DIFFERENTIAL EQUATIONS

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Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I've tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Outline

Here is a listing and brief description of the material in this set of notes.

Series Solutions – This section serves the same purpose as the Laplace Transform section. It is just here so we can say we've worked an example using series solutions for a differential equations of order higher than 2nd.

Boundary Value Problems & Fourier Series

Boundary Value Problems – In this section we'll define the boundary value problems as well as work some basic examples.
Eigenvalues and Eigenfunctions – Here we'll take a look at the eigenvalues and eigenfunctions for boundary value problems.
Periodic Functions and Orthogonal Functions – We'll take a look at periodic functions and orthogonal functions in section.
Fourier Sine Series – In this section we'll start looking at Fourier Series by looking at a special case : Fourier Sine Series.
Fourier Cosine Series – We'll continue looking at Fourier Series by taking a look at another special case : Fourier Cosine Series.
Fourier Series – Here we will look at the full Fourier series.
Convergence of Fourier Series – Here we'll take a look at some ideas involved in the just what functions the Fourier series converge to as well as differentiation and integration of a Fourier series.

Partial Differential Equations

The Heat Equation – We do a partial derivation of the heat equation in this section as well as a discussion of possible boundary values.
The Wave Equation – Here we do a partial derivation of the wave equation.
Terminology – In this section we take a quick look at some of the terminology used in the method of separation of variables.
Separation of Variables – We take a look at the first step in the method of separation of variables in this section. This first step is really the step that motivates the whole process.
Solving the Heat Equation – In this section we go through the complete separation of variables process and along the way solve the heat equation with three different sets of boundary conditions.
Heat Equation with Non-Zero Temperature Boundaries – Here we take a quick look at solving the heat equation in which the boundary conditions are fixed, non-zero temperature conditions.
Laplace's Equation – We discuss solving Laplace's equation on both a rectangle and a disk in this section.
Vibrating String – Here we solve the wave equation for a vibrating string.
Summary of Separation of Variables – In this final section we give a quick summary of the method of separation of variables.

Basic Concepts

Introduction

There isn't really a whole lot to this chapter it is mainly here so we can get some basic definitions and concepts out of the way. Most of the definitions and concepts introduced here can be introduced without any real knowledge of how to solve differential equations. Most of them are terms that we'll use throughout a class so getting them out of the way right at the beginning is a good idea.

During an actual class I tend to hold off on a couple of the definitions and introduce them at a later point when we actually start solving differential equations. The reason for this is mostly a time issue. In this class time is usually at a premium and some of the definitions/concepts require a differential equation and/or its solution so I use the first couple differential equations that we will solve to introduce the definition or concept.

Here is a quick list of the topics in this Chapter.

Definitions – Some of the common definitions and concepts in a differential equations course

Direction Fields – An introduction to direction fields and what they can tell us about the solution to a differential equation.

Final Thoughts – A couple of final thoughts on what we will be looking at throughout this course.

Definitions

Differential Equation
The first definition that we should cover should be that of **differential equation**. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

There is one differential equation that everybody probably knows, that is Newton's Second Law of Motion. If an object of mass *m* is moving with acceleration *a* and being acted on with force *F* then Newton's Second Law tells us.

$$F = ma \tag{1}$$

To see that this is in fact a differential equation we need to rewrite it a little. First, remember that we can rewrite the acceleration, *a*, in one of two ways.

$$a = \frac{dv}{dt} \qquad \text{OR} \qquad a = \frac{d^2u}{dt^2} \tag{2}$$

Where *v* is the velocity of the object and *u* is the position function of the object at any time *t*. We should also remember at this point that the force, *F* may also be a function of time, velocity, and/or position.

So, with all these things in mind Newton's Second Law can now be written as a differential equation in terms of either the velocity, *v*, or the position, *u*, of the object as follows.

$$m\frac{dv}{dt} = F\left(t,v\right) \tag{3}$$

$$m\frac{d^2u}{dt^2} = F\left(t,u,\frac{du}{dt}\right) \tag{4}$$

So, here is our first differential equation. We will see both forms of this in later chapters.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g\left(t\right) \tag{5}$$

$$\sin\left(y\right)\frac{d^2y}{dx^2} = \left(1-y\right)\frac{dy}{dx} + y^2\mathbf{e}^{-5y} \tag{6}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos\left(t\right) \tag{7}$$

$$\alpha^2\frac{\partial^2u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{8}$$

$$a^2u_{xx} = u_{tt} \tag{9}$$

$$\frac{\partial^3u}{\partial^2x\partial t} = 1 + \frac{\partial u}{\partial y} \tag{10}$$

Order
The **order** of a differential equation is the largest derivative present in the differential equation. In the differential equations listed above (3) is a first order differential equation, (4), (5), (6), (8),

and (9) are second order differential equations, (10) is a third order differential equation and (7) is a fourth order differential equation.

Note that the order does not depend on whether or not you've got ordinary or partial derivatives in the differential equation.

We will be looking almost exclusively at first and second order differential equations in these notes. As you will see most of the solution techniques for second order differential equations can be easily (and naturally) extended to higher order differential equations and we'll discuss that idea later on.

Ordinary and Partial Differential Equations

A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it. Likewise, a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has differential derivatives in it. In the differential equations above (3) - (7) are ode's and (8) - (10) are pde's.

The vast majority of these notes will deal with ode's. The only exception to this will be the last chapter in which we'll take a brief look at a common and basic solution technique for solving pde's.

Linear Differential Equations

A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n\left(t\right)y^{(n)}\left(t\right)+a_{n-1}\left(t\right)y^{(n-1)}\left(t\right)+\cdots+a_1\left(t\right)y'\left(t\right)+a_0\left(t\right)y\left(t\right)=g\left(t\right) \qquad \textbf{(11)}$$

The important thing to note about linear differential equations is that there are no products of the function, $y\left(t\right)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power.

The coefficients $a_0\left(t\right), \ldots, a_n\left(t\right)$ and $g\left(t\right)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y\left(t\right)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

In (5) - (7) above only (6) is non-linear, the other two are linear differential equations. We can't classify (3) and (4) since we do not know what form the function *F* has. These could be either linear or non-linear depending on *F*.

Solution

A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function $y\left(t\right)$ which satisfies the differential equation in question on the interval $\alpha < t < \beta$. It is important to note that solutions are often accompanied by intervals and these intervals can impart some important information about the solution. Consider the following example.

Example 1 Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2 y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2} x^{-\frac{5}{2}} \qquad\qquad y''(x) = \frac{15}{4} x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$\begin{aligned} 4x^2 \left(\frac{15}{4} x^{-\frac{7}{2}} \right) + 12x \left(-\frac{3}{2} x^{-\frac{5}{2}} \right) + 3 \left(x^{-\frac{3}{2}} \right) &= 0 \\ 15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} &= 0 \\ 0 &= 0 \end{aligned}$$

So, $y(x) = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution. Why then did I include the condition that $x > 0$? I did not use this condition anywhere in the work showing that the function would satisfy the differential equation.

To see why recall that

$$y(x) = x^{-\frac{3}{2}} = \frac{1}{\sqrt{x^3}}$$

In this form it is clear that we'll need to avoid $x = 0$ at the least as this would give division by zero.

Also, there is a general rule of thumb that we're going to run with in this class. This rule of thumb is : Start with real numbers, end with real numbers. In other words, if our differential equation only contains real numbers then we don't want solutions that give complex numbers. So, in order to avoid complex numbers we will also need to avoid negative values of *x*.

So, we saw in the last example that even though a function may symbolically satisfy a differential equation, because of certain restrictions brought about by the solution we cannot use all values of the independent variable and hence, must make a restriction on the independent variable. This will be the case with many solutions to differential equations.

In the last example, note that there are in fact many more possible solutions to the differential equation given. For instance all of the following are also solutions

$$\begin{aligned} y(x) &= x^{-\frac{1}{2}} \\ y(x) &= -9x^{-\frac{3}{2}} \\ y(x) &= 7x^{-\frac{1}{2}} \\ y(x) &= -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}} \end{aligned}$$

I'll leave the details to you to check that these are in fact solutions. Given these examples can you come up with any other solutions to the differential equation? There are in fact an infinite number of solutions to this differential equation.

So, given that there are an infinite number of solutions to the differential equation in the last example (provided you believe me when I say that anyway….) we can ask a natural question. Which is the solution that we want or does it matter which solution we use? This question leads us to the next definition in this section.

Initial Condition(s)
Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s when I'm feeling lazy…) are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to "nice enough" differential equations are unique and hence only one solution will meet the given conditions.

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Example 2 $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$, $y(4) = \frac{1}{8}$, and $y'(4) = -\frac{3}{64}$.

Solution As we saw in previous example the function is a solution and we can then note that

$$y(4) = 4^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{4}\right)^3} = \frac{1}{8}$$

$$y'(4) = -\frac{3}{2}4^{-\frac{5}{2}} = -\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5} = -\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4) = \frac{1}{8}$ and $y'(4) = -\frac{3}{64}$. In fact, $y(x) = x^{-\frac{3}{2}}$ is the only solution to this differential equation that satisfies these two initial conditions.

Initial Value Problem
An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example 3 The following is an IVP.

$$4x^2y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8}, \quad y'(4) = -\frac{3}{64}$$

Example 4 Here's another IVP.

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

As I noted earlier the number of initial conditions required will depend on the order of the differential equation.

Interval of Validity

The **interval of validity** for an IVP with initial condition(s)

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

is the largest possible interval on which the solution is valid and contains t_0. These are easy to define, but can be difficult to find, so I'm going to put off saying anything more about these until we get into actually solving differential equations and need the interval of validity.

General Solution

The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account.

Example 5 $y(t) = \frac{3}{4} + \frac{c}{t^2}$ is the general solution to

$$2t\,y' + 4y = 3$$

I'll leave it to you to check that this function is in fact a solution to the given differential equation. In fact, all solutions to this differential equation will be in this form. This is one of the first differential equations that you will learn how to solve and you will be able to verify this shortly for yourself.

Actual Solution

The **actual solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the actual solution to the following IVP?

$$2t\,y' + 4y = 3 \qquad y(1) = -4$$

Solution This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe my solution to this example…) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of *c* that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

From this last example we can see that once we have the general solution to a differential equation finding the actual solution is nothing more than applying the initial condition(s) and solving for the constant(s) that are in the general solution.

Implicit/Explicit Solution
In this case it's easier to define an explicit solution, then tell you what an implicit solution isn't, and then give you an example to show you the difference. So, that's what I'll do.

