Our curve represents $U(x)$ at each x. So, the kinetic energy for any x is represented by the distance between the E and the magnitude of $U(x)$. In our graph the KE of a particle at x_1 when its total energy is E_1 is indicated by "KE". It is clear that a particle with total energy E_1 can oscillate only between the points x_2 and x_3, since for $x > x_2$ or $x < x_3$ the potential energy would be greater than E_1. At x_2 and x_3 the total energy E_1 equals the potential energy, thus the kinetic energy is zero and the velocity of a particle is also zero. These points are called the <u>turning points</u> of the motion.

If the particle has energy $E = E_2$, there are four turning points. The particle can now move in only one of the two potential "valleys", depending on where it is initially. It cannot get from one valley to the other - for example at x_4 $U(x) > E_2$. For energy $E = E_3$, there is only one turning point at $x = x_5$ since $U(x) < E_3$ for all $x > x_5$.

We know the force F is related to the potential energy U by

$$F = -\frac{dU}{dx},$$

that is, the force is equal to the negative of the slope of the U - versus - x curve at any point.

At $x = x_0$ the slope is zero, so $F = 0$. At such a point the particle is said to be <u>in equilibrium</u>, since the net force on the particle is zero. So, its acceleration is zero. When the particle displaced slightly from $x = x_0$ it returns back its equilibrium point and the particle is said to be <u>in stable equilibrium.</u> <u>Any minimum in the potential energy curve represents a point of stable equilibrium.</u>

At point $x = x_4$ it is also $F = -dU/dx = 0$ a particle would also be in equilibrium. But the particle will not return to equilibrium if displaced slightly, it moves away from the equilibrium point. Points like x_4, where the potential energy curve has a maximum, are points of <u>unstable equilibrium.</u>

When a particle is in region of constant potential energy, such as $x = x_6$ in Fig. 4-8, the force is zero for any x of this region. The particle is said to be <u>in neutral equilibrium.</u>

5. MANY BODIES MECHANICS

Up to now we have been mainly concerned with the motion of a single particle. When we have dealt with an object, that is, a body that has size, we have assumed that it underwent only translational motion, we have assumed that our body approximated an ideal particle. Real bodies, however, can undergo rotational motion as well. A basic idea for study of such bodies is that of center of mass. Later in this chapter we discuss linear momentum and its conservation.

5 - 1 Center of Mass

General motion of a real body (or system of bodies) can be considered as the sum of the translational motion of its center of mass (cm) plus rotational motion about its center of mass.

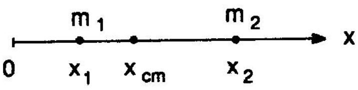

Figure 5 - 1

We can consider any body as being made up of many tiny particles. But let us first consider a system made up of only two particles of mass m_1 and m_2. Let both particle lie on the x axis at positions x_1 and x_2 (Fig. 5-1).

The center of mass of this system is defined to be at the position x_{cm}, given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1x_1 + m_2x_2}{M} ,$$

where $M = m_1 + m_2$ is the total mass of the system. The center of mass lies on the line joining m_1 and m_2 and is closer to the large mass.

It is clear, if the two masses are equal, x_{cm} is midway between them.

For system consisting of n particles of masses $m_1, m_2, \ldots, m_n$ at positions $x_1, x_2, \ldots, x_n$ on the x axis we define center of mass as follows

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{1}{M}\sum_{i=1}^{n} m_ix_i ,$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system.

If the particles are spread out in space, we define the coordinates of the center of mass as

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_ix_i , \qquad y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_iy_i , \qquad z_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_iz_i , \qquad (5\text{-}1)$$

where x_i, y_i, z_i are the coordinates of the particle of mass m_i and $M = \sum_{i=1}^{n} m_i$ is the total mass.

