# Physics 1

## Electric current

Ing. Jaroslav Jíra, CSc.

# Electric Current

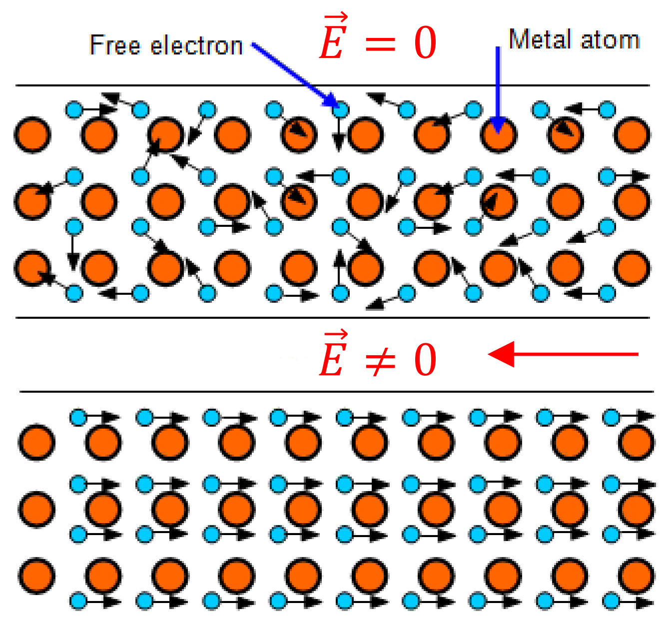
Valence electrons in an atom are the outermost electrons, the least bound to the central positive charge.

In a solid substance, there is often a crystalline structure of the atoms or molecules, the atoms or molecules form a lattice in space.

Valence electrons are free to move within the lattice, so the motion of such electrons without external electric field is similar to the motion of molecules in a gas.

If we connect a battery to the metallic conductor, an electric field $\vec{E}$ will be set up in the conductor.

This field will act on the electrons and will cause their motion in the direction of -$\vec{E}$. An **electric current** is established. The ability of conductors to carry an electric current is called conduction of free electrons.



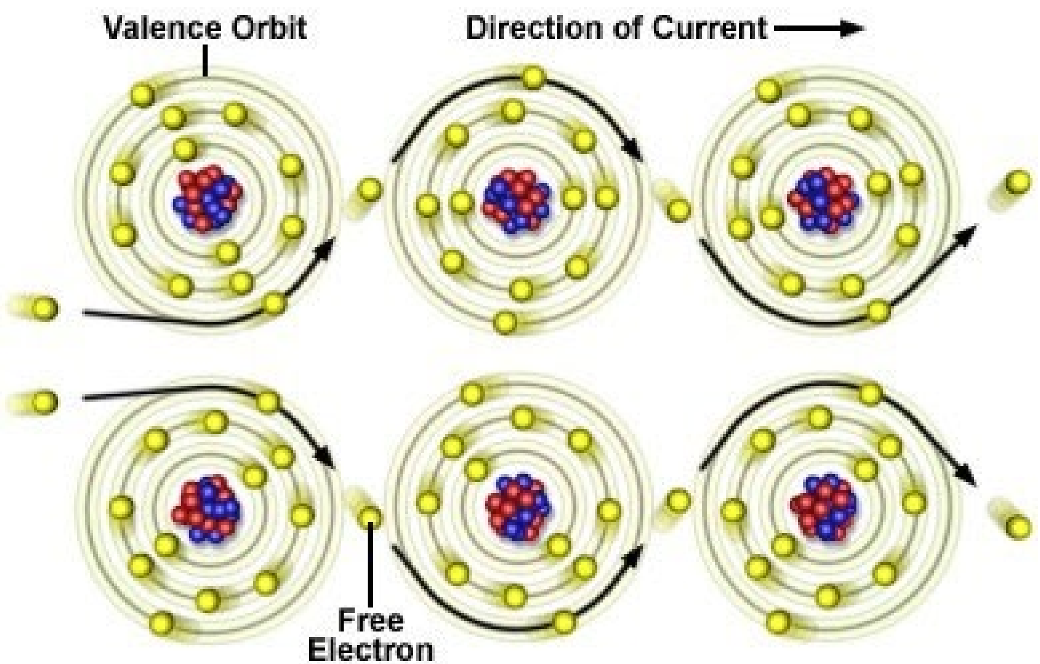
# Electric Current

The average electric current in a conductor is defined as $I = \dfrac{\Delta Q}{\Delta t}$