An **explicit solution** is any solution that is given in the form $y = y(t)$. In other words, the only place that *y* actually shows up is once on the left side and only raised to the first power. An **implicit solution** is any solution that isn't in explicit form. Note that it is possible to have either general implicit/explicit solutions and actual implicit/explicit solutions.

Example 7 $y^2 = t^2 - 3$ is the actual implicit solution to $y' = \dfrac{t}{y}, \quad y(2) = -1$

At this point I will ask that you trust me that this is in fact a solution to the differential equation. You will learn how to get this solution in a later section. The point of this example is that since there is a y^2 on the left side instead of a single $y(t)$ this is not an explicit solution!

Example 8 Find an actual explicit solution to $y' = \dfrac{t}{y}, \quad y(2) = -1$.

Solution We already know from the previous example that an implicit solution to this IVP is $y^2 = t^2 - 3$. To find the explicit solution all we need to do is solve for $y(t)$.

$$y(t) = \pm\sqrt{t^2 - 3}$$

Now, we've got a problem here. There are two functions here and we only want one and in fact only one will be correct! We can determine the correct function by reapplying the initial condition. Only one of them will satisfy the initial condition.

In this case we can see that the "-" solution will be the correct one. The actual explicit solution is then

$$y(t) = -\sqrt{t^2 - 3}$$

In this case we were able to find an explicit solution to the differential equation. It should be noted however that it will not always be possible to find an explicit solution.

Also, note that in this case we were only able to get the explicit actual solution because we had the initial condition to help us determine which of the two functions would be the correct solution.

We've now gotten most of the basic definitions out of the way and so we can move onto other topics.

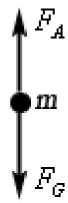
Direction Fields

This topic is given its own section for a couple of reasons. First, understanding direction fields and what they tell us about a differential equation and its solution is important and can be introduced without any knowledge of how to solve a differential equation and so can be done here before we get into solving them. So, having some information about the solution to a differential equation without actually having the solution is a nice idea that needs some investigation.

Next, since we need a differential equation to work with this is a good section to show you that differential equations occur naturally in many cases and how we get them. Almost every physical situation that occurs in nature can be described with an appropriate differential equation. The differential equation may be easy or difficult to arrive at depending on the situation and the assumptions that are made about the situation and we may not ever be able to solve it, however it will exist.

The process of describing a physical situation with a differential equation is called modeling. We will be looking at modeling several times throughout this class.

One of the simplest physical situations to think of is a falling object. So let's consider a falling object with mass *m* and derive a differential equation that, when solved, will give us the velocity of the object at any time, *t*. We will assume that only gravity and air resistance will act upon the object as it falls. Below is a figure showing the forces that will act upon the object.



Before defining all the terms in this problem we need to set some conventions. We will assume that forces acting in the downward direction are positive forces while forces that act in the upward direction are negative. Likewise, we will assume that an object moving downward (*i.e.* a falling object) will have a positive velocity.

Now, let's take a look at the forces shown in the diagram above. F_G is the force due to gravity and is given by $F_G = mg$ where *g* is the acceleration due to gravity. In this class I use *g* = 9.8 m/s^2 or *g* = 32 ft/s^2 depending on whether we will use the metric or British system. F_A is the force due to air resistance and for this example we will assume that it is proportional to the velocity, *v*, of the mass. Therefore the force due to air resistance is then given by $F_A = -\gamma v$, where $\gamma > 0$. Note that the "–" is required to get the correct sign on the force. Both γ and *v* are positive and the force is acting upward and hence must be negative. The "–" will give us the correct sign and hence direction for this force.

Recall from the previous section that Newton's Second Law of motion can be written as

$$m\frac{dv}{dt} = F\left(t,v\right)$$

where *F(t,v)* is the sum of forces that act on the object and may be a function of the time *t* and the velocity of the object, *v*. For our situation we will have two forces acting on the object gravity,

$F_G = mg$. acting in the downward direction and hence will be positive, and air resistance, $F_A = -\gamma v$, acting in the upward direction and hence will be negative. Putting all of this together into Newton's Second Law gives the following.

$$m\frac{dv}{dt} = mg - \gamma v$$

To simplify the differential equation let's divide out the mass, *m*.

$$\frac{dv}{dt} = g - \frac{\gamma v}{m} \qquad \textbf{(1)}$$

This then is a first order linear differential equation that, when solved, will give the velocity, *v* (in m/s), of a falling object of mass *m* that has both gravity and air resistance acting upon it.

In order to look at direction fields (that is after all the topic of this section....) it would be helpful to have some numbers for the various quantities in the differential equation. So, let's assume that we have a mass of 2 kg and that $\gamma = 0.392$. Plugging this into (1) gives the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad \textbf{(2)}$$

Let's take a geometric view of this differential equation. Let's suppose that for some time, *t*, the velocity just happens to be $v = 30$ m/s. Note that we're not saying that the velocity ever will be 30 m/s. All that we're saying is that let's suppose that by some chance the velocity does happen to be 30 m/s at some time *t*. So, if the velocity does happen to be 30 m/s at some time *t* we can plug $v = 30$ into (2) to get.

$$\frac{dv}{dt} = 3.92$$

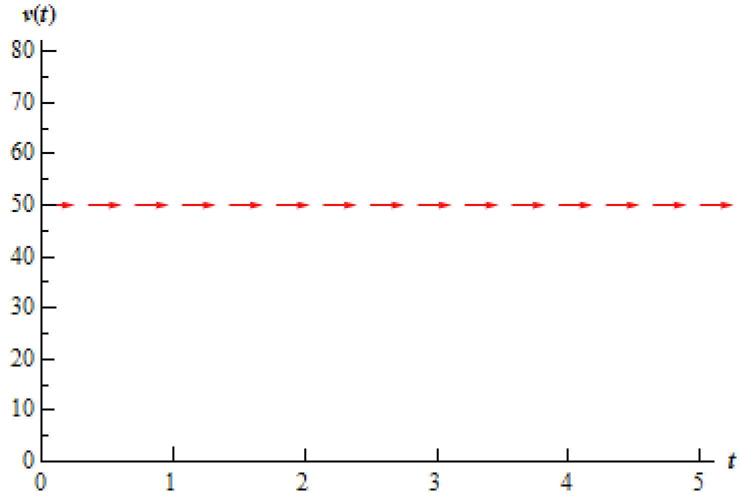
Recall from your Calculus I course that a positive derivative means that the function in question, the velocity in this case, is increasing, so if the velocity of this object is ever 30m/s for any time *t* the velocity must be increasing at that time.

Also, recall that the value of the derivative at a particular value of *t* gives the slope of the tangent line to the graph of the function at that time, *t*. So, if for some time *t* the velocity happens to be 30 m/s the slope of the tangent line to the graph of the velocity is 3.92.

We could continue in this fashion and pick different values of *v* and compute the slope of the tangent line for those values of the velocity. However, let's take a slightly more organized approach to this. Let's first identify the values of the velocity that will have zero slope or horizontal tangent lines. These are easy enough to find. All we need to do is set the derivative equal to zero and solve for *v*.

In the case of our example we will have only one value of the velocity which will have horizontal tangent lines, $v = 50$ m/s. What this means is that IF (again, there's that word if), for some time *t*, the velocity happens to be 50 m/s then the tangent line at that point will be horizontal. What the slope of the tangent line is at times before and after this point is not known yet and has no bearing on the slope at this particular time, *t*.

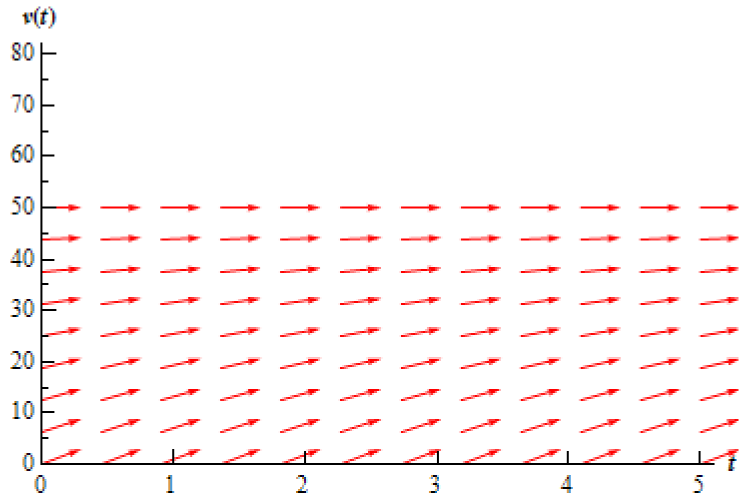
So, if we have $v = 50$, we know that the tangent lines will be horizontal. We denote this on an axis system with horizontal arrows pointing in the direction of increasing t at the level of $v = 50$ as shown in the following figure.



Now, let's get some tangent lines and hence arrows for our graph for some other values of v. At this point the only exact slope that is useful to us is where the slope horizontal. So instead of going after exact slopes for the rest of the graph we are only going to go after general trends in the slope. Is the slope increasing or decreasing? How fast is the slope increasing or decreasing? For this example those types of trends are very easy to get.

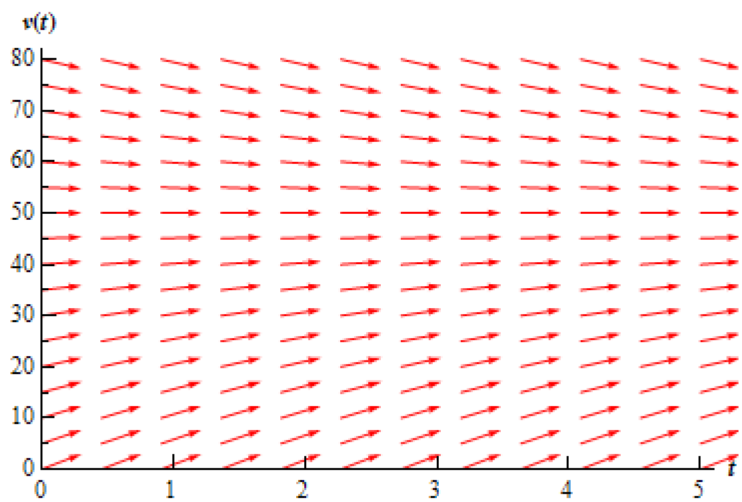
First, notice that the right hand side of (2) is a polynomial and hence continuous. This means that it can only change sign if it first goes through zero. So, if the derivative will change signs (no guarantees that it will) it will do so at $v = 50$ and the only place that it may change sign is $v = 50$. This means that for $v > 50$ the slope of the tangent lines to the velocity will have the same sign. Likewise, for $v < 50$ the slopes will also have the same sign. The slopes in these ranges may have (and probably will) have different values, but we do know what their signs must be.

