If a mass m of a substance of atomic weight M and of valence j is deposited, then the number of gram-equivalents is

$$\frac{m\,j}{M} \, .$$

Faraday's law may therefore be written as

$$\frac{m\,j}{M} = \frac{I\,t}{F} \, . \qquad (15\text{-}55)$$

The electrolysis is widely used in practice for example for refining of metals, in galvanic cells etc.

16. MAGNETIC FIELD

The science of magnetism grew from the observation that certain "stones" (magnetit) would attract bits of iron. The word magnetism comes from the district of Magnesia in Asia Minor, which is one of the places at which the stones were found. Today it is clear that magnetism and electricity are closely related. This relation was not discovered, however, untill 1820 when H. Ch. Oersted discovered that a current in a wire can also produce magnetic effects, namely that it can change the orientation of a compass needle. A compass needle placed near a straight section of current - carrying wire align itself so it is tangent to a circle drawn around the wire. Oersted had therefore found a connection between electricity - that is between movement of charges and magnetism.

16 - 1 Magnetic Field, Definition of $\vec{B}$

We define the space around a magnet or a current - carrying conductor as the site of a magnetic field, just as we defined the space near a charged rod as the site of an electric field.

Let us define the basic magnetic field vector $\vec{B}$, which is called the magnetic induction. For this let us place the test charge q in the electric and magnetic field. The force on this electric charge depends not only where it is, but also on how fast it is moving. Every point in space is characterized by two vector quantities which determine the force on any charge. First, there is the electric force which gives a force component independent of the motion of the charge.

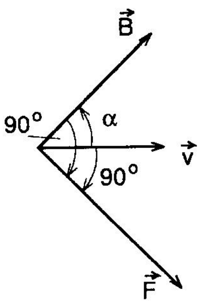

Figure 16 - 1

We describe it by the electric field $\vec{E}$. Second, there is an additional force component, called the magnetic force, which depends on the velocity of the charge. This magnetic force has a strange directional character: at every instant the force is always at right angles to the velocity vector (see Fig. 16-1).

It is possible to describe all of this behaviour by defining magnetic induction vector $\vec{B}$, which specifies both the unique direction in space and the constant of proportionality with the velocity, and to write the magnetic force as $q(\vec{v} \times \vec{B})$. The total electromagnetic force on a charge can be written as

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \, . \qquad (16\text{-}1)$$

This is called the Lorentz force.

The fact that the magnetic force is always at right angles to the direction of motion means that the work done by this force on the particle is zero. Thus a static magnetic field cannot change the kinetic energy of a moving charge; it can only deflect it sideways.

The unit of B that follows from Eq. (16-1) is given the SI name tesla (abbr. T) or weber/m^2 (abbr. Wb/m^2). Recalling that a coulomb/second is an ampere we have

$$1\ T = \frac{1\ Wb}{m^2} = \frac{1\ N}{A\ m} .$$

An earlier unit for B, still in common use is the gauss, the relationship is 1 T = = 10^4 gauss.

16 - 2. Lines of Induction, Magnetic Flux

Just as we represented the electric field by lines of force we can represent magnetic field by lines of induction. The magnetic induction vector is related to the lines of induction in this way:

1. The tangent to a line of induction at any point gives the direction of $\vec{B}$ at that point.
2. The lines of induction are drawn so that the number of lines per unit cross-sectional area (perpendicular to the lines) is proportional to the magnitude of B. Where the lines are close together $\vec{B}$ is large and where they are far apart B is small.

For the magnetic field, the magnetic induction vector B is of fundamental importance; the lines of induction simply giving a graphic representation of the way $\vec{B}$ varies throughout a certain region of space.

There is however a great difference between lines of force which represent the electric field and lines of induction. There is no magnetic analog of an electric charge, that is there are no magnetic charges from which lines of B can emerge. If we think in terms of lines of induction they can never start and they never stop. Since these lines do not begin or end, they will close back on themselves, making closed loops. But they will never diverge from points. No magnetic charge have ever been discovered.

The above mentioned fact can be used to obtain so called Gauss's law of magnetism. For this let us define the flux ϕ_B for a magnetic field in exact analogy with the flux for the electric field (see Section 14-5), namely

$$\phi_B = \iint \vec{B}.\vec{dS} \qquad (16\text{-}2)$$

in which the integral is taken over the surface (closed or open) for which ϕ_B is defined. The SI unit of ϕ_B is weber (abbr. Wb).

Let us determine the flux ϕ_B through a closed surface S which is immersed in a magnetic field (see Fig. 15-2).

As far as the induction lines make closed loops then the number of lines entering the closed surface equals to the number of lines leaving the surface. Thus the magnetic flux through a closed surface equals to zero, or

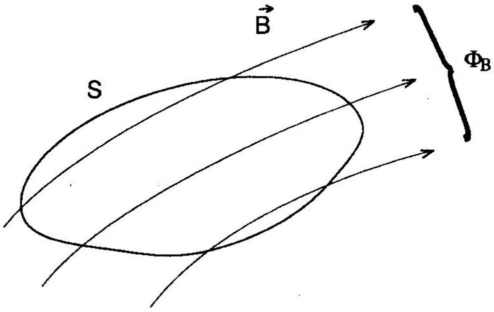

Figure 16 - 2

$$\oint\!\!\oint_S \vec{B}.\vec{dS} = 0 \,. \quad (16\text{-}3)$$

This equation which is called Gauss's law of magnetism is one of the basic equations not only for magnetostatics but even for dynamic fields.

16 - 3 Magnetic Force on a Current

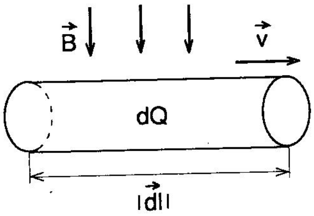

Figure 16 - 3

A current is an assembly of moving charges.Because a magnetic field exerts a sideways force on a moving charge, we expect that it will also exert a sideways force on a wire carrying a current. Figure 16-3 shows a differential element of a wire of length $|\vec{d\ell}|$ carrying a steady current I and placed in a magnetic field $\vec{B}$ only. The vector $\vec{d\ell}$ points in the direction of positive current flow.

The magnetic force acting on a differential length $d\ell$ of the wire can be obtained directly from definition of $\vec{B}$ in Eq. (16-1) which we can rewrite for the force on a charge element dQ which is moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ as

$$\vec{dF} = dQ(\vec{v} \times \vec{B}) \,. \quad (16\text{-}4)$$

From the definition of current in terms of charge transport we know that $dQ = I\,dt$ and $\vec{v} = \frac{\vec{d\ell}}{dt}$. Thus we can write

$$\vec{dF} = I\,dt(\frac{\vec{d\ell}}{dt} \times \vec{B})$$

or

$$\vec{dF} = I(\vec{d\ell} \times \vec{B}) \,. \quad (16\text{-}5)$$

Equation (16-5) expresses the force which magnetic field exerts on a differential length $d\ell$ of the conductor carrying a steady current I. By integrating this formula in an appropriate way we can find the force $\vec{F}$ on a conductor which is not straight:

$$\vec{F} = \int_\ell I(\vec{d\ell} \times \vec{B}) \,. \quad (16\text{-}6)$$

16 - 4 Torque on a Current Loop, Magnetic Dipole Moment

Let us consider now the rectangular loop of wire see Fig. 16-4 of height h and width ℓ placed in a uniform magnetic field $\vec{B}$. The loop, which carries a current I is suspended so that it is free to rotate about the axis x - x'.

The orientation of the loop with respect to the magnetic induction $\vec{B}$ is given by the angle α between $\vec{B}$ and unit vector $\vec{n_0}$ of the normal to the loop. The orientation of vector $\vec{n_0}$ is given by the right-hand rule (cup your right-hand so that your fingers wrap around the loop in the direction of current flow, then your thumb points in the direction of $\vec{n_0}$).

The net force on the loop is the resultant of the forces on the four sides of the loop. The magnetic force on each side can be determined from Eq. (46-5). Note that the orientation of vector $\vec{d\ell}$ is the same as the orientation of current.