where $\Delta Q$ is the net amount of charge that passes through a cross section of the conductor during the time interval $\Delta t$.

If the current is not constant in time, we can define instantaneous current as a limit

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$




The unit of electric current is Ampere [A]

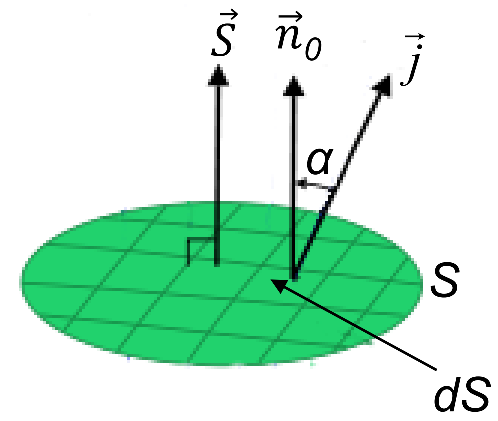
We could also write inversely

$$Q = \int_0^t I \cdot dt$$

# Current Density

To describe motion of charges it the 3D volume, we need another quantity, called **current density** $\vec{j}$. Unit of the current density is [A/m$^2$].

The current passing through an elementary surface area *dS* is

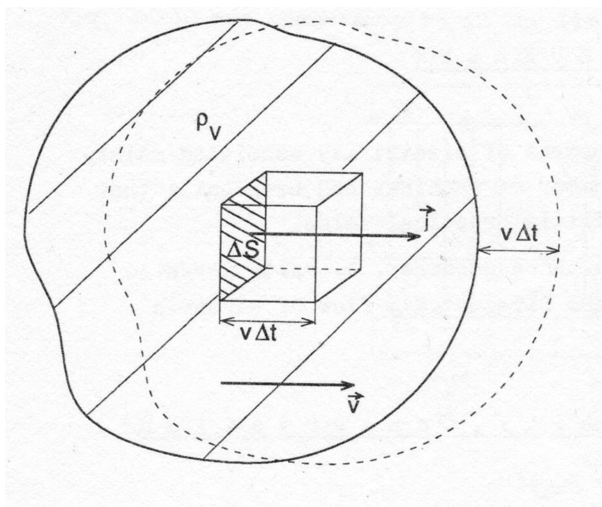



$$dI = \vec{j} \cdot d\vec{S} = j \cdot dS \cos\alpha; \quad d\vec{S} = \vec{n}_0 \cdot dS$$

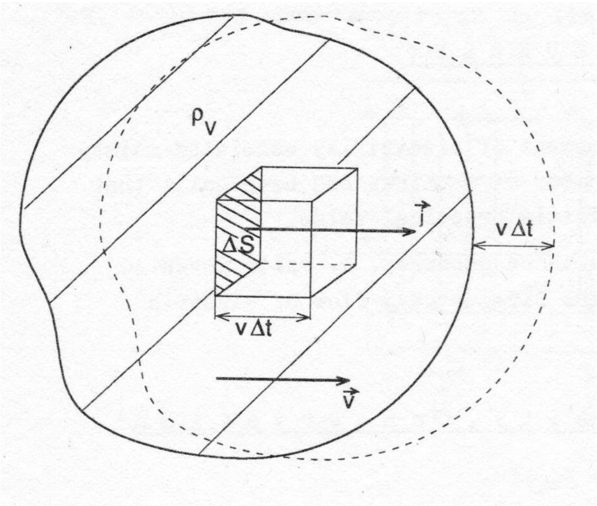
The current passing through an area *S* is

$$I = \iint_S \vec{j} \cdot d\vec{S}$$

Let us suppose that the average motion of charges is a drift with velocity $\vec{v}$ passing through a surface element *ΔS*. The charge *ΔQ* passing through *ΔS* in a time *Δt* is equal to the charge contained in a prism of the volume *ΔS·v·Δt*.