Let's start by looking at $v < 50$. We saw earlier that if $v = 30$ the slope of the tangent line will be 3.92, or positive. Therefore, for all values of $v < 50$ we will have positive slopes for the tangent lines. Also, by looking at (2) we can see that as v approaches 50, always staying less than 50, the slopes of the tangent lines will approach zero and hence flatten out. If we move v away from 50, staying less than 50, the slopes of the tangent lines will become steeper. If you want to get an idea of just how steep the tangent lines become you can always pick specific values of v and compute values of the derivative. For instance, we know that at $v = 30$ the derivative is 3.92 and so arrows at this point should have a slope of around 4. Using this information we can now add in some arrows for the region below $v = 50$ as shown in the graph below.



Now, let's look at $v > 50$. The first thing to do is to find out if the slopes are positive or negative. We will do this the same way that we did in the last bit, ***i.e.*** pick a value of v, plug this into (2) and see if the derivative is positive or negative. Note, that you should NEVER assume that the derivative will change signs where the derivative is zero. It is easy enough to check so you should always do so.

We need to check the derivative so let's use $v = 60$. Plugging this into (2) gives the slope of the tangent line as -1.96, or negative. Therefore, for all values of $v > 50$ we will have negative slopes for the tangent lines. As with $v < 50$, by looking at (2) we can see that as v approaches 50, always staying greater than 50, the slopes of the tangent lines will approach zero and flatten out. While moving v away from 50 again, staying greater than 50, the slopes of the tangent lines will become steeper. We can now add in some arrows for the region above $v = 50$ as shown in the graph below.



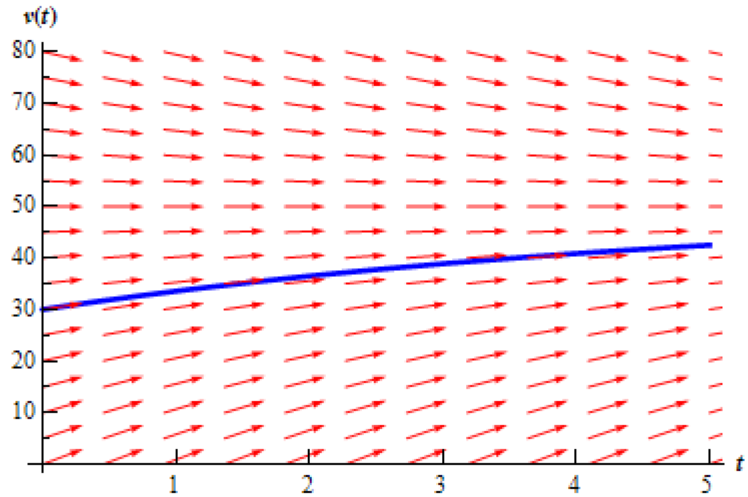
This graph above is called the **direction field** for the differential equation.

So, just why do we care about direction fields? There are two nice pieces of information that can be readily found from the direction field for a differential equation.

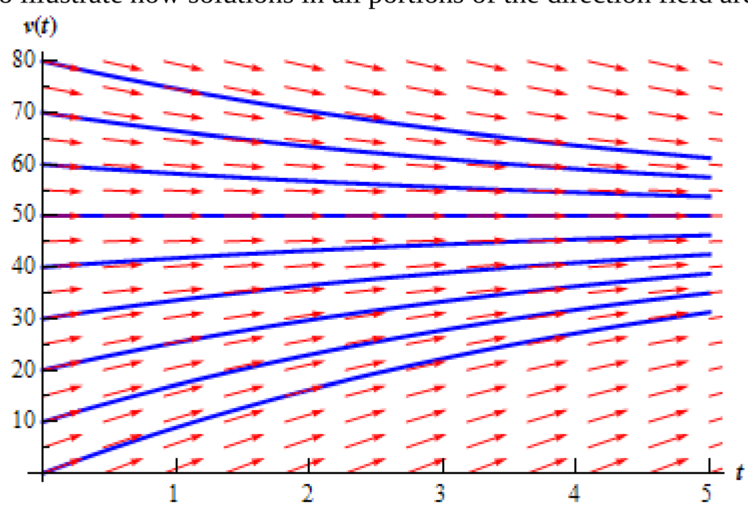
1. **Sketch of solutions**. Since the arrows in the direction fields are in fact tangents to the actual solutions to the differential equations we can use these as guides to sketch the graphs of solutions to the differential equation.

2. **Long Term Behavior**. In many cases we are less interested in the actual solutions to the differential equations as we are in how the solutions behave as t increases. Direction fields, if we can get our hands on them, can be used to find information about this long term behavior of the solution.

So, back to the direction field for our differential equation. Suppose that we want to know what the solution that has the value $v(0) = 30$ looks like. We can go to our direction field and start at 30 on the vertical axis. At this point we know that the solution is increasing and that as it increases the solution should flatten out because the velocity will be approaching the value of $v = 50$. So we start drawing an increasing solution and when we hit an arrow we just make sure that we stay parallel to that arrow. This gives us the figure below.



To get a better idea of how all the solutions are behaving, let's put a few more solutions in. Adding some more solutions gives the figure below. The set of solutions that we've graphed below is often called the **family of solution curves** or the set of **integral curves**. The number of solutions that is plotted when plotting the integral curves varies. You should graph enough solution curves to illustrate how solutions in all portions of the direction field are behaving.



Now, from either the direction field, or the direction field with the solution curves sketched in we can see the behavior of the solution as t increases. For our falling object, it looks like all of the solutions will approach $v = 50$ as t increases.

We will often want to know if the behavior of the solution will depend on the value of $v(0)$. In this case the behavior of the solution will not depend on the value of $v(0)$, but that is probably more of the exception than the rule so don't expect that.

Let's take a look at a more complicated example.

Example 1 Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation. Determine how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency.

$$y' = \left(y^2 - y - 2\right)\left(1 - y\right)^2$$

Solution

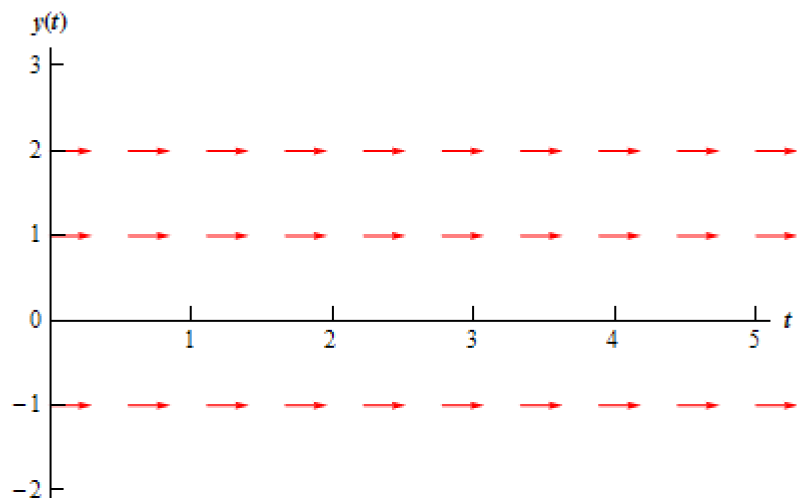
First, do not worry about where this differential equation came from. To be honest, we just made it up. It may, or may not describe an actual physical situation.

This differential equation looks somewhat more complicated than the falling object example from above. However, with the exception of a little more work, it is not much more complicated. The first step is to determine where the derivative is zero.

$$0 = \left(y^2 - y - 2\right)\left(1 - y\right)^2$$

$$0 = \left(y - 2\right)\left(y + 1\right)\left(1 - y\right)^2$$

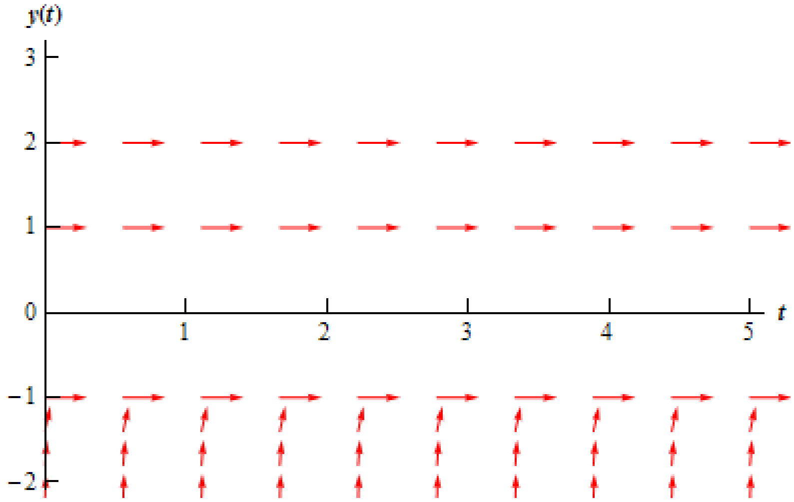
We can now see that we have three values of y in which the derivative, and hence the slope of tangent lines, will be zero. The derivative will be zero at $y = -1$, 1, and 2. So, let's start our direction field with drawing horizontal tangents for these values. This is shown in the figure below.



Now, we need to add arrows to the four regions that the graph is now divided into. For each of these regions I will pick a value of y in that region and plug it into the right hand side of the differential equation to see if the derivative is positive or negative in that region. Again, to get an accurate direction fields you should pick a few more over values over the whole range to see how the arrows are behaving over the whole range.

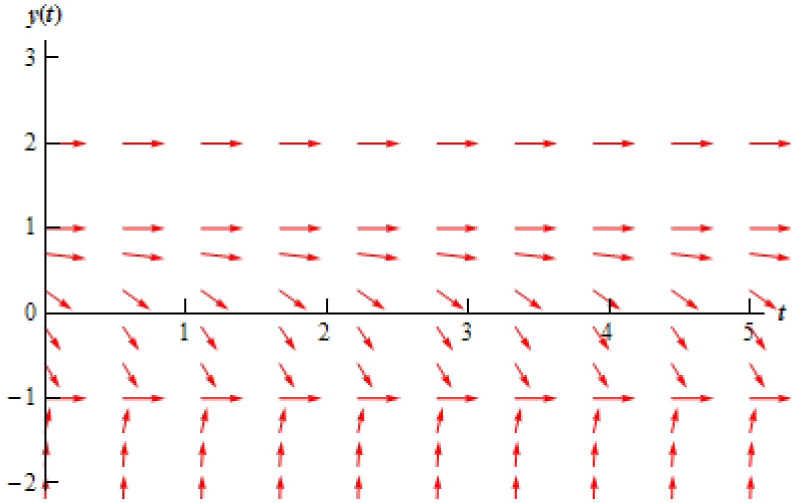
***y* < -1**

In this region we can use *y* = -2 as the test point. At this point we have $y' = 36$. So, tangent lines in this region will have very steep and positive slopes. Also as $y \to -1$ the slopes will flatten out while staying positive. The figure below shows the direction fields with arrows in this region.