Eq. 5-1 can be written in vector form. If $\vec{r}_i = x_i\,\vec{i} + y_i\,\vec{j} + z_i\,\vec{k}$ is the position vector of the i-th particle of the mass of m_i, and $\vec{r}_{cm} = x_{cm}\,\vec{i} + y_{cm}\,\vec{j} + z_{cm}\,\vec{k}$ will be the position vector of the center of mass, then

$$\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\,\vec{r}_i . \qquad (5\text{-}2)$$

A real body is assumed being made up of a continues distribution of matter. In this case we take the infinitesimal element of mass dm at point x, y, z (see Fig. 5-2) and the sums in Eqs. 5-1 and 5-2 become integrals:

$$x_{cm} = \frac{1}{M}\int x\,dm , \qquad y_{cm} = \frac{1}{M}\int y\,dm , \qquad z_{cm} = \frac{1}{M}\int z\,dm \qquad (5\text{-}3)$$

or

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm , \qquad (5\text{-}4)$$

where $M = \int dm$ is total mass of the object.

For symmetrical uniform body the center of mass must lie on a line of symmetry.

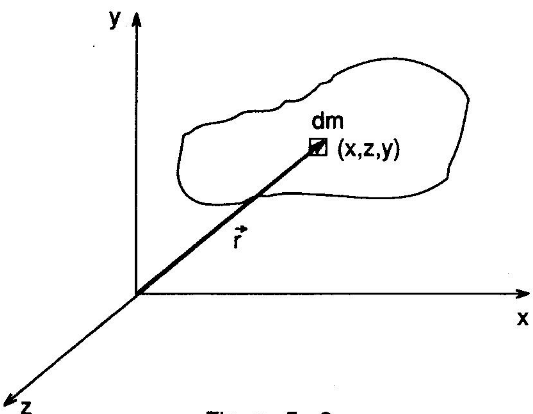

Figure 5 - 2

Example 1: Three particles, each of mass m are located at the corners of a right triangle whose sides are ℓ_1, and ℓ_2.

Determine the center of mass.

Solution:

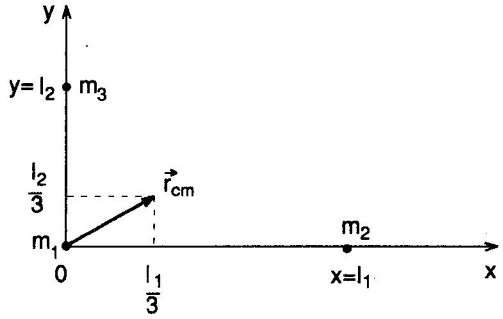

Figure 5 - 3

m_1 has coordinates $x_1 = y_1 = 0$, m_2 has coordinates $x_2 = \ell_1$, $y_2 = 0$, and m_3 has coordinates $x_3 = 0$, $y_2 = \ell_2$; $m_1 = m_2 = m_3 = m$, so $M = 3\,m$. Then,

$$x_{cm} = \frac{1}{3\,m}(m_1x_1 + m_2x_2 + m_3x_3) = \frac{1}{3\,m}\,m\,\ell_1 = \frac{\ell_1}{3}$$

$$y_{cm} = \frac{1}{3\,m}(m_1y_1 + m_2y_2 + m_3y_3) = \frac{1}{3\,m}\,m\,\ell_2 = \frac{\ell_2}{3}$$

Example 2: Determine the center of mass of a uniform cone of height h and radius R.

Solution: We choose the coordinate system so that the origin is at the tip of the cone and the z axis is along the line of symmetry (as in Fig. 5-3).

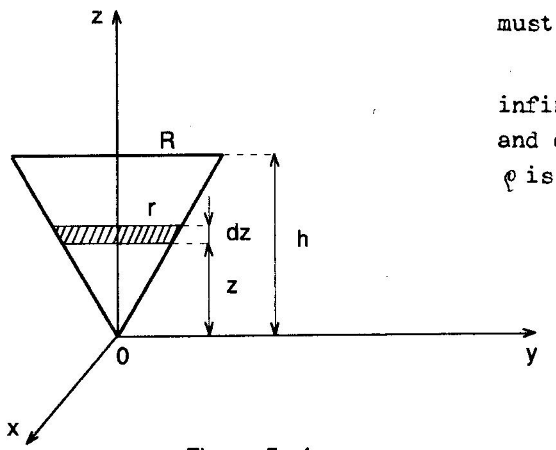

Figure 5 - 4

Then $x_{cm} = y_{cm} = 0$, since the center of mass must lie on the line of symmetry.