# Current Density




If we multiply the volume of the prism by the charge volume density $\rho_V$, we obtain

$$\Delta Q = \rho_V \cdot v \cdot \Delta t \Delta S$$

The charge per unit volume is an electric current, so by introducing differentials instead of deltas we obtain

$$dI = \frac{dQ}{dt} = \rho_V \cdot v \cdot dS = \rho_V \cdot \vec{v} \cdot d\vec{S}$$

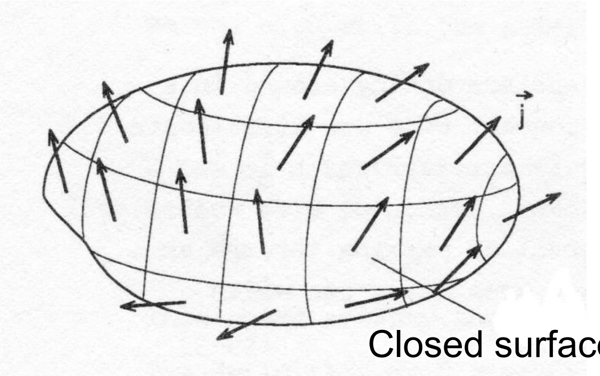
Considering $dI = \vec{j} \cdot d\vec{S}$ we can write $\vec{j} = \rho_V \cdot \vec{v}$

If there are $n$ charges in the unit volume, each with unit charge $e$ and they are moving with mean velocity $\vec{v}$, we can rewrite the formula for the current density to

$$\boxed{\vec{j} = e \cdot n \cdot \vec{v}}$$

# Current Density

The current *I* out of a closed surface represents the rate at which the charge leaves the volume *V* enclosed by *S*. Since the charge cannot be created or destroyed the amount of charge inside V must decrease. We can write the law of conservation of charge inside $Q_{in}$.




$$\oiint_S \vec{j}\cdot d\vec{S} = -\frac{dQ_{in}}{dt} \quad \text{where} \quad Q_{in} = \iiint_V \rho_V \, dV$$

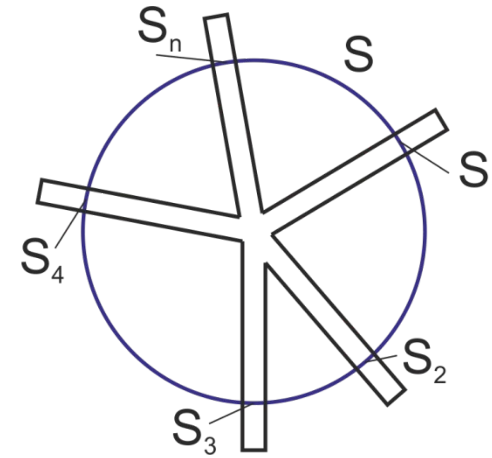
$$\boxed{\oiint_S \vec{j}\cdot d\vec{S} = -\frac{d}{dt}\iiint_V \rho_V \, dV}$$

By shrinking down the volume around a point (x,y,z) we obtain

$$div\,\vec{j} = -\frac{\partial \rho_V}{\partial t}$$

*No charge can flow away from a place without diminishing the amount of charge that is there*.