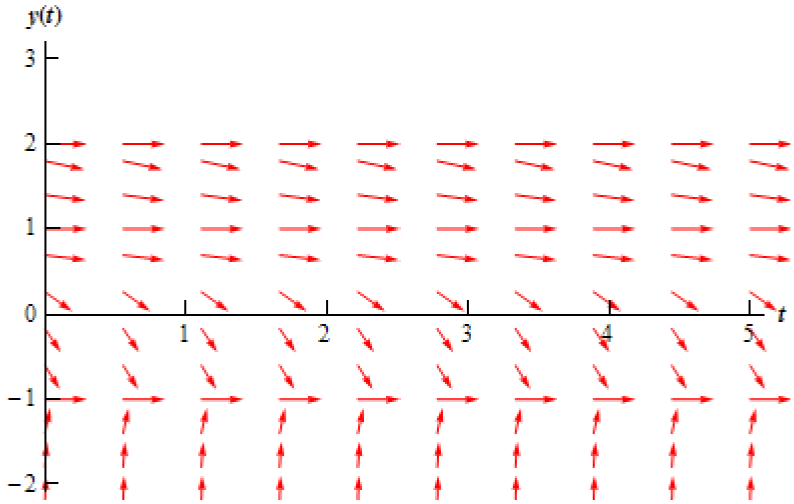
-1 < *y* < 1

In this region we can use *y* = 0 as the test point. At this point we have $y' = -2$. Therefore, tangent lines in this region will have negative slopes and apparently not be very steep. So what do the arrows look like in this region? As $y \to 1$ staying less than 1 of course, the slopes should be negative and approach zero. As we move away from 1 and towards -1 the slopes will start to get steeper (and stay negative), but eventually flatten back out, again staying negative, as $y \to -1$ since the derivative must approach zero at that point. The figure below shows the direction fields with arrows added to this region.



1 < *y* < 2

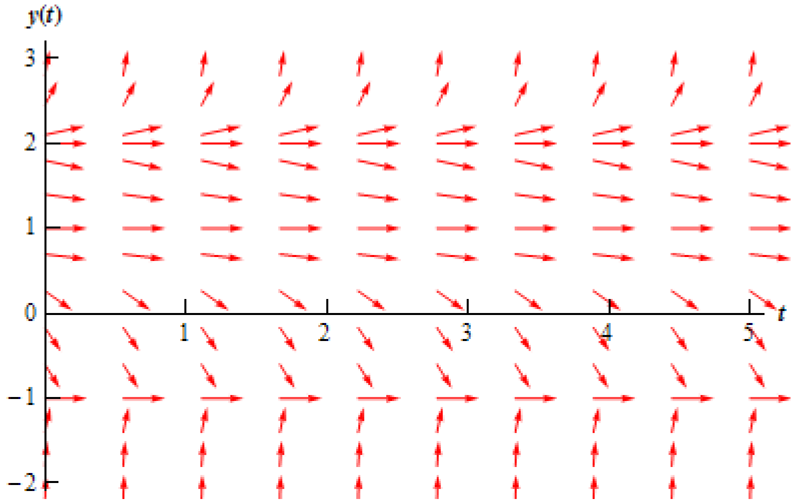
In this region we will use $y = 1.5$ as the test point. At this point we have $y' = -0.3125$. Tangent lines in this region will also have negative slopes and apparently not be as steep as the previous region. Arrows in this region will behave essentially the same as those in the previous region. Near $y = 1$ and $y = 2$ the slopes will flatten out and as we move from one to the other the slopes will get somewhat steeper before flattening back out. The figure below shows the direction fields with arrows added to this region.



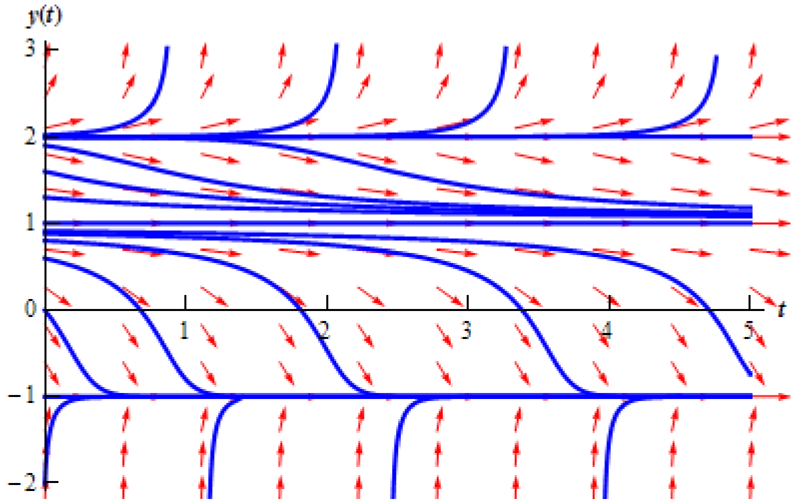
***y* > 2**

In this last region we will use $y = 3$ as the test point. At this point we have $y' = 16$. So, as we saw in the first region tangent lines will start out fairly flat near $y = 2$ and then as we move way from $y = 2$ they will get fairly steep.

The complete direction field for this differential equation is shown below.



Here is the set of integral curves for this differential equation. Note that due to the steepness of the solutions in the lowest region and the software used to generate these images I was unable to include more than one solution curve in this region.



Finally, let's take a look at long term behavior of all solutions. Unlike the first example, the long term behavior in this case will depend on the value of *y* at *t* = 0. By examining either of the previous two figures we can arrive at the following behavior of solutions as $t \to \infty$.

Value of *y*(0)	**Behavior as** $t \to \infty$
$y(0) < 1$	$y \to -1$
$1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$

Do not forget to acknowledge what the horizontal solutions are doing. This is often the most missed portion of this kind of problem.

In both of the examples that we've worked to this point the right hand side of the derivative has only contained the function and NOT the independent variable. When the right hand side of the differential equation contains both the function and the independent variable the behavior can be much more complicated and sketching the direction fields by hand can be very difficult. Computer software is very handy in these cases.

In some cases they aren't too difficult to do by hand however. Let's take a look at the following example.

Example 2 Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation.

$$y' = y - x$$

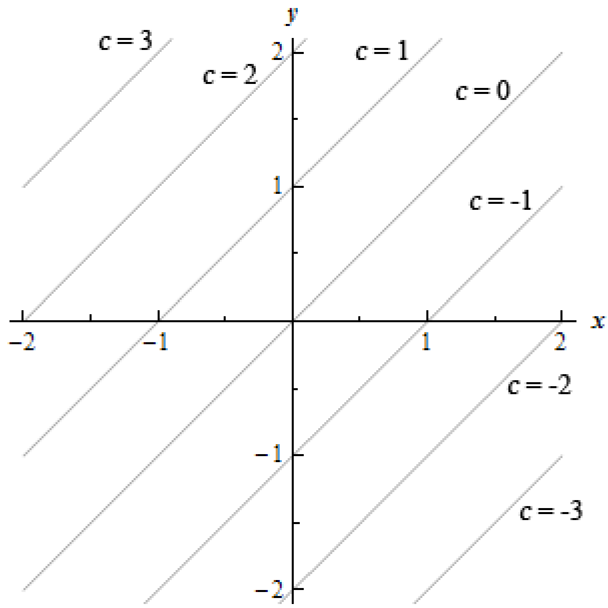
Solution

To sketch direction fields for this kind of differential equation we first identify places where the derivative will be constant. To do this we set the derivative in the differential equation equal to a constant, say *c*. This gives us a family of equations, called **isoclines**, that we can plot and on each of these curves the derivative will be a constant value of *c*.

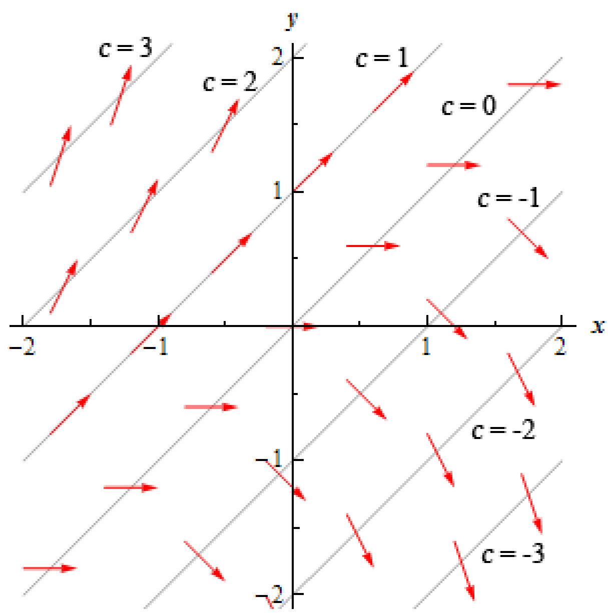
Notice that in the previous examples we looked at the isocline for *c* = 0 to get the direction field started. For our case the family of isoclines is.

$$c = y - x$$

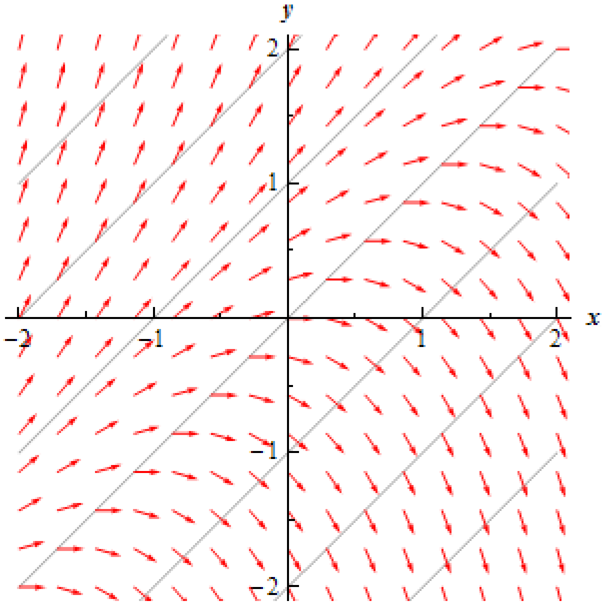
The graph of these curves for several values of *c* is shown below.



