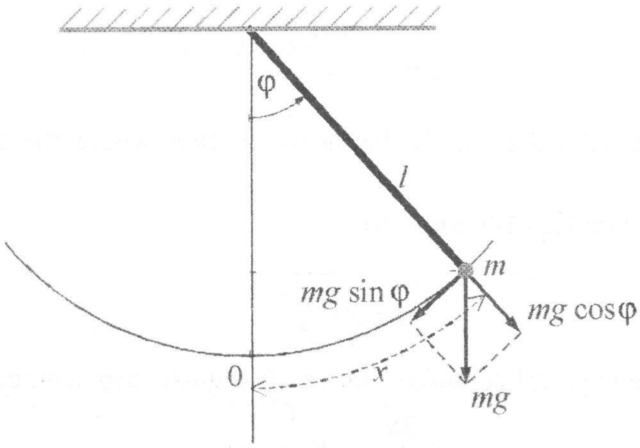

Figure 1-6

1.5 Physical Pendulum

The physical pendulum refers to any real body which oscillates around its equilibrium position. An example of a physical pendulum is shown in Fig.1-7.

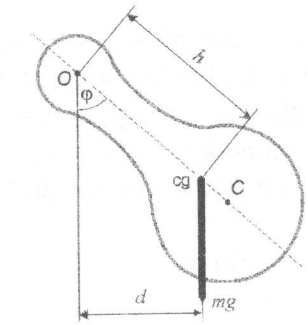

Figure 1-7

The body is suspended from the point O. The force of gravity acts at the centre of gravity (cg) whose distance from the point O let be h. To analyse the motion of the physical pendulum we use the equation of rotational motion.

First, we calculate the torque on a physical pendulum about point O

$$\tau = -mgh\sin\varphi \quad .$$

We know, that the equation for rotational motion states that

$$\tau = J\frac{d^2\varphi}{dt^2} \quad ,$$

where J is the moment of inertia of the body.

Thus we have

$$J\frac{d^2\varphi}{dt^2} = -mgh\sin\varphi$$

or

$$\frac{d^2\varphi}{dt^2} + \frac{mgh}{J}\sin\varphi = 0 \quad ,$$

where J is calculated about the axis passing point O. The last equation can be reduced for small angular amplitudes into form

$$\frac{d^2\varphi}{dt^2} + \frac{mgh}{J}\varphi = 0 \quad .$$

This is just equation for the simple harmonic motion where the term $\frac{mgh}{J}$ replaces $\omega^2 = \frac{k}{m}$ (compare with Eqs.1-3 and 1-5).

Thus,

$$\omega = \sqrt{\frac{mgh}{J}} \quad ,$$

and the period of the physical pendulum for small angular displacement is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{J}{mgh}} \quad . \qquad (1\text{-}24)$$

Using this result we have an easy way to measure the moment of inertia of an object about any axis. To do this we measure the period of oscillation about that axis.

1.6 Damped Harmonic Motion

The amplitude of any real oscillating object decreases in time. The damping is generally due to resistance of air and friction. Fig.1-8 shows a typical course of such an oscillating motion that is called damped harmonic motion.

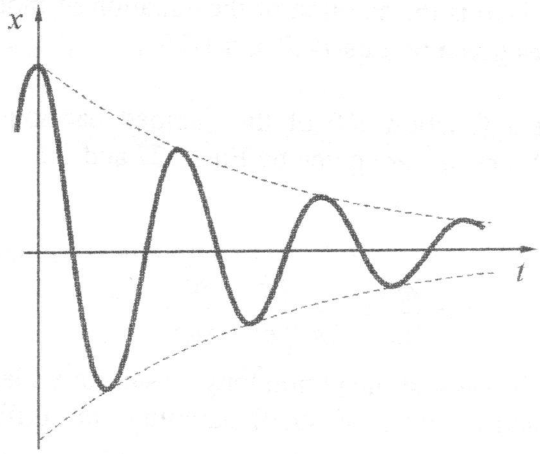

Figure 1-8

Let us look at this motion in more detail. The damping force opposes the motion and in many cases can be considered to be directly proportional to the speed:

$$F_{damp} = -b\upsilon \ ,$$

where b is a constant. For the object oscillating on the end of a spring the restoring force of the spring is known to be equal $F = -kx$.

The Newton's second law states now

$$m\,a = -k\,x - b\,\upsilon$$

or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \ , \qquad (1\text{-}25)$$

which is the equation of motion for the damped harmonic oscillator.

The solution of this equation can be written in form

$$x(t) = Ae^{-\delta t}\cos\omega' t \ , \qquad (1\text{-}26)$$

where A, δ and ω' are assumed to be constants and $A = x(t)$ at $t = 0$. To determine the constants δ and ω' we take the first and second derivatives of Eq.1-26 and substitute them into Eq.1-25. After reorganising we obtain:

$$Ae^{-\delta t}\left[\left(m\delta^2 - m\omega'^2 - b\delta + k\right)\cos\omega' t + (2\delta\omega' m - b\omega')\sin\omega' t\right] = 0 \ .$$

This equation must be equal zero for all times t.