# First Kirchhoff's Law




Let us suppose stationary case when a volume charge density is constant. The law of charge conservation will change to

$$\oiint_S \vec{j} \cdot d\vec{S} = 0$$

This equation can be rewritten according to particular currents flowing through areas $S_1$, $S_2$, … $S_n$ into the volume surrounded by $S$.

$$\oiint_S \vec{j} \cdot d\vec{S} = \iint_{S_1} \vec{j} \cdot d\vec{S} + \iint_{S_2} \vec{j} \cdot d\vec{S} + \ldots + \iint_{S_n} \vec{j} \cdot d\vec{S} = I_1 + I_2 + \ldots + I_n = 0$$

$$\sum_{k=1}^{n} I_k = 0$$

**The first Kirchhoff's law** can be formulated: *the algebraic sum of all currents into the junction point must be zero.*

# Ohm's Law, Resistance, Conductivity

If we want to establish an electric current in a circuit, we need a potential difference. G. S. Ohm established experimentally that the current *I* in metal wire is proportional to the potential difference *V*.

$$I = \frac{V}{R} \quad or \quad V = IR$$

where *R* is the resistance of the wire.

Unit of *R* is Ohm [Ω]

The Ohm's law expressed in words: *current through a metal conductor is proportional to the applied voltage*. This means that *R* is considered a constant, which is valid in case of metal wires or resistors, but we cannot use the Ohm's law for semiconductor elements, light bulbs and other devices, which are called nonohmic or nonlinear.

Another experimental finding is that the resistance of uniform metal wire is directly proportional to its length $l$ and inversely proportional to its cross-sectional area *S*.

$$R = \rho \frac{l}{S}$$

where *ρ* is **resistivity** in [Ω·m] and it depends on the material used

# Ohm's Law, Resistance, Conductivity

The resistivity depends on the material used and also on the temperature.

$$\rho_T = \rho_0(1 + \alpha\,\Delta T)$$

where $\rho_T$ is resistivity at temperature $T$, $\rho_0$ is resistivity at a standard temperature $T_0$, and $\alpha$ [$K^{-1}$] is the temperature coefficient of resistivity.

The reciprocal value of resistivity is called **conductivity** $\gamma$

$$\gamma = \frac{1}{\rho} \qquad \left[\frac{1}{\Omega \cdot m}\right]$$

Let us deduce the Ohm's law in differential form now. We will start with the formula for electric field $E$ in the wire of length $l$ and potential difference $V$ at its ends.

$$V = E \cdot l$$

If we substitute for $V$ from the Ohm's law

$$RI = E \cdot l$$

Now we substitute for $R$ with resistivity formula and realize that $I = j \cdot S$

$$\rho \frac{l}{S} j\,S = E \cdot l$$

**The Ohm's law in differential form**

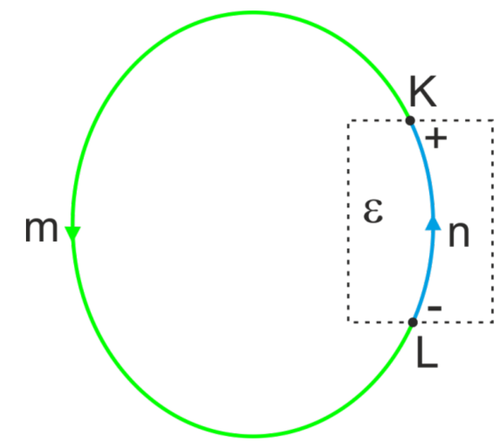
$$\vec{j} = \frac{1}{\rho}\vec{E} \quad \text{or} \quad \vec{j} = \gamma \vec{E}$$

# Properties of Conducting Materials

| **Material** | **Resistivity ρ** [μΩ·cm] | **Temp. coef α** [$mK^{-1}$] |
|---|---|---|
| Silver | 1.63 | 3.8 |
| Copper | 1.75 | 6.8 |
| Gold | 2.35 | 3.7 |
| Aluminum | 2.83 | 4.9 |
| Zinc | 5.9 | 3.8 |
| Brass | 7.5 | 2-7 |
| Iron | 9.8 | 6 |
| Platinum | 10.9 | 3.9 |
| Carbon | 33-185 | -6 to +1.2 |
| Tin | 11.5 | 4.2 |
| Constantan (Cu+Ni+Mn) | 49 | -0,03 |