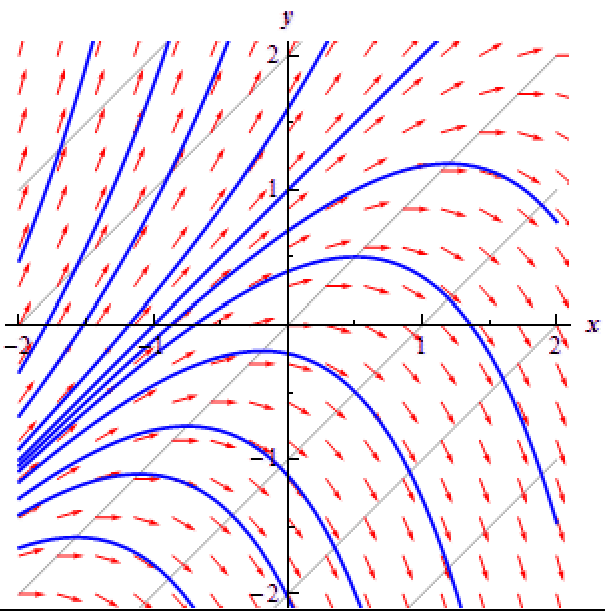
Now, on each of these lines, or isoclines, the derivative will be constant and will have a value of *c*. On the *c* = 0 isocline the derivative will always have a value of zero and hence the tangents will all be horizontal. On the *c* = 1 isocline the tangents will always have a slope of 1, on the *c* = -2 isocline the tangents will always have a slope of -2, ***etc.*** Below is a few tangents put in for each of these isoclines.



To add more arrows for those areas between the isoclines start at say, $c = 0$ and move up to $c = 1$ and as we do that we increase the slope of the arrows (tangents) from 0 to 1. This is shown in the figure below.



We can then add in integral curves as we did in the previous examples. This is shown in the figure below.



Final Thoughts

Before moving on to learning how to solve differential equations we want to give a few final thoughts. Any differential equations course will concern itself with answering one or more of the following questions.

1. **Given a differential equation will a solution exist?**
   Not all differential equations will have solutions so it's useful to know ahead of time if there is a solution or not. If there isn't a solution why waste our time trying to find something that doesn't exist?

   This question is usually called the **existence question** in a differential equations course.

2. **If a differential equation does have a solution how many solutions are there?**
   As we will see eventually, it is possible for a differential equation to have more than one solution. We would like to know how many solutions there will be for a given differential equation.

   There is a sub question here as well. What condition(s) on a differential equation are required to obtain a single unique solution to the differential equation?

   Both this question and the sub question are more important than you might realize. Suppose that we derive a differential equation that will give the temperature distribution in a bar of iron at any time t. If we solve the differential equation and end up with two (or more) completely separate solutions we will have problems. Consider the following situation to see this.

   If we subject 10 identical iron bars to identical conditions they should all exhibit the same temperature distribution. So only one of our solutions will be accurate, but we will have no way of knowing which one is the correct solution.

   It would be nice if, during the derivation of our differential equation, we could make sure that our assumptions would give us a differential equation that upon solving will yield a single unique solution.

   This question is usually called the **uniqueness question** in a differential equations course.

3. **If a differential equation does have a solution can we find it?**
   This may seem like an odd question to ask and yet the answer is not always yes. Just because we know that a solution to a differential equations exists does not mean that we will be able to find it.

In a first course in differential equations (such as this one) the third question is the question that we will concentrate on. We will answer the first two questions for special, and fairly simple, cases, but most of our efforts will be concentrated on answering the third question for as wide a variety of differential equations as possible.

First Order Differential Equations

Introduction

In this chapter we will look at solving first order differential equations. The most general first order differential equation can be written as,

$$\frac{dy}{dt} = f\left(y,t\right) \tag{1}$$

As we will see in this chapter there is no general formula for the solution to (1). What we will do instead is look at several special cases and see how to solve those. We will also look at some of the theory behind first order differential equations as well as some applications of first order differential equations. Below is a list of the topics discussed in this chapter.

Linear Differential Equations

The first special case of first order differential equations that we will look is the linear first order differential equation. In this case, unlike most of the first order cases that we will look at, we can actually derive a formula for the general solution. The general solution is derived below. However, I would suggest that you do not memorize the formula itself. Instead of memorizing the formula you should memorize and understand the process that I'm going to use to derive the formula. Most problems are actually easier to work by using the process instead of using the formula.

So, let's see how to solve a linear first order differential equation. Remember as we go through this process that the goal is to arrive at a solution that is in the form $y = y\left(t\right)$. It's sometimes easy to lose sight of the goal as we go through this process for the first time.

In order to solve a linear first order differential equation we MUST start with the differential equation in the form shown below. If the differential equation is not in this form then the process we're going to use will not work.

$$\frac{dy}{dt} + p\left(t\right)y = g\left(t\right) \tag{1}$$

Where both *p(t)* and *g(t)* are continuous functions. Recall that a quick and dirty definition of a continuous function is that a function will be continuous provided you can draw the graph from left to right without ever picking up your pencil/pen. In other words, a function is continuous if there are no holes or breaks in it.

Now, we are going to assume that there is some magical function somewhere out there in the world, $\mu\left(t\right)$, called an **integrating factor**. Do not, at this point, worry about what this function is or where it came from. We will figure out what $\mu\left(t\right)$ is once we have the formula for the general solution in hand.

So, now that we have assumed the existence of $\mu\left(t\right)$ multiply everything in (1) by $\mu\left(t\right)$. This will give.

$$\mu\left(t\right)\frac{dy}{dt} + \mu\left(t\right)p\left(t\right)y = \mu\left(t\right)g\left(t\right) \tag{2}$$

Now, this is where the magic of $\mu\left(t\right)$ comes into play. We are going to assume that whatever $\mu\left(t\right)$ is, it will satisfy the following.

$$\mu\left(t\right)p\left(t\right) = \mu'\left(t\right) \tag{3}$$

Again do not worry about how we can find a $\mu\left(t\right)$ that will satisfy (3). As we will see, provided *p(t)* is continuous we can find it. So substituting (3) into (2) we now arrive at.

$$\mu\left(t\right)\frac{dy}{dt} + \mu'\left(t\right)y = \mu\left(t\right)g\left(t\right) \tag{4}$$

At this point we need to recognize that the left side of (4) is nothing more than the following product rule.

$$\mu(t)\frac{dy}{dt}+\mu'(t)\,y=\left(\mu(t)\,y(t)\right)'$$

So we can replace the left side of (4) with this product rule. Upon doing this (4) becomes

$$\left(\mu(t)\,y(t)\right)'=\mu(t)\,g(t) \tag{5}$$

Now, recall that we are after *y(t)*. We can now do something about that. All we need to do is integrate both sides then use a little algebra and we'll have the solution. So, integrate both sides of (5) to get.

$$\begin{aligned}\int\left(\mu(t)\,y(t)\right)'\,dt&=\int\mu(t)\,g(t)\,dt\\ \mu(t)\,y(t)+c&=\int\mu(t)\,g(t)\,dt\end{aligned} \tag{6}$$

Note the constant of integration, *c*, from the left side integration is included here. It is vitally important that this be included. If it is left out you will get the wrong answer every time.

The final step is then some algebra to solve for the solution, *y(t)*.

$$\begin{aligned}\mu(t)\,y(t)&=\int\mu(t)\,g(t)\,dt-c\\ y(t)&=\frac{\int\mu(t)\,g(t)\,dt-c}{\mu(t)}\end{aligned}$$

Now, from a notational standpoint we know that the constant of integration, *c*, is an unknown constant and so to make our life easier we will absorb the minus sign in front of it into the constant and use a plus instead. This will NOT affect the final answer for the solution. So with this change we have.

$$y(t)=\frac{\int\mu(t)\,g(t)\,dt+c}{\mu(t)} \tag{7}$$

Again, changing the sign on the constant will not affect our answer. If you choose to keep the minus sign you will get the same value of *c* as I do except it will have the opposite sign. Upon plugging in *c* we will get exactly the same answer.

There is a lot of playing fast and loose with constants of integration in this section, so you will need to get used to it. When we do this we will always to try to make it very clear what is going on and try to justify why we did what we did.

So, now that we've got a general solution to (1) we need to go back and determine just what this magical function $\mu(t)$ is. This is actually an easier process than you might think. We'll start with (3).

$$\mu(t)\,p(t)=\mu'(t)$$

Divide both sides by $\mu(t)$,

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

Now, hopefully you will recognize the left side of this from your Calculus I class as nothing more than the following derivative.

$$\left(\ln \mu(t)\right)' = p(t)$$

As with the process above all we need to do is integrate both sides to get.

$$\ln \mu(t) + k = \int p(t)\,dt$$
$$\ln \mu(t) = \int p(t)\,dt + k$$

You will notice that the constant of integration from the left side, *k*, had been moved to the right side and had the minus sign absorbed into it again as we did earlier. Also note that we're using *k* here because we've already used *c* and in a little bit we'll have both of them in the same equation. So, to avoid confusion we used different letters to represent the fact that they will, in all probability, have different values.

Exponentiate both sides to get $\mu(t)$ out of the natural logarithm.

$$\mu(t) = \mathbf{e}^{\int p(t)\,dt + k}$$

Now, it's time to play fast and loose with constants again. It is inconvenient to have the *k* in the exponent so we're going to get it out of the exponent in the following way.

$$\begin{aligned}\mu(t) &= \mathbf{e}^{\int p(t)\,dt + k} \\ &= \mathbf{e}^{k}\mathbf{e}^{\int p(t)\,dt} \qquad\qquad \text{Recall } x^{a+b} = x^a x^b!\end{aligned}$$

Now, let's make use of the fact that *k* is an unknown constant. If *k* is an unknown constant then so is $\mathbf{e}^k$ so we might as well just rename it *k* and make our life easier. This will give us the following.

$$\mu(t) = k\,\mathbf{e}^{\int p(t)\,dt} \tag{8}$$

So, we now have a formula for the general solution, (7), and a formula for the integrating factor, (8). We do have a problem however. We've got two unknown constants and the more unknown constants we have the more trouble we'll have later on. Therefore, it would be nice if we could find a way to eliminate one of them (we'll not be able to eliminate both….).

This is actually quite easy to do. First, substitute (8) into (7) and rearrange the constants.

$$
\begin{aligned}
y(t) &= \frac{\int k\,\mathbf{e}^{\int p(t)\,dt} g(t)\,dt + c}{k\,\mathbf{e}^{\int p(t)\,dt}} \\
&= \frac{k\int \mathbf{e}^{\int p(t)\,dt} g(t)\,dt + c}{k\,\mathbf{e}^{\int p(t)\,dt}} \\
&= \frac{\int \mathbf{e}^{\int p(t)\,dt} g(t)\,dt + \frac{c}{k}}{\mathbf{e}^{\int p(t)\,dt}}
\end{aligned}
$$

So, (7) can be written in such a way that the only place the two unknown constants show up is a ratio of the two. Then since both *c* and *k* are unknown constants so is the ratio of the two constants. Therefore we'll just call the ratio *c* and then drop *k* out of (8) since it will just get absorbed into *c* eventually.

