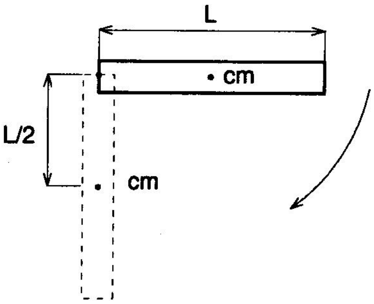

Figure 6 - 19

Solution: We can use the work-energy theorem here; the work done is due to gravity. The work done by gravity is, of course, equal to the change in gravitational potential energy of the rod. Since the cm of the rod drops a vertical distance L/2, the work done by gravity is

$$W = Mg\,\frac{L}{2}\,.$$

The initial KE is zero. Hence, from the work--energy theorem,

$$\frac{1}{2}\,I\omega^2 = Mg\,\frac{L}{2}\,.$$

Since $I = \frac{1}{3}ML^2$ for a rod pivoted about its end, we can solve for ω :

$$\omega = \sqrt{\frac{3g}{L}}\,.$$

The tip of the rod will have a linear speed

$$v = L\omega = \sqrt{3gL}\,.$$

7. EQUILIBRIUM AND ELASTICITY

7 - 1 Center of Gravity

We consider any body as made up of many particles, each of mass m_i. Although gravity acts on each of these particles, we can show that the sum of all these individual gravitational forces has the equivalent effect of a single force which acts at a single point called the center of gravity (cg). This force is equal to Mg, where $M = \Sigma m_i$ is the total mass of the body and g is acceleration due to gravity. If g has the same value at all parts of the body (which is the usual case), the position of the center of gravity is the same as that of the center of mass.

The total force of gravity on a body made up of n particles of masses $m_1, m_2, \ldots, m_n$ is

$$\vec{F} = m_1\vec{g} + m_2\vec{g} + \ldots + m_n\vec{g} = \Sigma\, m_i\vec{g} = M\vec{g}\,. \qquad (7\text{-}1)$$

So, a single force $\vec{F} = M\vec{g}$ will have the same effect on the translational motion of the body as does the sum of all the gravitational forces acting on the particles of the body.

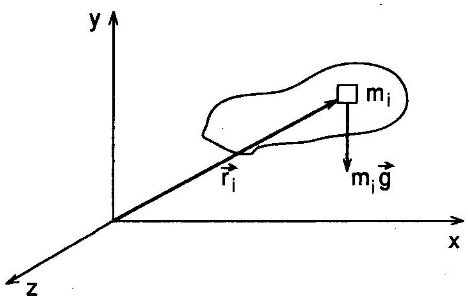

Figure 7 - 1

The position of the force $\vec{F}$ is given by condition so that the rotational motion of the body is the same as does the sum of all the forces of gravity acting on the particles. To determine this, we calculate the sum of all the torques on the body about some arbitrary point O, as shown in Fig. 7-1.

If $\vec{r}_i$ is the position vector of the i-th particle relative to O, then the sum of all the torques due to gravity acting on the particles of the body is

$$\vec{\tau} = \vec{r}_1 \times m_1 \vec{g} + \vec{r}_2 \times m_2 \vec{g} + \dots + \vec{r}_n \times m_n \vec{g} = \left(\sum m_i \vec{r}_i\right) \times \vec{g} \,. \qquad (7\text{-}2)$$

The force $M\vec{g}$ must apply the same torque. If it acts at a point whose position vector is $\vec{r}_{cg}$, then it must hold

$$\vec{\tau} = \vec{r}_{cg} \times M\vec{g} \,. \qquad (7\text{-}3)$$

By comparing Eqs. (7-2) and (7-3), the vector $\vec{r}_{cg}$ is given by

$$\vec{r}_{cg} = \frac{\sum m_i \vec{r}_i}{M} \,. \qquad (7\text{-}4)$$

The equation (7-4) is the definition of center of gravity. But it is the same relation as for center of mass; hence $\vec{r}_{cg} = \vec{r}_{cm}$. Final, we have shown that the force of gravity acting on all the particles of a body has the same translational and rotational effects on the body as a single force $\vec{F} = M\vec{g}$, which acts at the center of mass.

7 - 2 The Conditions for Equilibrium

A body, or system of bodies, on which the total net force is zero is said to be in equilibrium. The study of the forces acting on a body at rest in equilibrium is called statics. There are two conditions for a body to be in equilibrium. For a body to be in equilibrium it must hold:

a) The vector sum of all external forces acting on the body must be zero

$$\sum \vec{F} = 0 \qquad (7\text{-}5)$$

or

$$\sum F_x = 0 \,, \quad \sum F_y = 0 \,, \quad \sum F_z = 0 \,.$$

Under this condition the center of mass of an object does not accelerate. If an object is at rest initially, it will remain at rest.

b) The vector sum of all external torques acting on the body must be also zero

$$\sum \vec{\tau} = 0 \qquad (7\text{-}6)$$

or

$$\sum \tau_x = 0 \,, \quad \sum \tau_y = 0 \,, \quad \sum \tau_z = 0 \,,$$

where τ_x, τ_y and τ_z are the components along any three chosen axes. Under this condition the angular acceleration about any point will be zero. If the body is not rotating initially it will not start rotating.