To find z_{cm} we divide the cone into an infinite number of cylinders of thickness dz and of the mass $dm = \rho\,dV = \rho\,\pi r^2\,dz$, where ρ is the density (mass per unit volume). Volume of such infinitesimal cylinder is $dV = \pi r^2\,dz$.

By Eq. 5-3 we write

$$z_{cm} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int_0^h z\,\rho\,\pi r^2\,dz\,.$$

The radius of each infinitesimal cylinder can be expressed from the ratio $r/z = R/h$, so $r = Rz/h$.

Then we have

$$z_{cm} = \frac{1}{M}\int_0^h z\,\rho\,\pi\,\frac{R^2z^2}{h^2}\,dz = \frac{\rho\,\pi\,R^2}{Mh^2}\int_0^h z^3\,dz = \frac{\rho\,\pi\,R^2h^2}{4M}\,.$$

The total mass M of the cone is equal to the density ρ times the total volume of the cone $V = \pi R^2h/3$, that is, $M = \rho\,\pi R^2h/3$. Thus we have

$$z_{cm} = \frac{\rho \pi R^2 h^2}{4} \; \frac{3}{\rho \pi R^2 h} = \frac{3}{4} h .$$

So, the center of mass of the cone is $\frac{3}{4}$ h from the tip of the cone, or $\frac{1}{4}$ h from its base.

5 - 2 Center of Mass and Translational Motion

We examine the motion of a system of n particles of total mass M. From Eq. 5-2 we have

$$M \vec{r}_{cm} = \sum_{i=1}^{n} m_i \vec{r}_i .$$

We differentiate this equation with respect to time:

$$M \frac{d \vec{r}_{cm}}{dt} = \sum_{i=1}^{n} m_i \frac{d \vec{r}_i}{dt}$$

or

$$M \vec{v}_{cm} = \sum_{i=1}^{n} m_i \vec{v}_i , \qquad (5\text{-}5)$$

where $\vec{v}_i$ is the velocity of the i-th particle of mass m_i and $\vec{v}_{cm}$ is the velocity of the cm.

We take the derivative with respect to time again:

$$M \vec{a}_{cm} = \sum_{i=1}^{n} m_i \vec{a}_i , \qquad (5\text{-}6)$$

where $\vec{a}_i$ is the acceleration of the i-th particle and $\vec{a}_{cm}$ is the acceleration of the cm.

By the second law of motion we may write $m_i \vec{a}_i = \vec{F}_i$, where $\vec{F}_i$ is the net force on the i-th particle.

Eq. 5-7 can be now rewritten into form

$$M \vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i . \qquad (5\text{-}7)$$

This equation says that the vector sum of all the forces acting on the system is equal to the total mass of the system times the acceleration of its center of mass. (Here, the system may be a system of n particles or a real body.)

By Eq. 5-6 we may conclude that

> the center of mass of a system of particles (or of real body of total mass M) moves like a single particle of mass M which is acted on by the same net force.

5 - 3 Linear Momentum

Linear momentum $\vec{p}$ of a particle is defined as the product of its mass m and its vector velocity $\vec{v}$:

$$\vec{p} = m \vec{v} . \qquad (5\text{-}8)$$

It is clear that linear momentum is a vector. Its direction is the direction of the velocity vector $\vec{v}$ and its magnitude is $p = mv$.

The unit of linear momentum in SI units is kg m/s or N s.

If we assume the mass m is constant, we can write the second law of motion as follows

$$\vec{F} = m\,\vec{a} = m\,\frac{d\vec{v}}{dt} = \frac{d}{dt}(m\,\vec{v}) = \frac{d\vec{p}}{dt},$$

and thus

$$\vec{F} = \frac{d\vec{p}}{dt}. \qquad (5\text{-}9)$$

Eq. 5-9 is more general than $\vec{F} = m\,\vec{a}$ because it includes situation in which the mass may change.

Eq. 5-9 applies to a single particle. Let us now consider a system of n particles of total mass $M = m_1 + m_2 + \dots + m_n$. Let us assume the particles have linear momentum $\vec{p}_1 = m_1\,\vec{v}_1$, $\vec{p}_2 = m_2\,\vec{v}_2$, ..., $\vec{p}_n = m_n\,\vec{v}_n$, where $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ are the velocities of the particles.