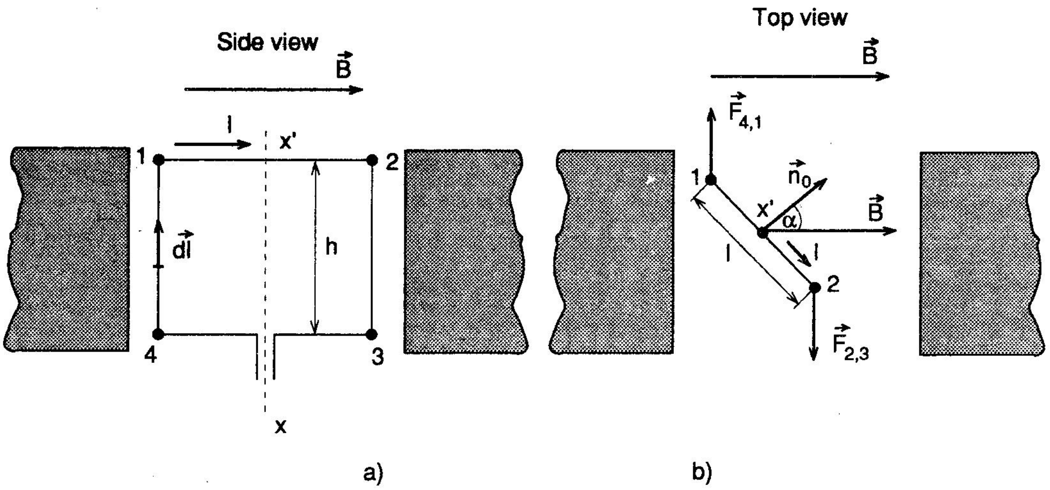

Figure 16 - 4

Using the Eq. (46-5) we can see that the magnetic forces on the loop's arms (sides 1 - 2 and 3 - 4) are of equal magnitudes and in opposite directions, hence they cancel and produce no net force. As far as they have the same line of action the net torque due to these forces is also zero.

The common magnitude of forces $\vec{F}_{4,1}$ and $\vec{F}_{2,3}$ is

$$F_{4,1} = F_{2,3} = IhB \tag{16-7}$$

since these sections are perpendicular to $\vec{B}$. The direction of $\vec{F}_{4,1}$ and $\vec{F}_{2,3}$ is shown in Fig. 16-4b. These forces do not act along the same line, hence can produce a torque. This torque is equal

$$\vec{\tau}_m = \vec{\ell} \times \vec{F} ,$$

where $|\vec{F}| = |\vec{F}_{4,1}| = |\vec{F}_{2,3}| =$ IhB. Magnitude of the torque is

$$\tau_m = \ell F \sin\alpha = \ell (IhB) \sin\alpha = I(\ell h)B \sin\alpha = ISB \sin\alpha , \tag{16-8}$$

where $S = \ell h$ is area of the loop shown in Fig. 16-4. This formula derived here for a rectangular loop is valid for any shape of the flat loop. The quantity:

$$\vec{\mu} = IS\,\vec{n}_o \tag{16-9}$$

is called the magnetic dipole moment of the loop. With this definition of $\vec{\mu}$ we can rewrite Eq. (16-8) in vector form

$$\vec{\tau}_m = \vec{\mu} \times \vec{B} \tag{16-10}$$

which gives the magnitude and direction for $\vec{\tau}_m$.

Since a torque acts on a current loop, or other magnetic dipole, when it is placed in an external magnetic field, it follows that work (positive or negative) must be done by an external agent to change the orientation of such a dipole. Thus a magnetic dipole has potential energy associated with its orientation in an external magnetic field.

We can assume that the magnetic energy W is zero when $\vec{\mu}$ and $\vec{B}$ are at right angles, that is, when $\alpha = 90^o$. This choice of a zero - energy configuration for W is arbitrary because we are interested only in the changes in energy that occur when the dipole is rotated.

The magnetic potential energy in any position α is defined as the work that an external agent must do to turn the dipole from its zero-energy position ($\alpha = 90^o$) to the given position α. Thus

$$U = W = \int_{90^\circ}^{\alpha} \tau_m \, d\alpha = \int_{90^\circ}^{\alpha} ISB \sin\alpha \, d\alpha = \mu B \int_{90^\circ}^{\alpha} \sin\alpha \, d\alpha = -\mu B \cos\alpha$$

in which Eq. (16-8) is used to substitute for τ_m. In vector symbolism we can write this relation as

$$U = - \vec{\mu} . \vec{B} . \qquad (16\text{-}11)$$

This relation is an equivalent of expression for the energy of an electric dipole in an external electric field (see Eq. 14-23).

The torque produced by the magnetic force on the current is the basic principle behind a number of important practical devices, including motors and meters.

16 - 5 Moving Electric Charge in a Magnetic Field

Let us imagine a positively charged particle Q introduced at time t = 0 with velocity $\vec{v}_o$ into a uniform magnetic field $\vec{B}$ at $x_o = y_o = z_o = 0$, see Fig. 16-5. We assume that $\vec{v}_o$ is at right angles to $\vec{B}$ and thus lies entirely in the plane of the figure that is in the plane x - y.

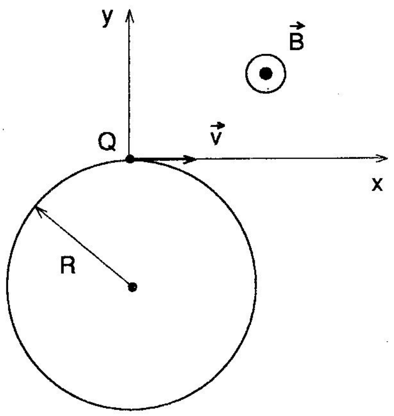

Figure 16 - 5

Let us analyze motion of the particle and let us determine the equation of the trajectory of the particle.

The initial conditions could be written as follows:

$$t = 0 \qquad x_o = y_o = z_o = 0$$

$$\vec{v}_o = \vec{v}_o(v_{ox}, 0, 0)$$

Magnetic induction vector has the components

$$\vec{B} = \vec{B}(0, 0, B).$$

From Eq. (16-1) we can obtain the expression for a force which exerts the magnetic field on an object of charge Q as

$$\vec{F} = Q(\vec{v} \times \vec{B}) . \qquad (16\text{-}12)$$

From the Newton's second law we have

$$Q(\vec{v} \times \vec{B}) = m \frac{d^2\vec{r}}{dt^2} , \qquad (16\text{-}13)$$

where the position vector $\vec{r}$ has the components x and y.

Let us write the single vector Eq. (16-13) as the two scalars equations, that is

$$m \frac{d^2x}{dt^2} = Q v_y B = Q \frac{dy}{dt} B , \qquad (16\text{-}14)$$

$$m \frac{d^2y}{dt^2} = - Q v_x B = - Q \frac{dx}{dt} B . \qquad (16\text{-}15)$$

After integration of Eqs. (16-14) and (16-15) with respect to initial conditions we have

$$v_x = \frac{Q}{m} y B + v_{ox} , \qquad (16\text{-}16)$$

$$v_y = -\frac{Q}{m} x B . \quad (16\text{-}17)$$

Integrating Eqs. (16-16) and (16-17) we obtain the components of the position vector as

$$x = \frac{Q}{m} yBt + v_{ox} t , \quad (16\text{-}18)$$

$$y = -\frac{Q}{m} xBt . \quad (16\text{-}19)$$

Eliminating t we obtain equation of the trajectory of the particle in a magnetic field. For this we use the equation

$$v_x^2 + v_y^2 = v^2 \quad (16\text{-}20)$$

Substituting Eqs. (16-16) and (16-17) into Eq. (16-20) we have after some rearrangement

$$x^2 + \left(y + \frac{mv_{ox}}{QB}\right)^2 = \left(\frac{mv_{ox}}{QB}\right)^2 \quad (16\text{-}21)$$

for the trajectory of the charged particle in a magnetic field. It is obvious that Eq. (16-21) is the equation of a circle with a radius

$$R = \frac{mv_{ox}}{QB} . \quad (16\text{-}22)$$

Thus the path of the positively charged particle Q which is introduced perpendiculary into the magnetic field is a circle with the coordinates of its center of rotation (see Eq. 16-21)

$$x_c = 0 ,$$

$$y_c = -\frac{mv_{ox}}{QB} .$$

The time T required for a particle of a charge Q to make one circular revolution in a uniform magnetic field $\vec{B}$ is

$$T = \frac{2\pi R}{v_{ox}} , \quad (16\text{-}23)$$

where $2\pi R$ is the circumference of its circular path. For the frequency of rotation we can, with respect to Eq. (16-22), obtain

$$f = \frac{1}{T} = \frac{QB}{2\pi m} . \quad (16\text{-}24)$$

This is often called the cycloctron frequency of a particle. Note that this frequency does not depend on the speed of the particle. The cycloctron frequency is a characteristic frequency for a charged particle in the magnetic field and may be compared for example to the characteristic frequency of an oscillating mass-spring system.