First, we choose $t = 0$, then $\sin\omega' t = 0$ and the above relation reduces to

$$m\delta^2 - m\omega'^2 - b\delta + k = 0 \ . \qquad (a)$$

Second, we choose $t = \pi / 2\omega'$, then $\cos\omega' t = 0$ and above relation reduces to

$$2\delta m - b = 0 \ . \qquad (b)$$

From the relation (b) we have

$$\delta = \frac{b}{2m} \tag{1-27}$$

and from (a)

$$\omega' = \sqrt{\delta^2 - \frac{b\delta}{m} + \frac{k}{m}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \,. \tag{1-28}$$

<u>We can conclude</u>: The Eq.1-26 is the solution of the equation of motion 1-25 as long as δ and ω' have specific values given by Eqs.1-27 and 1-28.

Summary, the displacement function $x(t)$ of the damped harmonic motion is given by Eq.1-26 where constants δ and ω' are given by Eqs.1-27 and 1-28.

The frequency f is equal

$$f = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \,. \tag{1-29}$$

We see that the frequency is less and the period longer than for undamped simple harmonic motion. (But in most practical cases of small damping, ω' differs only slightly from $\omega = \sqrt{k/m}$.) The constant $\delta = b/2m$ is the measure of how quickly the oscillations decrease to zero. The larger b the quickly the oscillations go away. The time $t_1 = 2m/b$ is the time taken for the oscillations to drop to $1/e$ of the original amplitude and is called mean lifetime of the oscillations.

The solution given in Eq.1-26 is not valid if b is so large that $b^2 > 4mk$. In this case ω' becomes imaginary and the system does not oscillate at all but returns directly to its equilibrium position.

We can make the following conclusion:

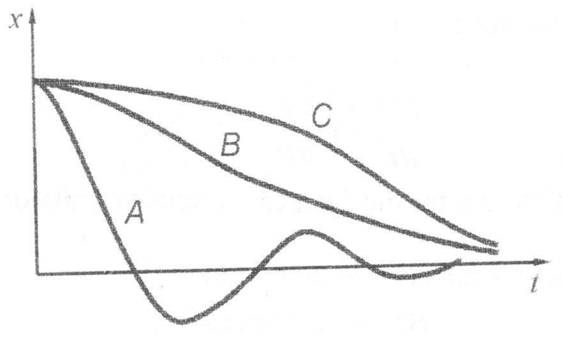

Figure 1-9

a) if $b^2 > 4\ mk$, the damping is so large and the system takes a long time to reach equilibrium – the system is overdamped (curve C);
b) if $b^2 < 4\ mk$, the system makes several swings before coming to rest (curve A);
c) if $b^2 = mk$, the equilibrium is reached in the shortest time – this case is called the critical damping (curve B).

1.7 Forced Oscillations - Resonance

Let us suppose that an object may oscillate at the frequency f of the external force acting on it, even if this frequency is different from the natural frequency of the undamped object. In this case we talk about **forced oscillations**. To distinguish the frequency f of the external force from the natural frequency of the system, we denote the latter by f_0 and the natural angular frequency by ω_0 .

We shall discuss the important case when an external force can be represented by

$$F_{ext} = F_0 \cos \omega t \quad ,$$

where $\omega = 2\pi f$ is the angular frequency and F_0 is the amplitude of the applied external force.

Thus, the equation of motion (assuming damping) is

$$ma = -kx - bv + F_0 \cos \omega t$$

or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos \omega t \ . \qquad (1\text{-}30)$$

The solution of this differential equation is

$$x(t) = A_0 \sin(\omega t + \varphi_0) \ , \qquad (1\text{-}31)$$

where

$$A_0 = \frac{F_0}{m\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + b^2\omega^2 / m^2}} \qquad (1\text{-}32)$$

and

$$\varphi_0 = \text{arctg}\frac{\omega_0^2 - \omega^2}{\omega b / m} \ .$$

The amplitude of the forced harmonic motion, A_0 , depends strongly on the difference between the frequency of the applied external force ω and the natural frequency ω_0.