# Electromotive Force

Let us suppose that we have an electric circuit containing a seat of emf $\varepsilon$ (e.g. a wire moving in magnetic field). The flow of charges can be described by the Ohm's law




$$\vec{j} = \gamma \vec{E} = \gamma(\vec{E}_S + \vec{E}_I)$$

where $E_S$ is an electric field causing the flow of charges outside the seat in the part m and $E_I$ is electric field inside the seat – part n.

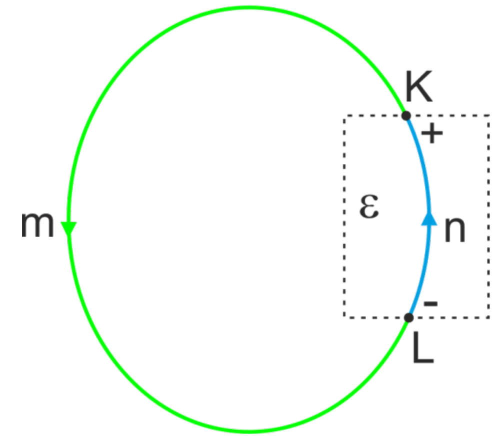
After small arrangements we obtain

$$\oint_{m+n} \frac{\vec{j}}{\gamma} \cdot d\vec{l} = \oint_{m+n} \vec{E} \cdot d\vec{l} = \oint_{m+n} (\vec{E}_S + \vec{E}_I) d\vec{l} \quad [1]$$

If we now realize that the motion of charges in the part n is influenced both by $E_S$ and $E_I$, while in the part m the value of $E_I = 0$, we can rewrite the right side

$$\oint_{m+n} (\vec{E}_S + \vec{E}_I) d\vec{l} = \int_m \vec{E}_S d\vec{l} + \int_n (\vec{E}_S + \vec{E}_I) d\vec{l} = \oint_{m+n} \vec{E}_S d\vec{l} + \int_n \vec{E}_I d\vec{l}$$

# Electromotive Force




Since we know that $$\oint_{m+n} \vec{E}_S d\vec{l} = 0$$

we can simplify the equation [1] to $$\oint_{m+n} \frac{\vec{j}}{\gamma} \cdot d\vec{l} = \int_L^K \vec{E}_I \cdot d\vec{l}$$

where the term $$\boxed{\int_L^K \vec{E}_I d\vec{l} = \varepsilon}$$ is the **electromotive force**

Now we can separate also the left part of the equation [1] $$\oint_{m+n} \frac{\vec{j}}{\gamma} \cdot d\vec{l} = \int_m \frac{\vec{j}}{\gamma} \cdot d\vec{l} + \int_n \frac{\vec{j}}{\gamma} \cdot d\vec{l}$$

Realizing that vectors $\vec{j}$ and $d\vec{l}$ are colinear and $$j = \frac{I}{S} \quad \text{and} \quad R = \frac{l}{\gamma S}$$

we can write

$$\int_m \frac{\vec{j}}{\gamma} \cdot d\vec{l} = \int_m \frac{j}{\gamma} dl = \int_m \frac{I}{S_m \gamma} dl = I \frac{l_m}{\gamma S_m} = I R$$