The <u>total linear momentum</u> $\vec{P}$ of our system is

$$\vec{P} = m_1\,\vec{v}_1 + m_2\,\vec{v}_2 + \dots + m_n\,\vec{v}_n = \sum_{i=1}^{n} m_i\,\vec{v}_i = \sum_{i-1}^{n} \vec{p}_i. \qquad (5\text{-}10)$$

Comparing Eq. 5-10 with Eq. 5-6 leads to

$$\vec{P} = M\,\vec{v}_{cm}. \qquad (5\text{-}11)$$

So, we may say:

> the total linear momentum of a system of particles is equal to the product of the total mass M and the velocity of the center of mass of the system. Or, the linear momentum of a body is the product of the body's mass and the velocity of its center of mass.

If we differentiate Eq. 5-11 with respect to time, we obtain (assuming the total mass M is constant)

$$\frac{d\vec{P}}{dt} = M\,\frac{d\,\vec{v}_{cm}}{dt} = M\,\vec{a}_{cm}, \qquad (5\text{-}12)$$

and by Eq. (5-7) we have

$$\frac{d\vec{P}}{dt} = \vec{F}, \qquad (5\text{-}13)$$

where $\vec{F}$ is the net external force acting on the system. Eq. 5-12 or 5-13 is the second law of motion for a system of particles.

If the net external force on a system of particles is zero, then from Eq. 5-13 we have

$$\frac{d\vec{P}}{dt} = 0, \quad \text{or} \quad \vec{P} = \text{constant}. \qquad (5\text{-}14)$$

And we may say:

> When the net external force on a system is zero, the total linear momentum remains constant.

This is the <u>law of conservation of linear momentum.</u> A system on which no external force acts is called <u>isolated system.</u> So, the law of conservation of linear momentum can also be stated as,

<u>the total linear momentum of an isolated system of bodies remains constant.</u>

5 - 4 Collisions and Impulse

Collisions are met in everyday life: a tennis racket striking a tennis ball, two billiard balls colliding, car striking another, a hammer hitting a nail. At the atomic level we study collisions between atoms.

By a collision we mean the interaction between two bodies that occurs over a short time interval and is so strong that other forces acting (such as force of gravity or air resistance) are insignificant compared to the forces each body exerts on the other during the collision. We shall assume the masses of particles remain constant and none of the speeds is close to the speed of light, so we ignore relativistic effects. A graph of the magnitude of the force one object exerts on the other during a collision, as a function of time, is like that shown in Fig. 5-5. The time interval $\Delta t = t_2 - t_1$ is usually very small (t_1 is time when the force starts acting, t_2 is time when the force stops acting). The force as shown in this figure is called the impulsive force.

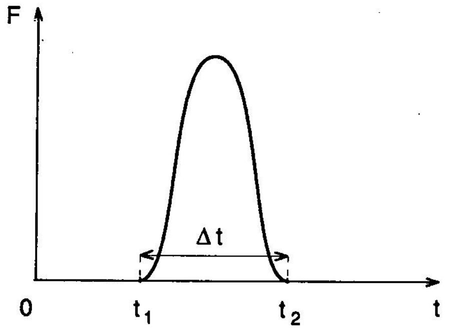

Figure 5 - 5

We write the second law of motion:

$$\frac{d\vec{p}}{dt} = \vec{F},$$

the rate of change of linear momentum of an object is equal to the net force on an object.

During the infinitesimal time interval dt, the linear momentum changes by

$$d\vec{p} = \vec{F}\,dt.$$

By integration over the duration of a collision

$$\vec{p}_2 - \vec{p}_1 = \int_{p_1}^{p_2} d\vec{p} = \int_{t_1}^{t_2} \vec{F}\,dt \,, \qquad (5\text{-}15)$$

where $\vec{p}_1$ and $\vec{p}_2$ are the momenta of the object before and after the collision.