The influence of magnetic field on the motion of charged particles is used for example in cycloctrons, synchrotrons, for deflection of electrons beams etc.

16 - 6 Ampère's Law

One class of problems involving magnetic fields, dealt with in preceding sections, concerns the forces exerted by a magnetic field on a moving charge or on a current-carrying conductor and the torque exerted on a magnetic dipole. A second class of problems concerns the production of a magnetic field by a current-carrying conductor. This section deals with problems of this second class.

We will now discuss how magnetic induction is determined for simple situation that is for a long straight wire carrying the current.

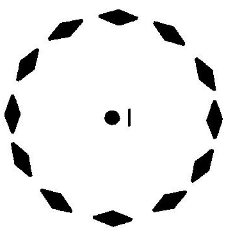

Figure 16 - 6

Imagine a wire surrounded by a number of small compass needles. If there is non current in the wire, all compass needles are aligned with the horizontal component of the earth´s magnetic field. Where a strong current I is present the needles point so as to suggest that the lines of induction form closed circles around the wire (see Fig. 16-6).

We might expect that the magnetic induction at a given point would be greater if the current flowing in the wire were greater, and the field would be less at points further from the wire.

Careful experiments show that the magnetic induction B at a point near the wire is directly proportional to the current I in the wire and inversely proportional to the distance r from the wire:

$$B \sim \frac{I}{r} .$$

We can convert this proportionality into an equality by inserting proportionality constant. We do not write this constant simply as, say, K but in a more complex form, namely $\mu_0/2\pi$ in which μ_0 is called the permeability of free space and its value is

$$\mu_0 = 4\pi.10^{-7}\ H.m^{-1}.$$

The constant is chosen like this so that Ampère´s law, which we shall see, has a simple and elegant form. Thus we can write the relation between the current in a long straight wire and a magnetic induction as

$$B = \frac{\mu_0 I}{2\pi r} . \qquad (16-25)$$

This equation is valid for a long straight wire only. The following question arises: is there a general relation between a current in a wire of any shape and the magnetic field around it? The answer on this question found A. M. Ampère.

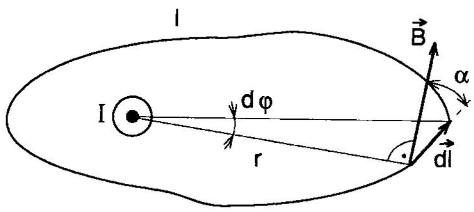

Figure 16 -7

Consider any arbitrary closed path around a straight conductor carrying a current I as shown in Fig. 16-7. Let us now try to determine the magnitude of the line integral $\oint \vec{B}.\vec{d\ell}$ around a closed path ℓ, where $\vec{d\ell}$ is an infinitesimal length vector. From Fig. 16-7 we can see:

$$d\ell \cos\varphi = r\, d\varphi .$$

The dot product $\vec{B}.\vec{d\ell}$ is:

$$\vec{B}.\vec{d\ell} = B\, d\ell \cos\alpha = Br\, d\varphi .$$

As far as the magnetic induction at a distance r from a current carrying straight wire is given by Eq.(16-25) we can insert this induction into the $\oint \vec{B}.\vec{d\ell}$ and integrate it over the whole closed path. Thus we have:

$$\oint \vec{B}.\vec{d\ell} = \oint Br\, d\varphi = \int_0^{2\pi} \frac{\mu_0 I}{2\pi r} r\, d\varphi = \mu_0 I$$

or

$$\oint \vec{B}.\vec{d\ell} = \mu_0 I . \qquad (16\text{-}26)$$

This equation is known as Ampère´s law. To determine the direction of magnetic induction near a wire carrying a current it is possible to use a right hand rule.

Practical value of Ampère´s law as a mean to calculate the magnetic field is limited to simple cases. Its real importance lies in the fact that it relates a magnetic field to the current in a direct way. This law is thus considered one of the basic laws of electricity and magnetism.

16 - 7 Biot - Savart Law

We can use Ampère´s law to calculate magnetic fields only if the symmetry of the current distribution is high enough to permit the easy evaluation of the line integral $\oint \vec{B}.\vec{d\ell}$. This requirement limits the usefulness of the law in practical problems. The law does not fail, it simply becomes difficult to apply in a useful way. This is much like the electric case, where Gauss´s law is also considered fundamental but is limited in its use for actually calculating $\vec{E}$; so we must often determine the electric field $\vec{E}$ by summing over contributions due to infinitesimal charge elements via Coulomb´s law. A magnetic equivalent to this infinitesimal form of Coulomb´s law is Biot - Savart law.

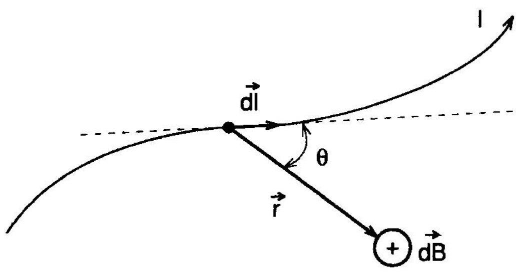

Figure 16 - 8

According to Biot and Savart, a current I flowing in any path can be considered as many infinitesimal current elements, such as in the wire of Fig. 16-8.

If $\vec{d\ell}$ represents any infinitesimal length along which the current is flowing, then the magnetic induction $\vec{dB}$ at any point in space due to this element of current is given by

$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{d\ell} \times \vec{r}_0}{r^2} , \qquad (16\text{-}27)$$

where $\vec{r}_0$ is the unit vector pointing from the element $\vec{d\ell}$ to the given point.

Equation (16-27) is known as the Biot - Savart law. The magnitude of $\vec{dB}$ is

$$dB = \frac{\mu_0 I\, d\ell \sin\theta}{4\pi r^2} , \qquad (16\text{-}28)$$

where θ is the angle between $\vec{d\ell}$ and $\vec{r}$. The total magnetic induction at given point is then found by integrating over all current elements

$$\vec{B} = \int \vec{dB} , \qquad (16\text{-}29)$$

where the integral is a vector integral.

The Biot - Savart law is the magnetic equivalent of Coulomb´s law in its infinitesimal form.

16 - 8 Applications of Ampère's and Biot - Savart Laws

In this section we show how to use Ampère's and Biot - Savart law for the simplest example that is for a long straight wire carrying the current I.

Example 1: Derive an expression for $\vec{B}$ at a distance R from the center of a long cylindrical wire. The wire carries a current I. For the solution use Ampère's law.

Solution: We know from the Section 16-6 that the lines of induction around the straight wire carrying a current are circles with the wire at their center. So to apply Eq. (16-26) we choose as our path of integration a circle of radius R. We choose this path because at any point on this path, $\vec{B}$ will be tangent to this circle. Thus for any short element of the circle, $\vec{B}$ will be parallel to that segment, see Fig. 16-9.

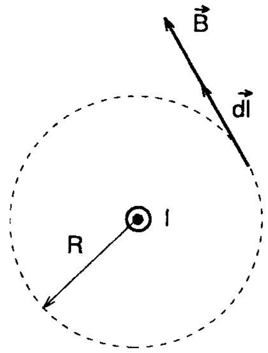

Figure 16 - 9

The central dot in this figure suggests a current I in the wire emerging from the page. Note that the angle between $\vec{B}$ and $\vec{d\ell}$ is zero so that $\vec{B}\,\vec{d\ell} = B\,d\ell$.

So we have

$$\oint \vec{B}.\vec{d\ell} = \oint B\,d\ell = B \oint d\ell = B\,2\pi R = \mu_o I \,.$$

We solve for B and obtain

$$B = \frac{\mu_o I}{2\pi R} \,.$$

Example 2: Derive an expression for $\vec{B}$ at a distance r from the center of a long cylindrical wire which carries a current I. For the solution use Biot - - Savart law.

Solution: Figure 16-10, a side view of the wire, shows a typical current element oriented by the infinitesimal vector $\vec{d\ell}$.