# Electromotive Force

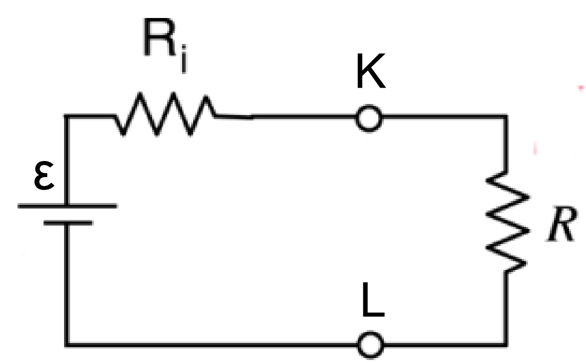
Similarly we could deduce
$$\int_n \frac{\vec{j}}{\gamma} \cdot d\vec{l} = I \frac{l_n}{\gamma S_n} = I R_i$$

so
$$\int_m \frac{\vec{j}}{\gamma} \cdot d\vec{l} + \int_n \frac{\vec{j}}{\gamma} \cdot d\vec{l} = IR + IR_i$$

The equation [1] can be then rewritten to
$$\varepsilon = I(R + R_i)$$

We can imagine it as a simple electric circuit with voltage source $\varepsilon$, internal resistance $R_i$ and load $R$. The emf seat includes both the source and $R_i$. The voltage on its terminals KL can be written as

$$V_{KL} = \varepsilon - I R_i$$

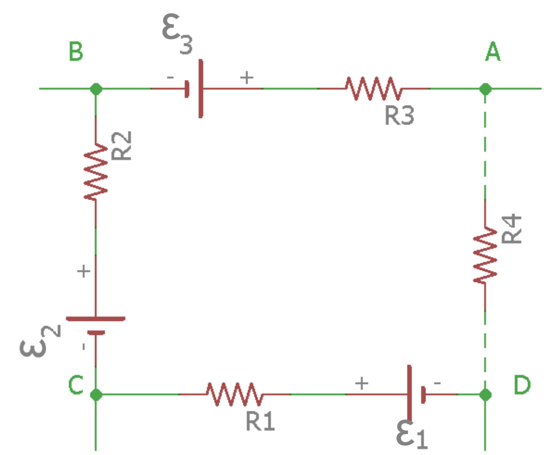



Example: ε=10V, $R_i$=0.5 Ω, R=5Ω.
$$I = \frac{\varepsilon}{R + R_i} = 1.82\,\mathrm{A}$$

$$V_{KL} = \varepsilon - I R_i = 9.1\,\mathrm{V}$$

# Second Kirchhoff's Law

We have a closed electric circuit consisting of junctions A-B-C-D and branches connecting them consisting of resistors and emf forces.




We already know that

$$\oint \vec{E}\cdot d\vec{l} = \oint \vec{E}_I \cdot d\vec{l} \qquad [2]$$

We can now rewrite integrals on both sides.

$$\oint \vec{E}\cdot d\vec{l} = \int_A^B \vec{E}\cdot d\vec{l} + \int_B^C \vec{E}\cdot d\vec{l} + \int_C^D \vec{E}\cdot d\vec{l} + \int_D^A \vec{E}\cdot d\vec{l}$$

$$\oint \vec{E}_I\cdot d\vec{l} = \int_A^B \vec{E}_I\cdot d\vec{l} + \int_B^C \vec{E}_I\cdot d\vec{l} + \int_C^D \vec{E}_I\cdot d\vec{l} + \int_D^A \vec{E}_I\cdot d\vec{l}$$

$$\oint \vec{E}\cdot d\vec{l} = R_1 I_1 + R_2 I_2 + R_3 I_3 + R_4 I_4$$

$$\oint \vec{E}_I\cdot d\vec{l} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

# Second Kirchhoff's Law

Both sides of the equation [2] can be simplified into the general form

$$\sum_{k=1}^{n} \varepsilon_k = \sum_{k=1}^{n} R_k I_k$$

or

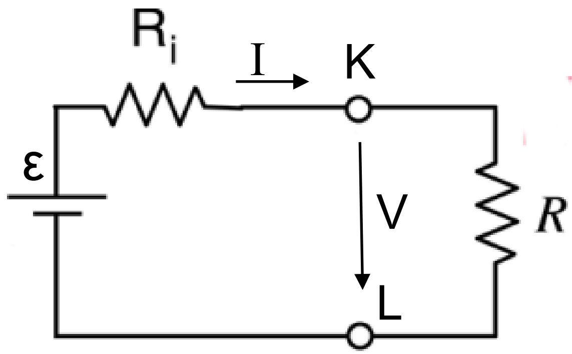
$$\sum_{k=1}^{n} \varepsilon_k - \sum_{k=1}^{n} R_k I_k = 0$$

Where *n* is the number of branches.