The integral of the force over the time interval during which it acts is called the impulse $\vec{J}$:

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}\,dt \,. \qquad (5\text{-}16)$$

From Eq. 5-15 we conclude that the change in linear momentum of an object $\Delta\vec{p} = \vec{p}_2 - \vec{p}_1$ is equal to the impulse acting on it:

$$\Delta\vec{p} = \vec{p}_2 - \vec{p}_1 = \int_{t_1}^{t_2} \vec{F}\,dt = \vec{J} \,. \qquad (5\text{-}17)$$

The unit for impulse is the same as for linear momentum, kg m/s or N s in SI.

5 - 5 Conservation of Momentum and Energy in Collision

Let us now consider two bodies of mass m_1 and m_2 that have momenta $\vec{p}_1$ and $\vec{p}_2$ before they collide and $\vec{p}_1'$ and $\vec{p}_2'$ after they collide (as in Fig. 5-6).

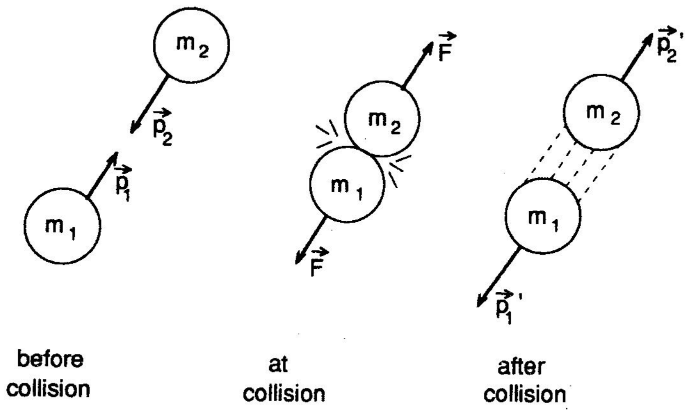

Figure 5-6

During the collision we suppose the force exerted by body m_1 on body m_2 is $\vec{F}$. By the third law of motion, the force exerted by body m_2 on body m_1 is $-\vec{F}$. During the very short collision time, the impulsive force $\vec{F}$ is assumed to be much greater than any other external forces acting, and so $\vec{F}$ represents the net force to a very good approximation.

So, the change in momentum of body m_2 is (see Eq. 5-16)

$$\Delta\vec{p}_2 = \vec{p}_2' - \vec{p}_1 = \int_{t_1}^{t_2} \vec{F}\, dt$$

and for body m_1

$$\Delta\vec{p}_1 = \vec{p}_1' - \vec{p}_1 = -\int_{t_1}^{t_2} \vec{F}\, dt .$$

By comparing we have

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2$$

or

$$\vec{p}_1' - \vec{p}_1 = -(\vec{p}_2' - \vec{p}_2)$$

and thus

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' . \qquad (5\text{-}18)$$

We have proved the total linear momentum of our two objects before the collision is the same as the total linear momentum after the collision. Linear momentum is conserved. To apply the law of conservation of linear momentum we must be sure that the impulsive forces are much greater than the other external forces.

By the law of conservation of energy the total energy of objects will also be conserved during a collision. Because energy can take many forms, this law has not always to take the practical use. There are, however, collisions in which the total kinetic energy of the two objects is conserved. Such collisions are called elastic collisions and for the collision of this type we can write

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 , \qquad (5\text{-}19)$$

where v_1 and v_1 are velocities of the objects before the collision, v_1' and v_2' are velocities after the collision.

Eq. 5-19 tells us that the total KE before the collision is the same as the total KE after the collision.

At the atomic level, the collisions of atoms or elementary particles may be very often considered as an elastic collision. But in the macroscopis world an elastic collision is an ideal case, because at least a little thermal or sound and other form energy is always produced during a collision.

5 - 6 Elastic Collisions in One Dimension

We consider that two particles of mass m_1 and m_2 are moving with velocities v_1 and v_2 along the x axis as in Fig. 5-7. When the velocity any particle is positive, it is moving the right (increasing x), when the velocity any particle is negative, it is moving the left (decreasing x). After the collision their velocities are v_1' and v_2'.