The magnitude of the contribution $\vec{dB}$ at point P of this element is found from Eq. (16-28), or

$$dB = \frac{\mu_o I}{4\pi}\,\frac{d\ell \sin\theta}{r^2} \,.$$

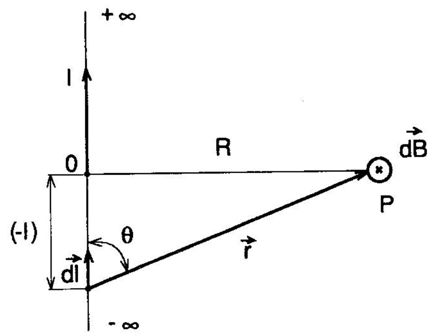

Figure 16 - 10

The directions of the contributions $\vec{dB}$ at point P for all elements are the same, namely, into the plane of the figure at right angles to the page. Thus the vector integral of Eq. (16-29) reduces to a scalar integral, or

$$B = \int dB = \frac{\mu_o I}{4\pi} \int_{x=-\infty}^{x=+\infty} \frac{\sin\theta\,d\ell}{r^2} \,. \tag{16-30}$$

Now ℓ, θ and r are not independent, being related (see Fig. 16-10) by

$$r = \frac{R}{\sin \Theta} \qquad (16\text{-}31)$$

and

$$\ell = - R \operatorname{cotg} \Theta .$$

Thus we have

$$d\ell = \frac{R}{\sin^2 \Theta} d\Theta . \qquad (16\text{-}32)$$

Substituting Eqs. (16-31) and (16-32) into Eq. (16-30) we have

$$B = \frac{\mu_0 I}{4\pi R} \int_0^{\pi} \sin \Theta \, d\Theta = \frac{\mu_0 I}{2\pi R} .$$

This is the result that we arrived at earlier for this problem. The Biot - Savart law will always yield results that are consistent with Ampère´s law and with experiments.

16 - 9 Force Between Two Parallel Wires

Figure 16-11 shows two long parallel wires separated by a distance d carrying currents I_1 and I_2 of the same orientation. It is an experimental fact that two such conductors attract each other.

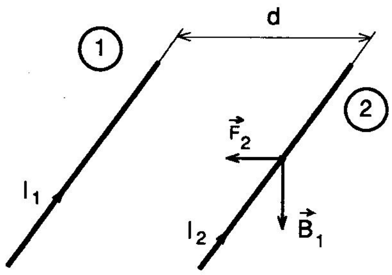

Figure 16 - 11

First wire will produce a magnetic field whose magnitude at the site of the second wire is, from Eq. (16-26),

$$B_1 = \frac{\mu_0 I_1}{2\pi d} . \qquad (16\text{-}33)$$

The right-hand rule shows that the direction of $\vec{B}_1$ at wire 2 is down, as shown in the figure.

Wire 2, which carries a current I_2, finds itself immersed in an external magnetic field $\vec{B}_1$. A length ℓ of this wire will experience a sideways magnetic force, see Eq. (16-6), whose magnitude is

$$F_2 = I_2 \ell B_1 = \frac{\mu_0 \ell I_1 I_2}{2\pi d} . \qquad (16\text{-}34)$$

The right hand rule tells us that $\vec{F}_2$ lies in the plane of the wires and points to the left in Fig. 16-11.

We could have started with second wire, computed the magnetic field which it produces at the site of wire 1, and then computed the force on the first wire. The force on wire 1 would, for parallel currents, point to the right. We see that two parallel wires that carry parallel currents attract each other. The forces that two wires exert on each other are equal and opposite, as they must be according to Newton´s law of action and reaction. For antiparallel currents the two wires repel each other.

The attraction between two long parallel wires is used to define the current of 1 ampere. Suppose that the wires are one meter apart (d = 1,0 m) and the two currents are equal $I_1 = I_2 = I$. If this common current is adjusted until, by measurement, the force of attraction per unit length between the wires is 2.10^{-7} N/m, the current is defined to be one ampere. From Eq. (16-34) we have

$$\frac{F}{\ell} = \frac{\mu_0 I^2}{2\pi d} = \frac{4\pi.10^{-7}.1}{2\pi.1} = 2.10^{-7} \text{ N/m}$$

as we expected.

16 - 10 Electromagnetic Induction

The development of electrical engineering began with Faraday and Henry, who independently and at nearly the same time discovered the principles of induced electromotive force and the methods by which mechanical energy can be converted directly to electrical energy.

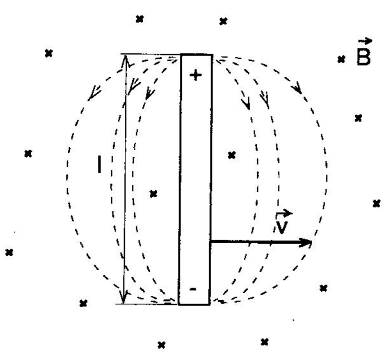

Figure 16 - 12

Let us consider a conductor of length ℓ in a uniform magnetic field, perpendicular to the plane of a Fig. 16-12 and directed away from us.

Let the conductor be set in motion toward the right with a velocity $\vec{v}$, perpendicular both to its own length and to the magnetic field. Every charged particle within the conductor experiences a force $F = qvB$ which is directed along the length of the conductor. Thus the electrons would collect at the lower end of the conductor, leaving the upper end positive. This distribution of charges establishes an electric field, which balances the force caused by the conductor in a magnetic field, or

$$q\vec{E} = q(\vec{v} \times \vec{B}) .$$

For so called induced electric field we obtain

$$\vec{E} = \vec{v} \times \vec{B} . \qquad (16\text{-}35)$$

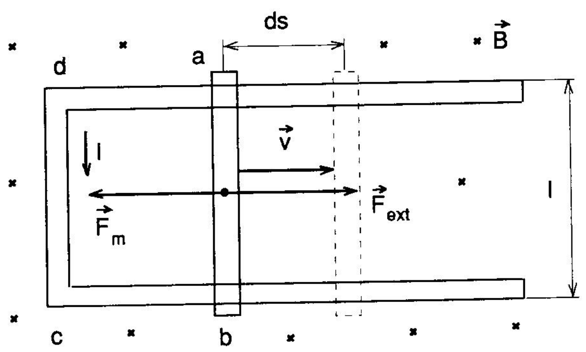

Figure 16 - 13

Imagine now that the moving conductor slides along a stationary U-shaped conductor as in Fig. 16-13. From the previous analysis it is clear that the current I will be established within the U-shaped conductor. As a result of this current the excess charges at the ends of the moving conductor are reduced, the electrostatic field within the moving conductor is weakened, and the magnetic forces cause a further displacement of the free electrons within it from a toward b. As long as the motion of the conductor is maintained there will be a current in the U-shaped conductor in a counterclockwise direction. The moving conductor corresponds to a seat of electromotive force and is said to have induced within it an electromotive force whose magnitude we now compute.

The definition of electromotive force (see Section 15-3) is the ratio of the work done on the circulating charge to the quantity of this charge. Let I be the current in the circuit in Fig. 16-13. Because of the existence of this current, the field exerts a force toward the left on the moving conductor, and therefore an external force provided by some working agent is needed to maintain the motion. The work done by this agent is the work done on the circulating charge - hence the direct conversion, in this device, of mechanical to electrical energy.

The force exerted by the magnetic field on the moving conductor, for our case, see Eq. (16-6), is

$$F_m = I\ell B .$$

The force exerted by an external agent has an equal magnitude but opposite direction (see Fig. 16-13), or

$$F_{ext} = - I\ell B ,$$

The distance traveled by the conductor in time dt is

$$ds = v\, dt$$

and thus the elementary work dW done by this force is

$$dW = F_{ext}\, ds = - I\ell B v\, dt .$$

The product (I dt) is the charge displaced in time dt. Hence

$$dW = - B\ell v\, dq .$$

Thus for the induced electromotive force we have

$$\mathcal{E} = \frac{dW}{dq} = - B\ell v . \qquad (16\text{-}36)$$

The induced emf in the circuit of Fig. 16-13 may be considered also from another point of view. While the conductor moves toward the right a distance ds, the area of the circuit a-b-c-d increases by

$$dS = \ell\, ds \qquad (16\text{-}37)$$

and the change in magnetic flux through the area bounded by the circuit is

$$d\phi_B = B\, dS = B\ell\, ds .$$

The induced emf given by Eq. (16-36) can be expressed as:

$$\mathcal{E} = - B\ell \frac{ds}{dt} .$$

Taking into account Eq. (16-37) we can write

$$\mathcal{E} = - \frac{B\, dS}{dt} = - \frac{d\phi_B}{dt}$$

or

$$\mathcal{E} = - \frac{d\phi_B}{dt} . \qquad (16\text{-}38)$$

This equation is known as Faraday's law which can be applied to any circuit through which the magnetic flux is caused to vary any means whatever. We see that the induced emf in the circuit is numerically equal to the rate of change of the flux through it. If we place a conducting loop in a time varying magnetic field, the flux through the loop will change and an induced emf will appear in the loop. This emf will set the charge carriers in motion, that is, it will induce a current. The direction of this current is determined by so called Lenz's law which states that the direction of an induced current is such as to oppose the cause producing it. The induced electric fields are just as real as electric fields set up by static charges and will exert a force on a charges.