The solution to a linear first order differential equation is then

$$
y(t) = \frac{\int \mu(t)\,g(t)\,dt + c}{\mu(t)} \tag{9}
$$

where,

$$
\mu(t) = \mathbf{e}^{\int p(t)\,dt} \tag{10}
$$

Now, the reality is that (9) is not as useful as it may seem. It is often easier to just run through the process that got us to (9) rather than using the formula. We will not use this formula in any of my examples. We will need to use (10) regularly, as that formula is easier to use than the process to derive it.

Solution Process
The solution process for a first order linear differential equation is as follows.

1. Put the differential equation in the correct initial form, (1).
2. Find the integrating factor, $\mu(t)$, using (10).
3. Multiply everything in the differential equation by $\mu(t)$ and verify that the left side becomes the product rule $\left(\mu(t)\,y(t)\right)'$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution *y(t)*.

Let's work a couple of examples. Let's start by solving the differential equation that we derived back in the Direction Field section.

Example 1 Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Solution

First we need to get the differential equation in the correct form.

$$\frac{dv}{dt} + 0.196v = 9.8$$

From this we can see that $p(t)=0.196$ and so $\mu(t)$ is then.

$$\mu(t) = \mathbf{e}^{\int 0.196\,dt} = \mathbf{e}^{0.196t}$$

Note that officially there should be a constant of integration in the exponent from the integration. However, we can drop that for exactly the same reason that we dropped the k from (8).

Now multiply all the terms in the differential equation by the integrating factor and do some simplification.

$$\mathbf{e}^{0.196t}\frac{dv}{dt} + 0.196\mathbf{e}^{0.196t}v = 9.8\mathbf{e}^{0.196t}$$

$$\left(\mathbf{e}^{0.196t}v\right)' = 9.8\mathbf{e}^{0.196t}$$

Integrate both sides and don't forget the constants of integration that will arise from both integrals.

$$\int \left(\mathbf{e}^{0.196t}v\right)' dt = \int 9.8\mathbf{e}^{0.196t}\,dt$$

$$\mathbf{e}^{0.196t}v + k = 50\mathbf{e}^{0.196t} + c$$

Okay. It's time to play with constants again. We can subtract k from both sides to get.

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c - k$$

Both c and k are unknown constants and so the difference is also an unknown constant. We will therefore write the difference as c. So, we now have

$$\mathbf{e}^{0.196t}v = 50\mathbf{e}^{0.196t} + c$$

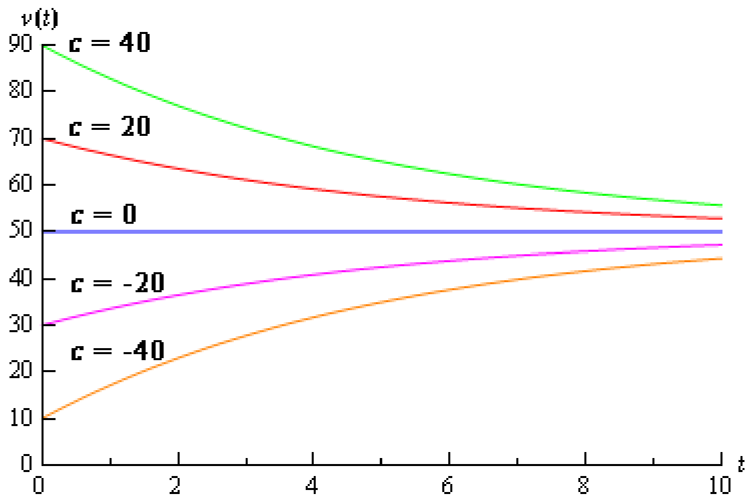
From this point on we will only put one constant of integration down when we integrate both sides knowing that if we had written down one for each integral, as we should, the two would just end up getting absorbed into each other.

The final step in the solution process is then to divide both sides by $\mathbf{e}^{0.196t}$ or to multiply both sides by $\mathbf{e}^{-0.196t}$. Either will work, but I usually prefer the multiplication route. Doing this gives the general solution to the differential equation.

$$v(t) = 50 + c\mathbf{e}^{-0.196t}$$

From the solution to this example we can now see why the constant of integration is so important in this process. Without it, in this case, we would get a single, constant solution, $v(t)=50$. With the constant of integration we get infinitely many solutions, one for each value of c.

Back in the direction field section where we first derived the differential equation used in the last example we used the direction field to help us sketch some solutions. Let's see if we got them correct. To sketch some solutions all we need to do is to pick different values of c to get a solution. Several of these are shown in the graph below.



So, it looks like we did pretty good sketching the graphs back in the direction field section.

Now, recall from the Definitions section that the Initial Condition(s) will allow us to zero in on a particular solution. Solutions to first order differential equations (not just linear as we will see) will have a single unknown constant in them and so we will need exactly one initial condition to find the value of that constant and hence find the solution that we were after. The initial condition for first order differential equations will be of the form

$$y(t_0) = y_0$$

Recall as well that a differential equation along with a sufficient number of initial conditions is called an Initial Value Problem (IVP).

Example 2 Solve the following IVP.

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad\qquad v(0) = 48$$

Solution

To find the solution to an IVP we must first find the general solution to the differential equation and then use the initial condition to identify the exact solution that we are after. So, since this is the same differential equation as we looked at in Example 1**,** we already have its general solution.

$$v = 50 + c\mathbf{e}^{-0.196t}$$

Now, to find the solution we are after we need to identify the value of c that will give us the solution we are after. To do this we simply plug in the initial condition which will give us an equation we can solve for c. So let's do this

$$48 = v(0) = 50 + c \qquad \Rightarrow \qquad c = -2$$

So, the actual solution to the IVP is.

$$v = 50 - 2\mathbf{e}^{-0.196t}$$

A graph of this solution can be seen in the figure above.

Let's do a couple of examples that are a little more involved.

Example 3 Solve the following IVP.

$$\cos(x)\,y' + \sin(x)\,y = 2\cos^3(x)\sin(x) - 1 \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \qquad 0 \le x < \frac{\pi}{2}$$

Solution :

Rewrite the differential equation to get the coefficient of the derivative a one.

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)\,y = 2\cos^2(x)\sin(x) - \sec(x)$$

Now find the integrating factor.

$$\mu(t) = \mathbf{e}^{\int \tan x\,dx} = \mathbf{e}^{\ln|\sec x|} = \mathbf{e}^{\ln \sec x} = \sec x$$

Can you do the integral? If not rewrite tangent back into sines and cosines and then use a simple substitution. Note that we could drop the absolute value bars on the secant because of the limits on *x*. In fact, this is the reason for the limits on *x*.

Also note that we made use of the following fact.

$$\mathbf{e}^{\ln f(x)} = f(x) \qquad \textbf{(11)}$$

This is an important fact that you should always remember for these problems. We will want to simplify the integrating factor as much as possible in all cases and this fact will help with that simplification.

Now back to the example. Multiply the integrating factor through the differential equation and verify the left side is a product rule. Note as well that we multiply the integrating factor through the rewritten differential equation and NOT the original differential equation. Make sure that you do this. If you multiply the integrating factor through the original differential equation you will get the wrong solution!

$$\sec(x)\,y' + \sec(x)\tan(x)\,y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$\left(\sec(x)\,y\right)' = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrate both sides.

$$\int \left(\sec(x)\, y(x)\right)' \, dx = \int 2\cos(x)\sin(x) - \sec^2(x)\, dx$$
$$\sec(x)\, y(x) = \int \sin(2x) - \sec^2(x)\, dx$$
$$\sec(x)\, y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

Note the use of the trig formula $\sin(2\theta) = 2\sin\theta\cos\theta$ that made the integral easier. Next, solve for the solution.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x)$$
$$= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x)$$

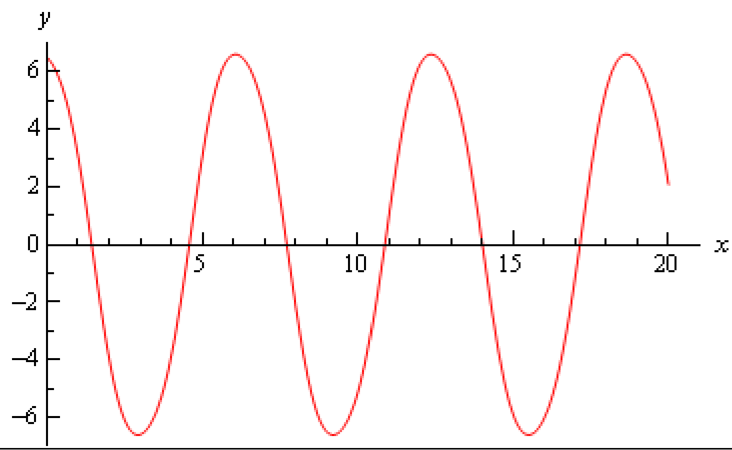
Finally, apply the initial condition to find the value of *c*.

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$
$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}$$
$$c = 7$$

The solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Below is a plot of the solution.



Example 4 **Find the solution to the following IVP.**

$$t\,y' + 2y = t^2 - t + 1 \qquad\qquad y(1) = \frac{1}{2}$$

Solution

First, divide through by the *t* to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|}$$

Now, we need to simplify $\mu(t)$. However, we can't use (11) yet as that requires a coefficient of one in front of the logarithm. So, recall that

$$\ln x^r = r\ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = \mathbf{e}^{2\ln|t|} = \mathbf{e}^{\ln|t|^2} = |t|^2 = t^2$$

We were able to drop the absolute value bars here because we were squaring the *t*, but often they can't be dropped so be careful with them and don't drop them unless you know that you can. Often the absolute value bars must remain.