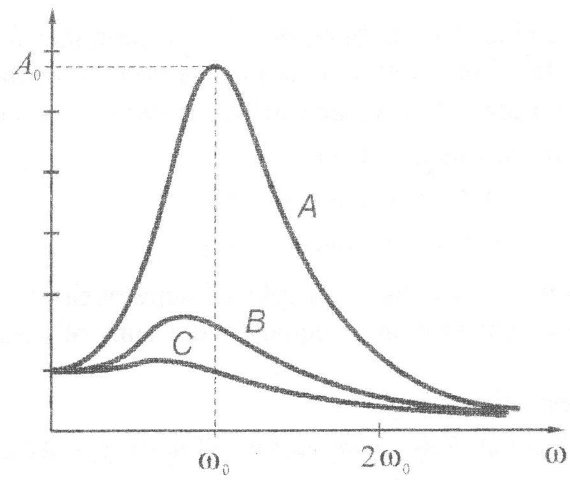

Figure 1-10

The dependence of A_0 (Eq.1-32) on the frequency ω of the applied external force is shown in Fig.1-10 for three specific values of the damping constant b.

Curve A ($b=\frac{1}{6}m\omega_0$) represents light damping, curve B ($b=\frac{1}{2}m\omega_0$) fairly heavy damping and curve C ($b=\sqrt{2}\,m\omega_0$) overdamped motion.
The amplitude can become very large when the frequency ω of the external force is near the natural frequency, $\omega \approx \omega_0$, as long as the damping is not too large (curve A). This is known as **resonance** and the natural frequency ω_0 of a system is called its **resonant frequency**. Exactly the resonant frequency is defined as the value of ω at which the amplitude has its maximum value and this depends somewhat on the damping constant. But except for very heavy damping this value is quite close to ω_0.
If b=0, resonance occurs at $\omega = \omega_0$ and the resonant peak of A_0 becomes infinity.
For real system, b is never zero and the resonant peak is finite and it does not occur precisely at $\omega = \omega_0$ (because of the term $b^2\omega^2/m^2$ in the denominator of Eq.1-32) but it is quite close to ω_0 unless the damping is very large. If the damping is large, there is little or no peak (curve C).
The oscillating system is often specified by its **quality factor** or **Q-value,** defined

$$Q=\frac{m\omega_0}{b} \quad .$$

For our examples in Fig.1-10, curve A has Q=6, curve B has Q=2 and curve C has Q=$1/\sqrt{2}$.
Thus, we can see that the smaller the damping constant b the larger Q value becomes and the higher the resonance peak.
The Q value also determines the narrowness of the resonance peak. The larger Q value the more narrow will be the resonance peak. So, a large Q value, representing a system of high quality, has a high narrow resonance peak.

1.8 Combination of Harmonic Motions

In this chapter we shall examine the combination of two harmonic motions of the same directions. We will assume that both simple harmonic motions have the same frequencies $\omega_1 = \omega_2 = \omega$, different amplitudes $A_1 \neq A_2$ and different phases $\varphi_1 \neq \varphi_2$. Thus, the first motion is described by the displacement function

$$x_1(t) = A_1 \cos(\omega t + \varphi_1)$$

and the second one by $x_2(t) = A_2 \cos(\omega t + \varphi_2)$.
To find the resultant motion we use the principle of superposition which tells us that the displacement of the resultant motion is equal to the sum of displacements of both partial motions.
Thus, we can write (after easy computing)

$$x(t) = x_1(t) + x_2(t) = (A_1 \cos\varphi_1 + A_2 \cos\varphi_2)\cos\omega t - (A_1 \sin\varphi_1 + A_2 \sin\varphi_2)\sin\omega t \ .$$

The terms in parentheses are constants. We see that the resultant motion is also harmonic with the same frequency ω but different amplitudes and phase angle. To find these unknown quantities we express the displacement function of the resultant motion in standard form:

$$x(t) = A\cos(\omega t + \varphi) = A\cos\varphi\cos\omega t - A\sin\varphi\sin\omega t \ ,$$

where A and φ represent the amplitude and the phase of the resultant motion, to be determined.
As the last equations must be identically equal at every instant of time, the coefficients at $\cos\omega t$ and $\sin\omega t$ must be the same.
Hence:

$$A\cos\varphi = A_1\cos\varphi_1 + A_2\cos\varphi_2 ,$$

$$A\sin\varphi = A_1\sin\varphi_1 + A_2\sin\varphi_2 .$$

Dividing of these equations gives us the phase angle of the resultant harmonic motion

$$tg\varphi = \frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2} .$$

To find the amplitude of the resultant motion we square these equations and then we add them. We have result:

$$A^2 = A_1^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1) + A_2^2 . \qquad (1\text{-}33)$$

We have obtained the important formula for the determination of the amplitude of the resultant harmonic motion. Let us have a look on this expression in detail:

1) If $\varphi_2 - \varphi_1 = 2k\pi$, where k=0, 1, 2, ...then $\cos(\varphi_2 - \varphi_1) = 1$ and the amplitude of the resultant oscillating motion will reach its maximum

$$A_{max} = A_1 + A_2 . \qquad (1\text{-}34)$$

2) If $\varphi_2 - \varphi_1 = (2k+1)\pi$, where k=0, 1, 2, ...then $\cos(\varphi_2 - \varphi_1) = -1$ and the amplitude of the resultant oscillating motion will reach its minimum

$$A_{min} = \left\langle \begin{matrix} A_1 - A_2 & if\ A_1 > A_2 \\ A_2 - A_1 & if\ A_1 < A_2 \end{matrix} \right. . \qquad (1\text{-}35)$$