We can write Faraday's law for a more general case, taking into account that

$$\mathcal{E} = \oint \vec{E}.d\vec{\ell} .$$

If Eq. (16-39) is combined with Eq. (16-38) we obtain

$$\int \vec{E}.\vec{d\ell} = -\frac{d\,\phi_B}{dt} .$$

Substituting into this expression Eq. (16-2) for magnetic flux we have

$$\oint \vec{E}.\vec{d\ell} = -\frac{d}{dt}\iint \vec{B}.\vec{dS} . \qquad (16\text{-}40)$$

This is Faraday's law of induction in the most general form. This law gives the value of induced emf no matter whether the change in magnetic flux is produced by moving a coil, moving a magnet, changing the magnetic induction, changing the shape of conducting loop or in other ways.

Equation (16-40) is also known as second Maxwell's equation.

There is a great number of devices using the principle of electromagnetic induction: generators, transformers, meters etc.

16 - 11 Inductance

We discussed in the last section how a changing magnetic flux through a circuit induces an emf in that circuit. We saw earlier that an electric current produces a magnetic field. Combining these two ideas, we expect that a changing current in one circuit ought to induce an emf in a second nearby circuit and even to induce an emf in itself. Now we will treat this effect in a more general way in terms of what we will call mutual inductance and self-inductance.

Let us consider a closed loop $\mathcal{L}$ carrying a current I, see Fig. 16-14.

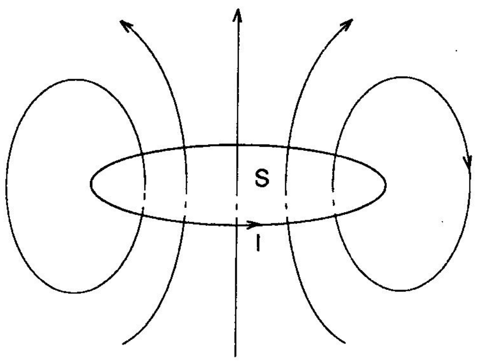

Figure 16 - 14

Magnetic flux ϕ_B through a surface S which is bounded by this loop is given as

$$\phi_B = \iint_S \vec{B}.\vec{dS} . \qquad (16\text{-}41)$$

To obtain magnetic induction $\vec{B}$ in a given point on the surface we can use Biot - Savart law, or

$$\vec{B} = \frac{\mu I}{4\pi} \oint_{\mathcal{L}} \frac{\vec{d\ell} \times \vec{r^o}}{r^2}$$

where $\vec{r^o}$ is a unit vector in the direction of $\vec{r}$ (see Fig. 16-8) and integration is taken over all loop.

Substituting Biot - Savart law in Eq. (16-41) we obtain for a magnetic flux through a surface S

$$\phi_B = \iint_S \frac{\mu I}{4\pi r^2} \oint_{\mathcal{L}} (\vec{d\ell} \times \vec{r^o}).\ \vec{dS} . \qquad (16\text{-}42)$$

Denoting

$$L = \iint_S \frac{\mu}{4\pi r^2} \oint_{\mathcal{L}} (\vec{d\ell} \times \vec{r^o}).\ \vec{dS} , \qquad (16\text{-}43)$$

we can write Eq. (16-42) as

$$\phi_B = LI , \qquad (16\text{-}44)$$

where constant of proportionality between ϕ_B and I, L is called self-inductance. As it is seen from expression (16-43) the magnitude of L depends on the geometry and on the permeability μ of space in which the loop is situated.

The SI unit of inductance can be obtained from Eq. (16-44) as

$$[L] = \frac{[\Phi]}{[I]} = \frac{Wb}{A} = \frac{V.s}{A} = H .$$

This unit has a special name, the henry (abbr. H).

Let us now imagine that a time varying current passes through the loop. In this case a changing magnetic flux is produced and this in turn induces an emf. This induced emf opposes the change in flux (Lenz's law - see preceding section). For example if the current in the loop is increasing the increasing magnetic flux induces an emf that opposes the original current and tends to retard its increase. If the current is decreasing, the decreasing flux induces an emf in the same direction as the current, thus tending to maintain the original current.

The value of emf induced in a loop of inductance L can be obtained from Faraday's law (Eq. 16-38) as

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(LI) = -L\frac{dI}{dt} . \qquad (16\text{-}45)$$

Now let us consider a case when loops 1 and 2 are placed near each other, as in Fig. 16-15.

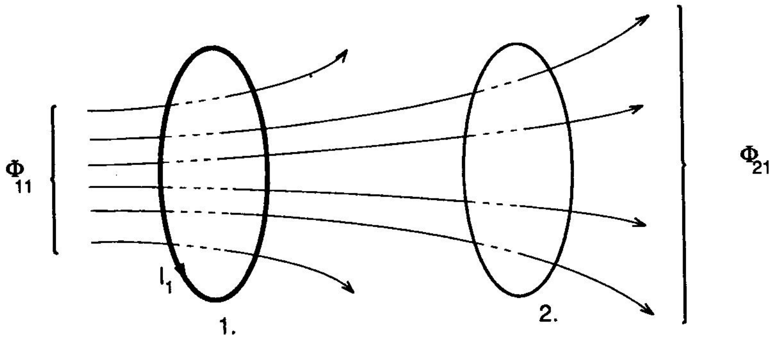

Figure 16 - 15

It is obvious that a changing current in one loop will induce an emf in the other. According to Faraday's law, the emf $\mathcal{E}_2$ induced in second loop is proportional to the rate of change of flux passing through it. This flux is due to the current I_1 in first loop, and it is often convenient to express the emf in loop 2 in terms of the current in loop 1.

We let Φ_{21} be the magnetic flux in the second loop due to the current I_1 in the first loop. If the two loops are fixed in space then Φ_{21} is proportional to the current I_1 in the first loop; the proportionality constant is called mutual inductance, M_{21}, defined by

$$M_{21} = \frac{\Phi_{21}}{I_1} . \qquad (16\text{-}46)$$

The emf $\mathcal{E}_2$ induced in second loop due to the changing current in the first loop is

$$\mathcal{E}_2 = -\frac{d\Phi_{21}}{dt} . \qquad (16\text{-}47)$$

Combining Eqs. (16-47) and (16-46) we obtain

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt} . \qquad (16\text{-}48)$$

This expression relates the current in the first loop to the emf it induces in the second loop. The mutual inductance of the second loop with respect to the first loop is a constant that depends on the size, shape, on the relative positions of the two loops and also on whether iron (or other ferromagnetic material) is present.

Suppose now, we consider the reverse situation, when a changing current in the second loop induces an emf in the first loop. In this case we have

$$\mathcal{E}_1 = - M_{12} \frac{dI_2}{dt} , \tag{16-49}$$

where M_{12} is the mutual inductance of the loop 1 with respect to the loop 2. It is possible to show, although we will not prove it here, that $M_{12} = M_{21}$. Hence, for a given arrangement of loops, we do not need the subscripts and we can let

$$M_{12} = M_{21} = M \tag{16-50}$$

so that

$$\mathcal{E}_1 = - M \frac{dI_2}{dt} \tag{16-51}$$

and

$$\mathcal{E}_2 = - M \frac{dI_1}{dt} . \tag{16-52}$$

The SI unit for mutual inductance is the henry (H).

16 - 12 Energy and Energy Density in the Magnetic Field

When we lift a stone we do work, which we can get back again by lowering the stone. When we pull two unlike charges apart we like to say that the work we do is stored in the electric field between the charges. We can get it back from the field by letting the charges move closer together again.

In the same way we can store energy in a magnetic field. For example, two long, rigid, parallel wires carrying current in the same direction attract each other and we must do work to pull them apart. We can get this stored energy back at any time by letting the wires move back to their original positions.

To derive a quantitative expression for the storage of energy in the magnetic field, let us consider Fig. 16-16a, which shows a source of emf $\mathcal{E}$ connected across a switch to a resistor R and an inductor L.