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here…) by the integrating factor.

$$\left(t^2 y\right)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^2 y &= \int t^3 - t^2 + t\,dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \\ y(t) &= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2} \end{aligned}$$

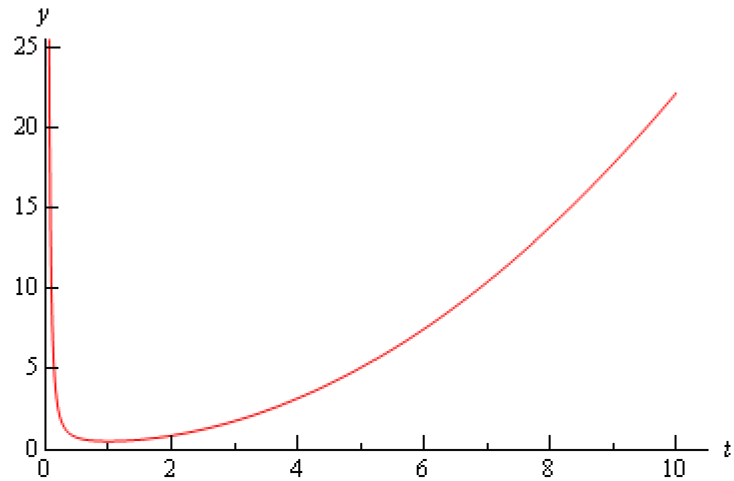
Finally, apply the initial condition to get the value of *c*.

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \qquad \Rightarrow \qquad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Here is a plot of the solution.



Example 5 Find the solution to the following IVP.

$$t\,y' - 2y = t^5 \sin(2t) - t^3 + 4t^4 \qquad y(\pi) = \frac{3}{2}\pi^4$$

Solution

First, divide through by t to get the differential equation in the correct form.

$$y' - \frac{2}{t}y = t^4 \sin(2t) - t^2 + 4t^3$$

Now that we have done this we can find the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{2}{t}dt} = \mathbf{e}^{-2\ln|t|}$$

Do not forget that the "-" is part of *p(t)*. Forgetting this minus sign can take a problem that is very easy to do and turn it into a very difficult, if not impossible problem so be careful!

Now, we just need to simplify this as we did in the previous example.

$$\mu(t) = \mathbf{e}^{-2\ln|t|} = \mathbf{e}^{\ln|t|^{-2}} = |t|^{-2} = t^{-2}$$

Again, we can drop the absolute value bars since we are squaring the term.

Now multiply the differential equation by the integrating factor (again, make sure it's the rewritten one and not the original differential equation).

$$\left(t^{-2}y\right)' = t^2 \sin(2t) - 1 + 4t$$

Integrate both sides and solve for the solution.

$$t^{-2}y(t) = \int t^2 \sin(2t)\,dt + \int -1 + 4t\,dt$$

$$t^{-2}y(t) = -\frac{1}{2}t^2\cos(2t) + \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t) - t + 2t^2 + c$$

$$y(t) = -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + ct^2$$

Apply the initial condition to find the value of *c*.

$$\frac{3}{2}\pi^4 = y(\pi) = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + c\pi^2 = \frac{3}{2}\pi^4 - \pi^3 + \frac{1}{4}\pi^2 + c\pi^2$$

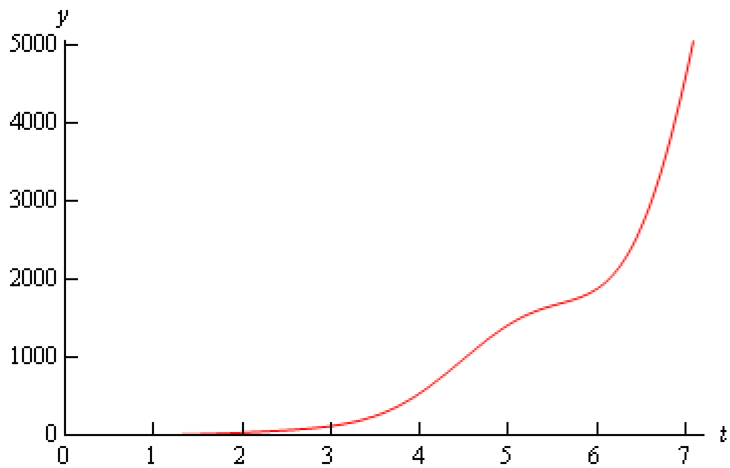
$$\pi^3 - \frac{1}{4}\pi^2 = c\pi^2$$

$$c = \pi - \frac{1}{4}$$

The solution is then

$$y(t) = -\frac{1}{2}t^4\cos(2t) + \frac{1}{2}t^3\sin(2t) + \frac{1}{4}t^2\cos(2t) - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Below is a plot of the solution.



Let's work one final example that looks more at interpreting a solution rather than finding a solution.

Example 6 Find the solution to the following IVP and determine all possible behaviors of the solution as $t \to \infty$. If this behavior depends on the value of y_0 give this dependence.

$$2y' - y = 4\sin(3t) \qquad y(0) = y_0$$

Solution

First, divide through by a 2 to get the differential equation in the correct form.

$$y' - \frac{1}{2}y = 2\sin(3t)$$

Now find $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int -\frac{1}{2}dt} = \mathbf{e}^{-\frac{t}{2}}$$

Multiply $\mu(t)$ through the differential equation and rewrite the left side as a product rule.

$$\left(\mathbf{e}^{-\frac{t}{2}}y\right)' = 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)$$

Integrate both sides and solve for the solution.

$$\mathbf{e}^{-\frac{t}{2}}y = \int 2\mathbf{e}^{-\frac{t}{2}}\sin(3t)\,dt + c$$

$$\mathbf{e}^{-\frac{t}{2}}y = -\frac{24}{37}\mathbf{e}^{-\frac{t}{2}}\cos(3t) - \frac{4}{37}\mathbf{e}^{-\frac{t}{2}}\sin(3t) + c$$

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + c\mathbf{e}^{\frac{t}{2}}$$

Apply the initial condition to find the value of c and note that it will contain y_0 as we don't have a value for that.

$$y_0 = y(0) = -\frac{24}{37} + c \qquad \Rightarrow \qquad c = y_0 + \frac{24}{37}$$

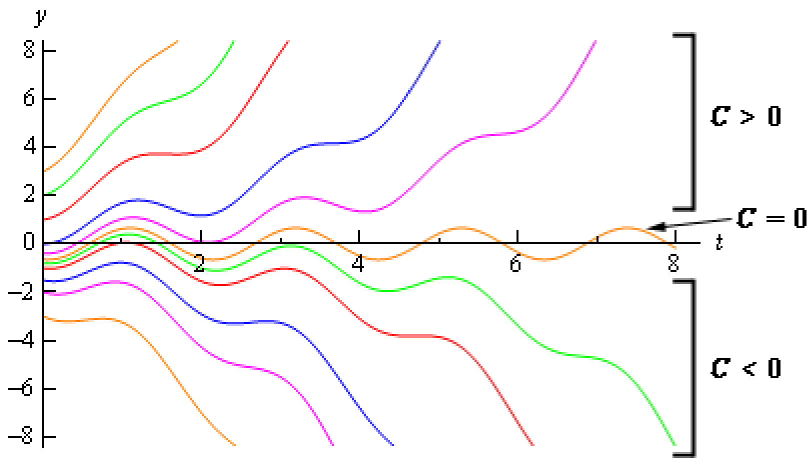
So the solution is

$$y(t) = -\frac{24}{37}\cos(3t) - \frac{4}{37}\sin(3t) + \left(y_0 + \frac{24}{37}\right)\mathbf{e}^{\frac{t}{2}}$$

Now that we have the solution, let's look at the long term behavior (*i.e.* $t \to \infty$) of the solution. The first two terms of the solution will remain finite for all values of t. It is the last term that will determine the behavior of the solution. The exponential will always go to infinity as $t \to \infty$, however depending on the sign of the coefficient c (yes we've already found it, but for ease of this discussion we'll continue to call it c). The following table gives the long term behavior of the solution for all values of c.

Range of c	Behavior of solution as $t \to \infty$
$c < 0$	$y(t) \to -\infty$
$c = 0$	$y(t)$ remains finite
$c > 0$	$y(t) \to \infty$

This behavior can also be seen in the following graph of several of the solutions.



Now, because we know how c relates to y_0 we can relate the behavior of the solution to y_0. The following table give the behavior of the solution in terms of y_0 instead of c.

Range of y_0	Behavior of solution as $t \to \infty$
$y_0 < -\frac{24}{37}$	$y(t) \to -\infty$
$y_0 = -\frac{24}{37}$	$y(t)$ remains finite
$y_0 > -\frac{24}{37}$	$y(t) \to \infty$

Note that for $y_0 = -\frac{24}{37}$ the solution will remain finite. That will not always happen.

Investigating the long term behavior of solutions is sometimes more important than the solution itself. Suppose that the solution above gave the temperature in a bar of metal. In this case we would want the solution(s) that remains finite in the long term. With this investigation we would now have the value of the initial condition that will give us that solution and more importantly values of the initial condition that we would need to avoid so that we didn't melt the bar.

Separable Differential Equations

We are now going to start looking at nonlinear first order differential equations. The first type of nonlinear first order differential equations that we will look at is separable differential equations.

A separable differential equation is any differential equation that we can write in the following form.

$$N\left(y\right)\frac{dy}{dx} = M\left(x\right) \tag{1}$$

Note that in order for a differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

Solving separable differential equation is fairly easy. We first rewrite the differential equation as the following

$$N\left(y\right)dy = M\left(x\right)dx$$

Then you integrate both sides.

$$\int N\left(y\right)dy = \int M\left(x\right)dx \tag{2}$$

So, after doing the integrations in (2) you will have an implicit solution that you can hopefully solve for the explicit solution, $y(x)$. Note that it won't always be possible to solve for an explicit solution.

Recall from the Definitions section that an implicit solution is a solution that is not in the form $y = y\left(x\right)$ while an explicit solution has been written in that form.

We will also have to worry about the interval of validity for many of these solutions. Recall that the interval of validity was the range of the independent variable, x in this case, on which the solution is valid. In other words, we need to avoid division by zero, complex numbers, logarithms of negative numbers or zero, ***etc.*** Most of the solutions that we will get from separable differential equations will not be valid for all values of x.

Let's start things off with a fairly simple example so we can see the process without getting lost in details of the other issues that often arise with these problems.

Example 1 Solve the following differential equation and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2x \qquad y\left(1\right) = \frac{1}{25}$$

Solution

It is clear, hopefully, that this differential equation is separable. So, let's separate the differential equation and integrate both sides. As with the linear first order officially we will pick up a constant of integration on both sides from the integrals on each side of the equal sign. The two can be moved to the same side and absorbed into each other. We will use the convention that puts the single constant on the side with the x's.

$$y^{-2}\,dy = 6x\,dx$$
$$\int y^{-2}\,dy = \int 6x\,dx$$
$$-\frac{1}{y} = 3x^2 + c$$

So, we now have an implicit solution. This solution is easy enough to get an explicit solution, however before getting that it is usually easier to find the value of the constant at this point. So apply the initial condition and find the value of *c*.

$$-\frac{1}{1/25} = 3(1)^2 + c \qquad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$
$$y(x) = \frac{1}{28 - 3x^2}$$

Now, as far as solutions go we've got the solution. We do need to start worrying about intervals of validity however.