Consider now a particle which undergoes simple harmonic motion along two perpendicular directions, say the x and y axes, and let frequency ω in both directions be equal:

$$x = A_x\cos(\omega t + \varphi_x) ,$$

$$y = A_y\cos(\omega t + \varphi_y) .$$

Examine the resultant motion for the following cases:

1) equal phases, $\varphi_x = \varphi_y = \varphi$;
2) phase difference $\varphi_y - \varphi_x = \pm\pi/2$ and the amplitudes are equal $A_x = A_y = A$;
3) phase difference $\varphi_y - \varphi_x = \pm\pi/2$ and the amplitudes are different $A_x \neq A_y$.

Solution:

1) Equal phases, $\varphi_x = \varphi_y = \varphi$:

Since $x = A_x\cos(\omega t + \varphi)$ and $y = A_y\cos(\omega t + \varphi) = \frac{A_y}{A_x}x$ which is the equation of a straight line of slope (A_y / A_x), the resultant motion will be a straight line in the xy plane. In Fig.1-11 there is shown this motion when $A_y = 2A_x$.

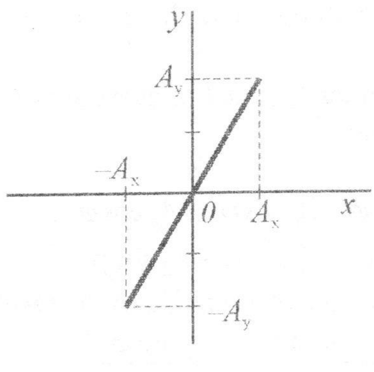

Figure 1-11

2) $\varphi_y - \varphi_x = \pm\pi/2,\ A_x = A_y = A$:

$$x = A\cos(\omega t + \varphi),$$

and
$$y = A\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = A\sin(\omega t + \varphi).$$

So, we can write
$$x^2 + y^2 = A^2 ,$$
which is the equation of a circle in the xy plane of radius A (Fig.1-12).

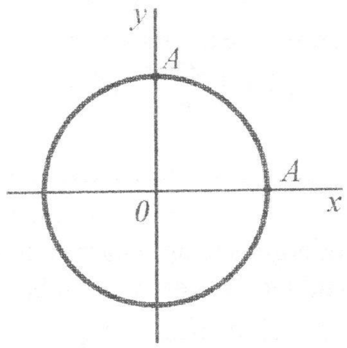

Figure 1-12

3) $\varphi_y - \varphi_x = \pm\pi/2,\ A_x \neq A_y$:

$$x = A_x\cos(\omega t + \varphi) ,$$

$$y = A_y\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = A_y\sin(\omega t + \varphi),$$

or
$$\frac{x}{A_x} = \cos(\omega t + \varphi) ,$$

$$\frac{y}{A_y} = \sin(\omega t + \varphi)$$

and thus
$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = 1$$

which is the equation of the ellipse with major and minor axes equal to $2A_x$ and $2A_y$.

In Fig. 1-13 $A_x = 2A_y$.

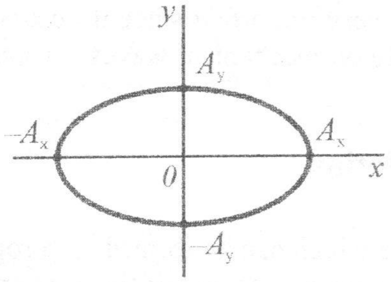

Figure 1-13

<u>Note</u>: when the frequencies are not equal ($\omega_x \neq \omega_y$) the resultant motion can be very complex. Generally the curve is not closed and thus is not periodic. However, if the ratio ω_x / ω_y is equal to the ratio of two integers the curve is closed and the motion is periodic one. These types of curves are called Lissajous figures. The example for $\omega_y = 2\omega_x$, $\varphi_y - \varphi_x = \pi / 4$, $A_x = A_y$ is shown in Fig. 1-14.

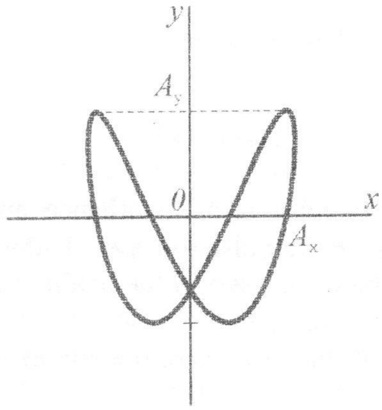

Figure 1-14