At the instant when the RL circuit is connected to the emf $\mathcal{E}$, the current starts to rise gradually as shown in Fig. 16-16b. Let us apply the second Kirchhoff's

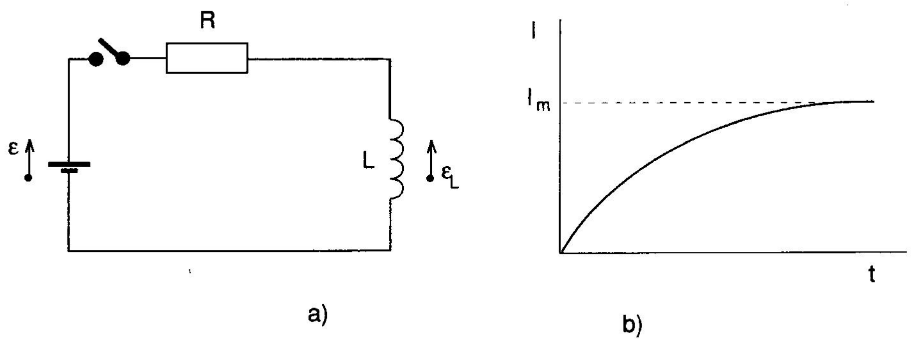

Figure 16 - 16

rule (see Eq. 15-45) to the circuit of Fig. 16-16a. The emf's in the circuit are the battery $\mathcal{E}$ and the induced emf $\mathcal{E}_L = -L\frac{dI}{dt}$. These must equal the potential drop across the resistor, or

$$\mathcal{E} + \mathcal{E}_L = RI$$

or

$$\mathcal{E} - L\frac{dI}{dt} = RI \,. \qquad (16\text{-}53)$$

If we multiply each side of this equation by I we obtain

$$\mathcal{E}I - LI\frac{dI}{dt} = RI^2 \,. \qquad (16\text{-}54)$$

Equation (16-54) represents a statement of the conservation of energy for LR circuit. The terms in Eq. (16-54) have the following physical interpretation:

- The term $\mathcal{E}I$ expresses the rate at which the seat of emf delivers energy to the circuit.

- The term RI^2 expresses the rate at which energy appears as thermal energy in the resistor.

- Energy that does not appear as thermal energy must be stored in the magnetic field. Thus the term $LI\frac{dI}{dt}$ must represent the rate $\frac{dU_m}{dt}$ at which energy is stored in the magnetic field, or

$$\frac{dU_m}{dt} = LI\frac{dI}{dt} \,,$$

we can write this as

$$dU_m = LI\,dI \,.$$

Integrating yields

$$U_m = \int_0^{I_m} LI\,dI = \frac{1}{2}LI_m^2 \,, \qquad (16\text{-}55)$$

which represents the total stored magnetic energy in an inductor of inductance L carrying a current I_m.

Just as the energy stored in a capacitor (see Eq. 14-61) can be considered to reside in the electric field between its plates, so the energy in an inductor can be considered to be stored in its magnetic field.

To write the energy in terms of the magnetic field, we can consider a straight wire carrying a current I. Let us choose a circular flux tube around the wire, see Fig. 16-17, with a cross section dS.

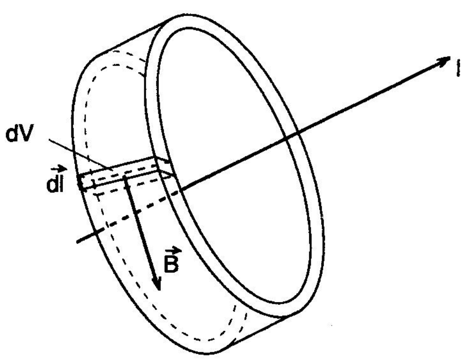

Figure 16 - 17

As a convenient starting point we use Eq. (16-55). Taking into account that $\Phi_B = LI$ we obtain:

$$U_m = \frac{1}{2}I\,\Phi_B \,. \qquad (16\text{-}56)$$

The lines of induction lie in the surface of the flux tube so that the tube encloses a certain amount of flux:

$$d\,\Phi_B = \vec{B}.\vec{dS} \,,$$

where $\vec{dS} = dS\,\vec{n_o}$.

Let us choose an elementary volume dV of the flux tube. Thus we can write the energy of the magnetic field enclosed in this elementary volume (see Eq. 16-56) as

$$dU_m = \frac{1}{2} I \, d\Phi_B . \qquad (16\text{-}57)$$

To express the current we can use Ampère's law, see Eq. (16-26). For vacuum we have $\vec{B} = \mu_0 \vec{H}$ (see 16-72), thus we have:

$$\oint \vec{B}.\vec{d\ell} = \mu_0 I \qquad\qquad \oint \mu_0 \vec{H}.\vec{d\ell} = \mu_0 I$$

or

$$\oint \vec{H}.\vec{d\ell} = I . \qquad (16\text{-}58)$$

Substituting Eq. (16-58) and expression for $d\Phi_B$ into Eq. (16-57) we obtain:

$$dU_m = \frac{1}{2} \oint \vec{H}.\vec{d\ell} \iint \vec{B}.\vec{dS} \qquad \text{where} \qquad \vec{d\ell}.\vec{dS} = dV$$

The energy of magnetic field enclosed in volume V is given by integration:

$$U_m = \frac{1}{2} \iiint_V \vec{H}.\vec{B} \, dV . \qquad (16\text{-}59)$$

We can define the energy density of magnetic field as the energy per unit volume:

$$w_m = \frac{dU_m}{dV} \qquad \text{or} \qquad dU_m = w_m \, dV .$$

From this expression we can obtain the energy of magnetic field enclosed in a volume V as

$$U_m = \iiint_V w_m \, dV .$$

From the comparison of previous relation and Eq. (16-59) we obtain for the energy density at any point in which there is a certain magnetic field:

$$w_m = \frac{1}{2} \vec{H}.\vec{B} \qquad [J.m^{-3}] \qquad (16\text{-}60)$$

16 - 13 Three Magnetic Vectors

In section 14-12 we saw that if a dielectric is placed in an electric field, polarization charges will appear on its surface. These surface charges, which find their origin in the elementary electric dipoles (permanent or induced) that make up the dielectric, set up a field of their own that modifies the original field.

In magnetism we find a similar situation. If magnetic materials are placed in an external magnetic field, the elementary magnetic dipoles (see section 16-4) will act to set up a field of their own that will modify the original field. To describe this situation we find it useful to introduce, except of magnetic induction vector $\vec{B}$, two other magnetic vectors, the magnetization $\vec{M}$ and the magnetic field strength $\vec{H}$.

Consider a torus, see Fig. 16-18a, carrying a current I_0 in its windings and designed so that its core, assumed to be iron, can be removed. The magnetic induction B will be much greater when the core is in place than when it is not, assuming that the current in the windings remains unchanged.

We can understand the large value of B in the iron core in terms of the alignement of the elementary magnetic dipoles in the iron. A hypothetical slice out of the iron core, as in Fig. 16-18b, has a magnetic dipole moment $\vec{d\mu}$ equal to the vector sum of all of the elementary magnetic dipoles contained in it.

We define our first subsidiary vector, <u>the magnetization $\vec{M}$</u>, as the magnetic moment per unit volume of the core material. For the slice of Fig. 16-18b we have

$$\vec{M} = \frac{d\vec{\mu}}{S\,d\ell} ,$$

where $(S.d\ell)$ is the volume of the slice, S being cross-section of the core.

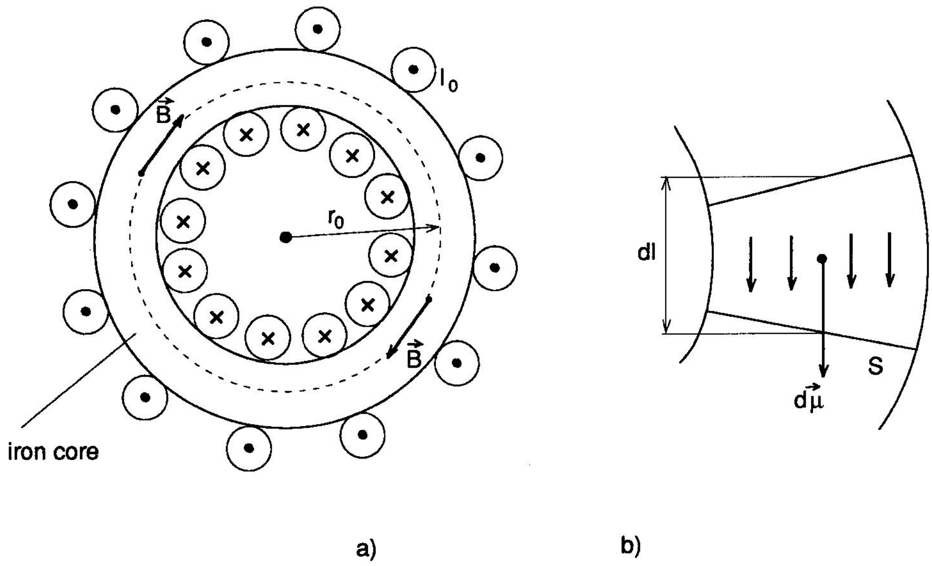

Figure 16 - 18

When we discussed Ampère's law, see Eq.(16-26), we assumed that no magnetic materials were present. If we apply this law to the circular path of integration shown in Fig. 16-18a, we have

$$B.2\pi r_0 = \mu_0 N I_0 \qquad (16\text{-}61)$$

in which r_0 is the mean radius of the core and N is the number of turns. We see at once that Ampère's law expressed by Eq. (16-61) is not valid when magnetic materials are present. This equation predicts that, since the right side is the same whether or not the core is in place, the magnetic induction should also be the same, a prediction not in accord with experiment.