When all the external forces are acting in a plane such as xy plane, we have only two forces equations

$$\sum F_x = 0 \,, \quad \sum F_y = 0$$

and one torque equation

$$\sum \tau_z = 0$$

and torque is calculated about an axis that is perpendicular to the xy plane.

The next two examples show how to calculate the forces on objects in equilibrium.

E x a m p l e 1 : Calculate the forces F_1 and F_2 in the two cords which are connected to the cord supporting the body of mass of m-kg as shown in Fig. 7-2.

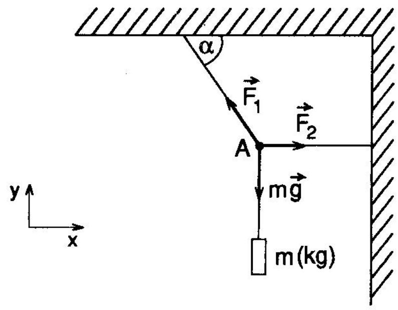

Figure 7 - 2

Solution: For junction point A we may write the force condition (7-5). First, we resolve all the forces:

$$\vec{F}_1 = (-F_1 \cos\alpha, F_1 \sin\alpha),$$
$$\vec{F}_2 = (F_2, 0),$$
$$m\vec{g} = (0, -mg).$$

We now can write two forces equations:

$$\sum F_x = -F_1 \cos\alpha + F_2 = 0 ,$$
$$\sum F_y = F_1 \sin\alpha - mg = 0 .$$

We have received two equations for two unknowns F_1 and F_2. From the second Eq. we have

$$F_1 = \frac{m}{\sin\alpha}\,(kg)g .$$

Substituting into first Eq. gives

$$F_2 = m\,\frac{\cos\alpha}{\sin\alpha}\,(kg)\,g .$$

The magnitudes of F_1 and F_2 determine the strength of cord. In our case, the used cord must be able to hold at least $\frac{m}{\sin\alpha}$ (kg).

E x a m p l e 2 : A uniform m_1-kg beam supports m_2-kg body. Calculate the force on each of the supports. Fig. 7-3.

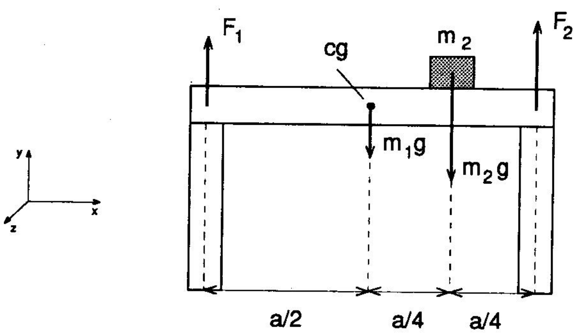

Figure 7 - 3

Solution: The weight of the beam acts at its center of gravity a/2 from either end. The force the beam exerts on the supports is equal and opposite to the forces exerted by the supports on the beam. It does not matter which point we choose as the axis for the torque equation, so we calculate the torques about the point of application of $\vec{F}_1$.

We now can write the torque and force equations:

$$\sum \tau_z = -m_1 g\,\frac{a}{2} - m_2 g\,\frac{3}{4}\,a + F_2 a = 0$$

$$\sum F_y = F_1 - m_1 g - m_2 g + F_2 = 0 .$$

From the first condition we have

$$F_2 = \left(\frac{m_1}{2} + \frac{3}{4}\,m_2\right)g .$$

Substituting into second condition gives

$$F_1 = \left(m_1 + m_2 - \frac{m_1}{2} - \frac{3}{4}\,m_2\right)g = \left(\frac{m_1}{2} + \frac{m_2}{4}\right)g .$$

To confirm that the results don't depend on where the axis is chosen, choose a different axis, say at the other end where F_2 acts and write the torque equation.

7 - 3 Elasticity and Elastic Moduli

In this section we study the case when any object changes shape (is deformated) under the action of applied forces.

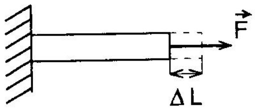

Figure 7 - 4

If a force is exerted on a fixed object such as metal bar shown in Fig. 7-4, the length of the object changes. The bar is said to be under tension or tensile stress.