**The second Kirchhoff's law** can be formulated: *the algebraic sum of changes in potential around any closed path of the circuit is zero.*

# Energy Transfers in an Electric Circuit

We have a simple circuit consisting of battery B with its internal resistance $R_i$ and between its terminals KL we have a resistor $R$ as a load. The electric current flowing through the circuit is $I$. The elementary energy transformed (consumed) by the resistor $R$ is




$$dU = dq \cdot V$$

where $V$ is the voltage on the resistor $R$ and $dq$ is elementary charge passing through the $R$.

If $dt$ is elementary time necessary for passing the $dq$ through the $R$, then the **power consumed** by the $R$ is

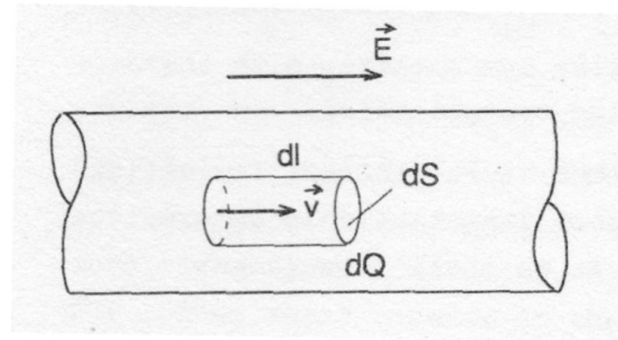
$$P = \frac{dU}{dt} = \frac{dq}{dt} V = I\,V$$

Using the Ohm's law we can rewrite the formula to the form of **Joule's law**.

$$P = I\,IR \qquad P = I^2 R \qquad P = \frac{V^2}{R} \quad [\mathrm{W}]$$

# Joule's Law From the Microscopic Point of View

We have a conductor connected to a battery. The electric field $\vec{E}$ acts on the charged particles so that they move with constant velocity $\vec{v}$. Now we choose an elementary volume *dV=dS·dl.* We can define the **power density**




$$p = \frac{dP}{dV} \quad \left[\mathrm{W/m^3}\right]$$

From the mechanics and electrostatics we know that $P = \vec{F}\cdot\vec{v}$ and $\vec{F} = Q\cdot\vec{E}$

so $$p = \frac{d}{dV}(\vec{F}\cdot\vec{v}) = \frac{d}{dV}(Q\cdot\vec{E}\cdot\vec{v})$$

For the stationary state we can consider $\vec{E}$ and $\vec{v}$ constant, so

$$p = \vec{E}\cdot\vec{v}\frac{dQ}{dV} = \vec{E}\cdot\vec{v}\cdot\rho_V$$ where $\rho_V$ is volume charge density

# Joule's Law From the Microscopic Point of View

From the current density chapter we know that

$$\rho_V = e \cdot n \qquad \text{and} \qquad \vec{j} = e \cdot n \cdot \vec{v}$$

where *n* is number of charges in the volume *dV* and *e* is elementary charge.

We can finally write for the power density

$$p = \vec{E} \cdot \vec{v} \cdot \rho_V = \vec{E} \cdot \vec{v} \cdot e \cdot n$$

$$\boxed{p = \vec{E} \cdot \vec{j}}$$

This relation is also called Joule's law from the microscopic point of view.