We can increase B in the absence of the iron core to the value that it has when the core is in place if we increase the current in the windings by an amount I_M. The magnetization of the iron core is thus equivalent in its effect on B to such a hypothetical current increase. It is possible to give reality to the magnetizing current by viewing it as a real current that flows arround the magnetic material at its surface, being the resultant macroscopic effect of all the microscopic current loops that constitute the atomic electron orbits.

We choose to modify Ampère's law by arbitrarily inserting a magnetizing current term I_M on the right, obtaining

$$\oint \vec{B}.d\vec{\ell} = \mu_0 (I_0 + I_M) . \qquad (16\text{-}62)$$

If we give I_M a suitable value it is clear that Ampère's law, in this new form, can remain valid. It remains to relate this hypothetical magnetizing current to something more physical, the magnetization vector $\vec{M}$.

Applying Eq. (16-62) to the iron ring of Fig. 16-18a yields

$$B.2\pi r_0 = \mu_0 (N I_0 + N I_{M,0}) . \qquad (16\text{-}63)$$

Magnetic moment of a magnetic dipole in the form of a current loop is given by Eq. (16-9). For the case when the loop has N turns, the loop area is S and the loop current is I than the magnitude of the magnetic dipole moment is:

$$\mu = NIS \qquad (16-64)$$

To relate $I_{M,o}$ to the magnetization $\vec{M}$ let us use Eq. (16-64) to find what increase $I_{M,o}$ in current in the windings around the slice of Fig. 16-18b, would produce a magnetic moment equivalent to that actually produced by the alignment of the elementary dipoles in the slice. We have

$$M(S\,d\ell) = \left(N\frac{d\ell}{2\pi r_o}\right) I_{M,o}\,S ,$$

when the quantity in the first parentheses on the right is the number of turns associated with the slice of thickness d . This equation reduces to

$$N\,I_{M,o} = M.2\pi r_o . \qquad (16-65)$$

Substituting this into Eq. (16-63) yields

$$B.2\pi r_o = \mu_o N I_o + \mu_o M.2\pi r_o . \qquad (16-66)$$

We now choose to generalize from the special case of the torus by writing Eq. (16-66) as

$$\oint \vec{B}.\vec{d\ell} = \mu_o I + \mu_o \oint \vec{M}.\vec{d\ell}$$

or

$$\oint \left(\frac{\vec{B} - \mu_o \vec{M}}{\mu_o}\right)\vec{d\ell} = I . \qquad (16-67)$$

The quantity in parentheses occurs so often in magnetic situations that we give it a special name, the <u>magnetic field strength</u> $\vec{H}$, or

$$\vec{H} = \frac{\vec{B} - \mu_o \vec{M}}{\mu_o}$$

which we write as

$$\vec{B} = \mu_o \vec{H} + \mu_o \vec{M} . \qquad (16-68)$$

Ampère's law can now be written in the simple form

$$\oint \vec{H}.\vec{d\ell} = I \qquad (16-69)$$

which holds in the presence of magnetic materials.

16 - 14 Magnetic Properties of Matter

All materials are magnetic to some extent. The material which has the most striking magnetic properties is iron. Similar magnetic properties are shared also by nickel, cobalt, and - at sufficiently low temperatures (below 16 °C) by gadolinium and some alloys. This kind of magnetism is called a ferromagnetism. However all ordinary substances do show some magnetic effects, although very small ones. This small magnetism is of two kinds. Let us imagine a strong electromagnet which has one sharply pointed pole piece and one flat pole piece, as drawn in Fig. 16-19.

The magnetic field is much stronger near the pointed pole than near the flat pole. If a small piece of material is fastened to a long string and suspended

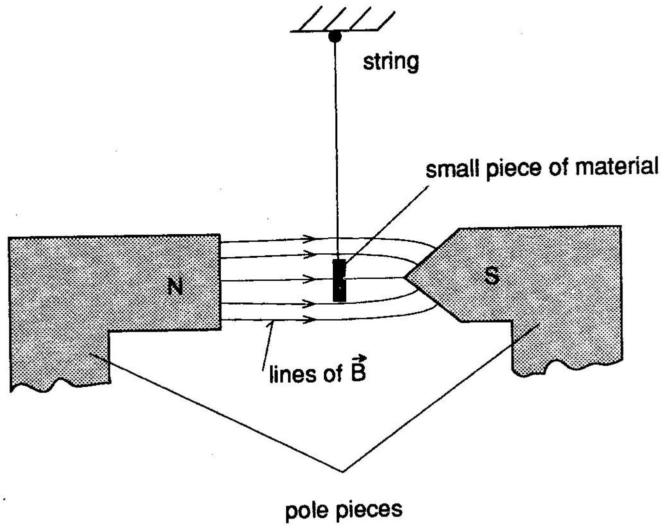

Figure 16 - 19

between the poles, there will be, in general a small force on it. This force can be seen by the slight displacement of the hanging material when the magnet is turned on. The ferromagnetic materials are attracted very strongly toward the pointed pole. All other materials feel only a very week force. Some are weakly attracted to the pointed pole; and some are weakly repelled.

The effect is most easily seen with a small cylinder of bismuth, which is repelled from the high-field region. Substances which are repelled in this way are called diamagnetic. If a small piece of aluminium is suspended between the poles, the aluminium is attracted toward the pointed pole. Substances like aluminium are called paramagnetic.

Let us assume that we have made measurements of $\vec{H}$, $\vec{B}$ and $\vec{M}$ for a wide variety of magnetic materials. For paramagnetic and diamagnetic materials we would find, as an experimental result, that $\vec{B}$ is directly proportional to $\vec{H}$, or

$$\vec{B} = \varkappa \mu_0 \vec{H} \qquad (16\text{-}70)$$

in which $\varkappa$, the permeability of the magnetic medium, is a constant for a given temperature. Eliminating $\vec{B}$ between Eqs. (16-70) and (16-68) allows us to write

$$\vec{M} = (\varkappa - 1)\vec{H} \qquad (16\text{-}71)$$

which is another expression of the linear character of paramagnetic and diamagnetic materials.

For a vacuum, in which there are no magnetic dipoles present, the magnetization $\vec{M}$ must be zero. Putting $\vec{M} = 0$ in Eq. (16-68) we have

$$\vec{B} = \mu_0 \vec{H}. \qquad (16\text{-}72)$$

Comparison with Eq. (16-70) shows that a vacuum must be described by $\varkappa = 1$. Equation (16-71) shows that the magnetization vanishes if we put $\varkappa$ equal to unity.

For paramagnetic materials $\varkappa$ is slightly greater than unity.

For diamagnetic material $\varkappa$ is slightly less than unity. From Eq. (16-71) it is seen that $\vec{M}$ and $\vec{H}$ are oppositely directed.

In ferromagnetic materials the relationship between $\vec{B}$ and $\vec{H}$ is far from linear. It was shown experimentally that $\varkappa$ is a function not only of the value of H but also, because of hysteresis, of the magnetic and thermal history of specimen.

The difference between paramagnetic and diamagnetic materials can be understood theoretically at the molecular level on the basis of whether or not the molecules (or atoms) have a permanent magnetic dipole moment. Paramagnetism occurs in materials whose molecules have a permanent magnetic dipole moment. In the absence of an external field, the molecules are randomly oriented and no magnetic effects are observed. However, when an external magnetic field is applied, then this field exerts a torque on the magnetic dipoles (see section 16-4) tending to align them parallel to the field. The total magnetic field (external plus that due to the aligned magnetic dipoles) will be slightly greater than the applied field. The thermal motion reduces the alignement, however.

Diamagnetic materials are made up of molecules that have no permanent magnetic dipole moment. When an external magnetic field is applied, magnetic dipoles are induced, but the induced magnetic dipole moment is in the direction opposite to that of the field. Hence the total field will be slightly less than the external field. Diamagnetism is present in all materials, but is weaker even than paramagnetism.