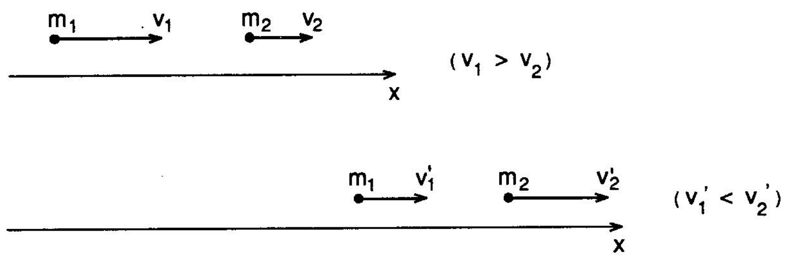

Figure 5 -7

By the law of conservation of linear momentum we have

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' . \qquad (5\text{-}20)$$

If the collision is assumed to be elastic, total kinetic energy is also conserved:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 . \qquad (5\text{-}21)$$

We have obtained two equations for two unknowns velocities v_1' and v_2' after the collision. We can solve them if we are given the masses m_1, m_2 and initial velocities v_1, v_2.

First, we rearrange both the momentum equation and the KE equation into form:

$$m_1(v_1 - v_1') = m_2(v_2' - v_2)$$
$$m_1(v_1^2 - v_1'^2) = m_2(v_2'^2 - v_2^2)$$

or

$$m_1(v_1 - v_1') = m_2(v_2' - v_2) \qquad (5\text{-}22a)$$

$$m_1(v_1 - v_1')(v_1 + v_1') = m_2(v_2' - v_2)(v_2' + v_2) . \qquad (5\text{-}22b)$$

Dividing Eq. 5-22b by Eq. 5-22a (assuming $v_1 \neq v_1'$ and $v_2 \neq v_2'$) we receive

$$v_1 + v_1' = v_2' + v_2$$

or

$$v_1 - v_2 = v_2' - v_1' . \qquad (5\text{-}23)$$

The result in Eq. 5-23 tells us that for any head-on collision the relative speed of the two particles after the collision is the same as before, no matter what the masses are.

Now we examine some special cases. We assume v_1, v_2, m_1 and m_2 are known and we want to determine velocities v_1' and v_2' after the collision.

1. Equal masses, $m_1 = m_2$:

From momentum equation (5-21) we have

$$v_1 + v_2 = v_1' + v_2' . \qquad (5\text{-}24)$$

If we add and subtract Eqs. 5-24 and 5-23:

$$v_2' = v_1 ,$$
$$v_1' = v_2 .$$

The particles exchange **velocities** as a result of the collision.

If particle m_2 is at rest before the collision, so that $v_2 = 0$, we have

$$v_1' = 0 \quad \text{and} \quad v_2' = v_1 ,$$

that is, particle m_1 is stopped after the collision and particle m_2 starts to move with a velocity equal to a velocity of particle m_1 before the collision.

2. $m_1 \neq m_2$, $v_2 = 0$; that is, the particle m_2 at rest initially:

Combining the momentum equation (5-22a) with Eq. 5-23 which now are

$$m_1(v_1 - v_1') = m_2 v_2'$$
$$v_1 = v_2' - v_1'$$

we obtain

$$v_2' = v_1 \frac{2m_1}{m_1 + m_2} ,$$
$$v_1' = v_1 \frac{m_1 - m_2}{m_1 + m_2} . \qquad (5\text{-}25)$$

Let us examine some special cases of this result:

a) $v_2 = 0$ and $m_1 = m_2$: from Eqs. 5-25 we have

$$v_2' = v_1 \quad \text{and} \quad v_1' = 0 .$$

We get the same case and thus the same result as above.

b) $v_2 = 0$ and $m_1 \gg m_2$: a very heavy moving object strikes a light object at rest. From Eqs. 5-25 we receive the result:

$$v_2' \approx 2v_1 ,$$
$$v_1' \approx v_1 .$$

The velocity of the heavy particle is practically unchanged, the light particle, originally at rest, takes twice the velocity of the heavy particle.

c) $v_2 = 0$ and $m_1 \ll m_2$: a moving light particle strikes a very heavy particle at rest.

From Eqs. 5-25 we obtain the result:

$$v_2' \approx 0 ,$$
$$v_1' \approx - v_1 .$$

The heavy particle remains at rest and the very light particle rebounds with its same speed but in the opposite direction.