If the elongation ΔL is small compared to the length of the object, experiments show that ΔL is proportional to the force exerted on the object:

$$L = kF , \qquad (7\text{-}7)$$

where ΔL is the increase in length, F represents the force pulling on the object and k is a proportionality constant. The Eq. (7-7) is found to be valid for almost any solid material.

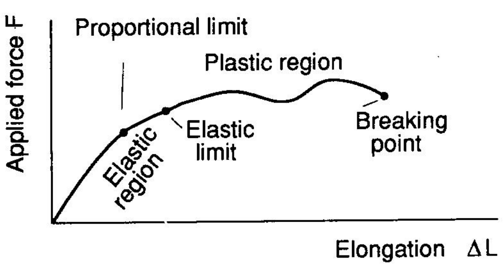

Figure 7- 5

Figure 7-5 shows a typical graph of elongation versus applied force. Up to a point called the proportional limit, the Eq. (7-7) is a good approximation for many materials and the curve is a straight line. Beyond this point the graph deviates from a straight line and no simple relationship exists between F and ΔL. Up to a point along the curve called the elastic limit, the object will return to its original length if the applied force is removed. The region from the origin to the elastic limit is called the elastic region. Beyond the elastic limit the object enters the plastic region; it does not return to the original length if the applied force is removed, but remains permanently deformed. The maximum elongation is reached at the breaking point.

The amount of elongation of an object, such as the bar in Fig. 7-4 depends not only on the applied force, but also on the material from which it is made and on its dimensions. Experimentally it was found that for the same applied force the amount of elongation (assumed small compared to the total length) is proportional to the original length and inversely proportional to the cross-sectional area. That is, the longer the object, the more it elongates for a given force. So, the Eq. (7-7) can be expressed as

$$\Delta L = \frac{1}{E} \frac{F}{A} L_o , \qquad (7\text{-}8)$$

where L_o is the original length of the object, A is the cross-sectional area, ΔL is the change in length due to the applied force F and E is a constant called the elastic modulus or Young's modulus, and its value depends only on the material of the object.

From Eq. (7-8) we see that the change in length of an object is directly proportional to the product of the object's length L_o and the force per unit area called the stress σ

$$\sigma = \frac{F}{A} . \qquad (7\text{-}9)$$

We define the strain ε as the ratio of the change in length to the original length

$$\varepsilon = \frac{\Delta L}{L_o} . \qquad (7\text{-}10)$$

It is the fractional change in length of an object. Now Eq. (7-8) can be rewritten as

$$\frac{F}{A} = E \frac{\Delta L}{L_o} ,$$

or

$$\sigma = E \varepsilon . \qquad (7\text{-}11)$$

We see that the strain is directly proportional to the stress. This dependence is called Hook's law.

Compressive stress is the exact opposite of tensile stress; the material is compressed. Eq. (7-11) apply equally well to compression and tension, and the values for E are also the same.

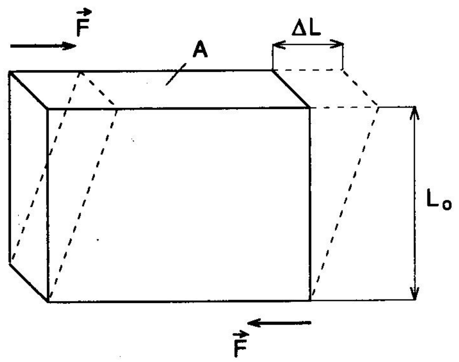

Figure 7 - 6

The third type of the deformation is shear stress. An object under shear stress has equal and opposite forces applied across its opposite faces. The shape of the object does change as shown in Fig. 7-6.

An equation similar to (7-8) can be applied to calculate shear strain:

$$\Delta L = \frac{1}{G} \frac{F}{A} L_o , \qquad (7\text{-}12)$$

where A is the area of the surface parallel to the applied force (not perpendicular as for tension) and ΔL is perpendicular to L_o. The constant of proportionality G is called the shear modulus and is generally one-third to one-half the value of the elastic modulus E.

As the forth type of deformation we assume a body submerged in a fluid. In this case the fluid exerts a pressure on the object in all directions, as we shall see in chapter 8. Pressure is defined as force per unit area and thus is the equivalent of stress. In this case the fractional change in volume dV/V_o of an object is proportional to the increase in the pressure dp:

$$\frac{dV}{V_o} = - \frac{1}{K} dp , \qquad (7\text{-}13)$$

where dV is the change of the volume, V_o is the original volume, dp is the increase in the pressure and K is proportionality constant called the bulk modulus. Since liquids and gases do not have a fixed shape, only the bulk modulus applies to them, and not the shear or Young's modulus. The minus sign in Eq. (7-13) indicates that the volume decreases with an increase in pressure.

8. FLUIDS AT REST

It is known that a liquid cannot keep a fixed shape. It takes on the shape of its container, but like a solid it is not readily compressible and its volume can