A microscopic examination of ferromagnetic materials reveals that these materials are made up of tiny regions known as domains. Each domain behaves like a tiny magnet. In an unmagnetized piece of iron, these domains are arranged randomly. The magnetic effects of the domains cancel each other out, so this piece of iron does not behave like a magnet. In a magnet, the domains are preferentially aligned in one direction. The magnetic field of the domains is caused by the "spin" magnetic moment. The name "spin" comes from an early suggestion that this intrinsic magnetic moment arises from the electron "spinning" on its axis to produce the extra field. In iron and other ferromagnetic materials, a complicated cooperative mechanism, known as "exchange coupling" operates; the result is that the electrons contributing to the ferromagnetism in a domain "spin" in the same direction.

However we have to point out that to understand the magnetic effects of materials quantitavely it is necessary to use quantum-mechanics.

17. MAXWELL´S EQUATIONS

In classical mechanics and thermodynamics we sought to identify the smallest, most compact set of equations or laws that would define the subject as completely as possible. Thus for example in mechanics we found this in Newton´s three laws of motion and in the associated force laws, such as Newton´s law of gravitation. In this section we seek to do the same thing for electromagnetism.

It was the Scottish physicist James Clerk Maxwell who showed that all electric and magnetic phenomena could be described using only four equations involving electric and magnetic field. These equations, known as Maxwell´s equations, are as fundamental as Newton´s laws. We have already arrived by different routes at various pieces of it which we shall now assemble in the traditional form.

Ampère´s law (see Eq. 16-26) which expresses the idea that a magnetic field is produced by any current, or

$$\oint \vec{B}.\vec{d\ell} = \mu_0 I \, . \qquad (17\text{-}1)$$

Faraday´s law of induction (see Eq. 16-40) which states that an electric field is produced by a changing magnetic field, or

$$\oint \vec{E}.\vec{d\ell} = -\frac{d\,\phi_B}{dt} \, . \qquad (17\text{-}2)$$

Gauss´s law of electricity (see Eq. 16-28), which relates the electric field to its sources, electric charges, or

$$\varepsilon_0 \oint \vec{E}.\vec{dS} = \Sigma Q_i . \qquad (17\text{-}3)$$

Gauss's law of magnetism (see Eq. 16-3) which expresses the fact, that the lines of induction are continuous - they do not begin or end (as electric field lines do on charges), or

$$\oint\!\!\oint \vec{B}.\vec{dS} = 0 . \qquad (17\text{-}4)$$

With this set of Maxwell's equations in so called integral form we sometimes write the equations expressing the properties of material, or

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$$

$$\vec{B} = \mu_r \mu_0 \vec{H} \qquad (17\text{-}5)$$

and

$$\vec{j} = \sigma \vec{E} .$$

Maxwell's equations can be also written in another, so called differential,form that is often more convenient than Eqs. (17-1) till (17-4).

To transform Maxwell's equations from integral in differential form we use two theorems known from vector analysis. The first is called Gauss's theorem which relates the integral over a surface of any vector function $\vec{F}$ to a volume integral over the volume enclosed by the surface:

$$\oint\!\!\oint_S \vec{F}.\vec{dS} = \iiint_V \nabla \vec{F}.dV , \qquad (17\text{-}6)$$

where the operator ∇ is the del operator, defined in Cartesian coordinates as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \quad \text{and the quantity} \quad \nabla \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

is called the divergance of $\vec{F}$.

The second theorem is Stokes's theorem which relates a line integral around a closed path to a surface integral over any surface enclosed by that path:

$$\oint \vec{F}.\vec{d\ell} = \iint_S (\nabla \times \vec{F})\, \vec{dS} .$$

The quantity $\nabla \times \vec{F}$ is called curl of $\vec{F}$. We now use these two theorems to obtain the differential form of Maxwell's equations in free space. We apply Stokes's theorem to Eq. (17-1) and write

$$\oint \vec{B}.\vec{d\ell} = \iint_S (\nabla \times \vec{B})\, \vec{dS} = \mu_0 I .$$

The current I can be written in terms of the current density, using Eq. (15-5), so that we obtain

$$\iint_S (\nabla \times \vec{B}).\vec{dS} = \mu_0 \iint_S \vec{j}.\vec{dS} .$$

For this to be true over any area S, whatever its size or shape,the integrands on each side of the equation must be equal:

$$\nabla \times \vec{B} = \mu_0 \vec{j} . \qquad (17\text{-}7)$$

Let us apply Stokes's theorem to the Eq. (17-2), or

$$\oint \vec{E}.\vec{d\ell} = \iint_S (\nabla \times \vec{E})\, \vec{dS} = -\frac{d\Phi_B}{dt} .$$

Since the magnetic flux $\phi_B = \iint \vec{B}.d\vec{S}$, we have

$$\iint_S (\nabla \times \vec{E})\, d\vec{S} = -\frac{\partial}{\partial t}\iint_S \vec{B}.d\vec{S} ,$$

where we use the partial derivative $\frac{\partial \vec{B}}{\partial t}$, since $\vec{B}$ may also depend on position.These are surface integrals over the same area, and to be true over any area, we must have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} . \qquad (17-8)$$

We now apply Gauss's theorem to Eq. (17-3):

$$\oint\!\!\oint_S \vec{E}.d\vec{S} = \iiint_V \nabla\vec{E}\, dV = \frac{\sum Q_i}{\varepsilon_o} .$$

The charge $\sum Q_i$ can be written as a volume integral over the charge density ϱ, or $\sum Q_i = \iiint \varrho\, dV$. Then

$$\iiint_V \nabla\vec{E}\, dV = \frac{1}{\varepsilon_o}\iiint_V \varrho\, dV .$$

Both sides contain volume integrals over the same volume, and for this to be true over any volume, whatever its size or shape, the integrands must be equal:

$$\nabla\vec{E} = \frac{\varrho}{\varepsilon_o} . \qquad (17-9)$$

Finally, the fourth Maxwell's equation, that is Eq. (17-4), is treated in the same way. We obtain

$$\nabla\vec{B} = 0 . \qquad (17-10)$$

Eqs. (17-7), (17-8), (17-9) and (17-10) are Maxwell's equations in differential form.

There is, however, very important thing which must be said in connection with first Maxwell's equation. Maxwell's noticed, that there was something strange about this Eq. (17-7). If one takes divergence of this equation, the left hand side will be zero, because the divergence of a curl is always zero. So this equation requires that the divergence of $\vec{j}$ also be zero. But if the divergence of $\vec{j}$ is zero, then the total flux of current out of any closed surface is also zero. However, as we know from Eq. (15-12), the flux of current from a closed surface is the decrease of the charge inside the surface. This certainly can not in general be zero because we know that the charges can be moved from one place to another.

Maxwell appreciated this difficulty and proposed that Eq. (17-7) should be rewritten in the form

$$\frac{1}{\mu_o}(\nabla \times \vec{B}) = \vec{j} ,$$

and we have to add the term $\varepsilon_o \frac{\partial \vec{E}}{\partial t}$ to its right hand side. He then obtained:

$$\frac{1}{\mu_o}(\nabla \times \vec{B}) = \vec{j} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} . \qquad (17-11)$$

From the analyse of dimensions it is clear that the term $\varepsilon_o \frac{\partial \vec{E}}{\partial t}$ expresses the current density of the so called displacement current, or

$$\vec{j}_D = \varepsilon_o \frac{\partial \vec{E}}{\partial t} \qquad (17-12)$$

or with respect to Eq. (14-81):

$$\vec{j}_D = \frac{\partial \vec{D}}{\partial t} . \qquad (17-13)$$

The total current density can be then expressed as sum of conduction as well as displacement current density. In this respect it should be pointed out that I in Eq. (17-1) expresses total current.

Finally let us summarize the complete set of Maxwell's equations in differential form:

1. $\frac{1}{\mu_0} (\nabla \times \vec{B}) = \vec{j} + \frac{\partial \vec{D}}{\partial t}$

$\frac{1}{\mu_0}$ (Line integral of $\vec{B}$ around a loop) = (conduction plus displacement current density)

2. $\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$

(Line integral of $\vec{E}$ around a loop) = $-\frac{d}{dt}$ (Flux of $\vec{B}$ through the loop)

3. $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

(Flux of $\vec{E}$ through a closed surface) = (charge inside) $\frac{1}{\varepsilon_0}$

4. $\nabla \cdot \vec{B} = 0$

(Flux of $\vec{B}$ through a closed surface equals to zero)

This is the set of Maxwell's equations in differential form which describe all the electric and magnetic phenomena. (Note that the commentaries given below the Maxwell's equations relate to their integral form.)