The total energy used by a device during time *t* can be written as

$$\boxed{W = \int_0^t P \cdot dt}$$

# Summary – what we have learnt

Relations between electric charge and current $$I = \frac{dQ}{dt}; \quad Q = \int_0^t I \cdot dt$$

Current density relations $$I = \iint_S \vec{j} \cdot d\vec{S}; \quad \vec{j} = e \cdot n \cdot \vec{v}; \quad div\,\vec{j} = -\frac{\partial \rho_V}{\partial t}$$

First Kirchhoff's law $$\sum_{k=1}^{n} I_k = 0$$

Second Kirchhoff's law $$\sum_{k=1}^{n} \varepsilon_k = \sum_{k=1}^{n} R_k I_k$$

Ohm's law $$V = IR; \quad \vec{j} = \gamma \vec{E}$$

Resistivity $$R = \rho \frac{l}{S}$$

Joule's law $$P = I^2 R; \quad p = \vec{E} \cdot \vec{j}$$

# Maxwell's Equations – integral form

1. Ampere's law

$$\oint_l \vec{H} \cdot d\vec{l} = \iint_S \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

2. Faraday's law

$$\oint_l \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S}$$

3. Gauss's law of electricity

$$\oiint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho \cdot dV$$

4. Gauss's law of magnetism

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

Constitutive relations

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}; \quad \vec{B} = \mu_r \mu_0 \vec{H}; \quad \vec{j} = \gamma \vec{E}$$

The term $\frac{\partial \vec{D}}{\partial t}$ in the Ampere's law represents a **displacement current**.

# Maxwell's Equations – differential form

To be able to transform Maxwell's equation into the differential form, we need two fundamental theorems of the vector calculus.

Gauss's theorem (divergence theorem)

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{F}) dV$$

Stokes's theorem

$$\oint_l \vec{F} \cdot d\vec{l} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

<u>Explanation of symbols</u>

**Nabla** (or del) operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

**Divergence** of the vector $\vec{F}$

$$\nabla \cdot \vec{F}$$

**Curl** of the vector $\vec{F}$

$$\nabla \times \vec{F}$$

# Maxwell's Equations – differential form

1. Transformation of the Ampere's law

$$\oint_l \vec{H}\cdot d\vec{l} = \iint_S \left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{S}$$

Using the Stokes's theorem we have

$$\oint_l \vec{H}\cdot d\vec{l} = \iint_S \left(\nabla\times\vec{H}\right)\cdot d\vec{S} = \iint_S \left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{S}$$

$$\boxed{\nabla\times\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}}$$

2. Transformation of the Faraday's law

$$\oint_l \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$

Using the Stokes's theorem we have

$$\oint_l \vec{E}\cdot d\vec{l} = \iint_S (\nabla\times\vec{E})d\vec{S} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{S}$$

$$\boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

# Maxwell's Equations – differential form

3. Transformation of the Gauss's law of electricity

$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = \iiint \rho \cdot dV$$

Using the Gauss's theorem we have

$$\oint\!\!\oint_S \vec{D}\cdot d\vec{S} = \iiint_V (\nabla\cdot\vec{D})dV = \iiint_V \rho\cdot dV$$

$$\boxed{\nabla\cdot\vec{D} = \rho}$$

4. Transformation of the Gauss's law of magnetism

$$\oint\!\!\oint_S \vec{B}\cdot d\vec{S} = 0$$

Using the Gauss's theorem we have

$$\oint\!\!\oint_S \vec{B}\cdot d\vec{S} = \iiint_V (\nabla\cdot\vec{B})dV = 0$$

$$\boxed{\nabla\cdot\vec{B} = 0}$$

# Maxwell's Equations in both forms

| Integral | Differential |
| --- | --- |
| $\oint_l \vec{H} \cdot d\vec{l} = \iint_S \left( \vec{j} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$ | $\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$ |
| $\oint_l \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{S}$ | $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ |
| $\oiint_S \vec{D} \cdot d\vec{S} = \iiint_V \rho \cdot dV$ | $\nabla \cdot \vec{D} = \rho$ |
| $\oiint_S \vec{B} \cdot d\vec{S} = 0$ | $\nabla \cdot \vec{B} = 0$ |