Recall that there are two conditions that define an interval of validity. First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition, $x = 1$ in this case.

So, for our case we've got to avoid two values of *x*. Namely, $x \neq \pm\sqrt{\tfrac{28}{3}} \approx \pm 3.05505$ since these will give us division by zero. This gives us three possible intervals of validity.

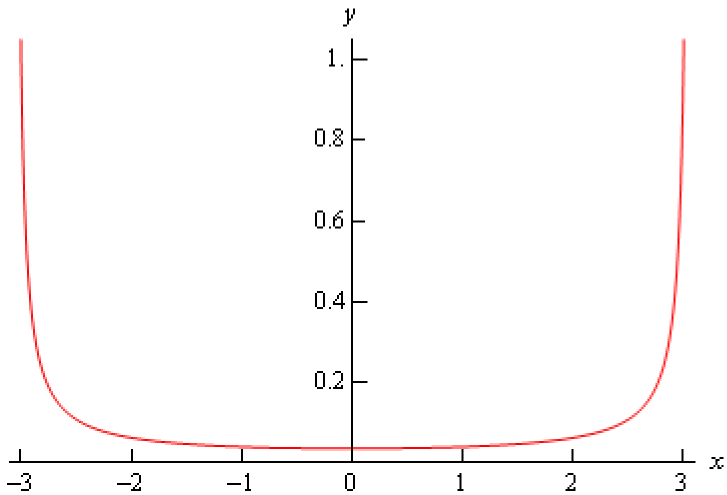
$$-\infty < x < -\sqrt{\frac{28}{3}} \qquad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \qquad \sqrt{\frac{28}{3}} < x < \infty$$

However, only one of these will contain the value of *x* from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

must be the interval of validity for this solution.

Here is a graph of the solution.



Note that this does not say that either of the other two intervals listed above can't be the interval of validity for any solution. With the proper initial condition either of these could have been the interval of validity.

We'll leave it to you to verify the details of the following claims. If we use an initial condition of

$$y(-4) = -\frac{1}{20}$$

we will get exactly the same solution however in this case the interval of validity would be the first one.

$$-\infty < x < -\sqrt{\frac{28}{3}}$$

Likewise, if we use

$$y(6) = -\frac{1}{80}$$

as the initial condition we again get exactly the same solution and in this case the third interval becomes the interval of validity.

$$\sqrt{\frac{28}{3}} < x < \infty$$

So, simply changing the initial condition a little can give any of the possible intervals.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \qquad y(1) = 3$$

Solution

This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$\left(2y-4\right)dy=\left(3x^2+4x-4\right)dx$$
$$\int\left(2y-4\right)dy=\int\left(3x^2+4x-4\right)dx$$
$$y^2-4y=x^3+2x^2-4x+c$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of c.

$$\left(3\right)^2-4\left(3\right)=\left(1\right)^3+2\left(1\right)^2-4\left(1\right)+c \qquad c=-2$$

The implicit solution is then

$$y^2-4y=x^3+2x^2-4x-2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First we need to rewrite the solution a little

$$y^2-4y-\left(x^3+2x^2-4x-2\right)=0$$

To solve this all we need to recognize is that this is quadratic in y and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant, but will in fact involve x's.

So, upon using the quadratic formula on this we get.

$$\begin{aligned} y\left(x\right)&=\frac{4\pm\sqrt{16-4\left(1\right)\left(-\left(x^3+2x^2-4x-2\right)\right)}}{2}\\ &=\frac{4\pm\sqrt{16+4\left(x^3+2x^2-4x-2\right)}}{2}\end{aligned}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2…) and then simplify a little.

$$\begin{aligned} y\left(x\right)&=\frac{4\pm2\sqrt{4+\left(x^3+2x^2-4x-2\right)}}{2}\\ &=2\pm\sqrt{x^3+2x^2-4x+2}\end{aligned}$$

We are almost there. Notice that we've actually got two solutions here (the "$\pm$") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3=y\left(1\right)=2\pm\sqrt{1+2-4+2}=2\pm1=3,1$$

In this case it looks like the "+" is the correct sign for our solution. Note that it is completely possible that the "–" could be the solution so don't always expect it to be one or the other.

The explicit solution for our differential equation is.

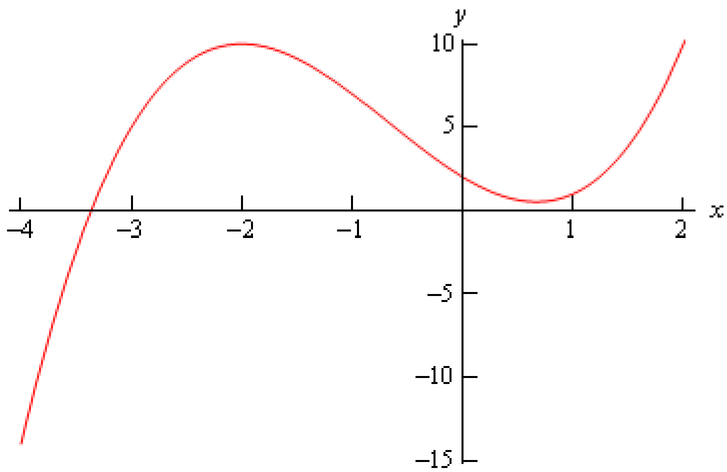
$$y(x) = 2 + \sqrt{x^3 + 2x^2 - 4x + 2}$$

To finish the example out we need to determine the interval of validity for the solution. If we were to put a large negative value of *x* in the solution we would end up with complex values in our solution and we want to avoid complex numbers in our solutions here. So, we will need to determine which values of *x* will give real solutions. To do this we will need to solve the following inequality.

$$x^3 + 2x^2 - 4x + 2 \geq 0$$

In other words, we need to make sure that the quantity under the radical stays positive.

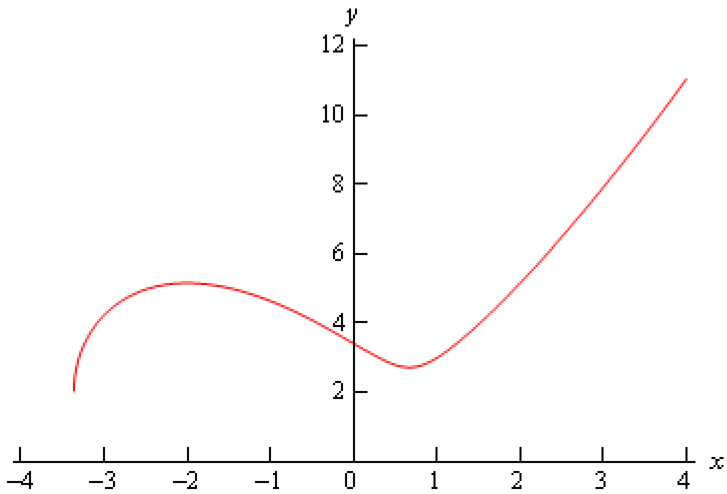
Using a computer algebra system like Maple or Mathematica we see that the left side is zero at *x* = –3.36523 as well as two complex values, but we can ignore complex values for interval of validity computations. Finally a graph of the quantity under the radical is shown below.



So, in order to get real solutions we will need to require $x \geq -3.36523$ because this is the range of *x*'s for which the quantity is positive. Notice as well that this interval also contains the value of *x* that is in the initial condition as it should.

Therefore, the interval of validity of the solution is $x \geq -3.36523$.

Here is graph of the solution.



Example 3 Solve the following IVP and find the interval of validity of the solution.

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad\qquad y(0) = -1$$

Solution

First separate and then integrate both sides.

$$y^{-3}dy = x\left(1+x^2\right)^{-\frac{1}{2}} dx$$

$$\int y^{-3}dy = \int x\left(1+x^2\right)^{-\frac{1}{2}} dx$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition to get the value of *c*.

$$-\frac{1}{2} = \sqrt{1} + c \qquad\qquad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for *y(x)*.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows us that the "–" is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1+x^2 \geq 0$ the "inner" root will not be a problem. Therefore all we need to worry about is division by zero and negatives under the "outer" root. We can take care of both be requiring

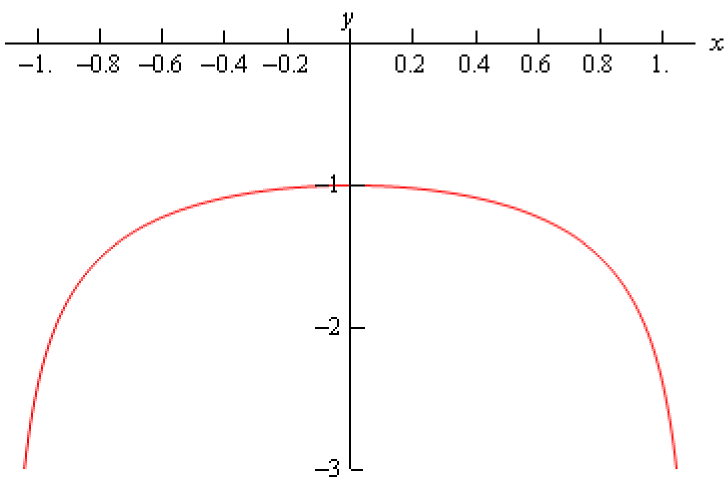
$$\begin{aligned}3-2\sqrt{1+x^2} &> 0\\ 3 &> 2\sqrt{1+x^2}\\ 9 &> 4\left(1+x^2\right)\\ \frac{9}{4} &> 1+x^2\\ \frac{5}{4} &> x^2\end{aligned}$$

Note that we were able to square both sides of the inequality because both sides of the inequality are guaranteed to be positive in this case. Finally solving for *x* we see that the only possible range of *x*'s that will not give division by zero or square roots of negative numbers will be,

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition *x=0*. This interval is therefore our interval of validity.

Here is a graph of the solution.



Example 4 Solve the following IVP and find the interval of validity of the solution.

$$y' = \mathbf{e}^{-y}\left(2x-4\right) \qquad\qquad y\left(5\right)=0$$

Solution

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$\mathbf{e}^{y}\,dy = \left(2x-4\right)dx$$

$$\int \mathbf{e}^{y}\,dy = \int \left(2x-4\right)dx$$

$$\mathbf{e}^{y} = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad\qquad c = -4$$

This then gives an implicit solution of.

$$\mathbf{e}^{y} = x^2 - 4x - 4$$

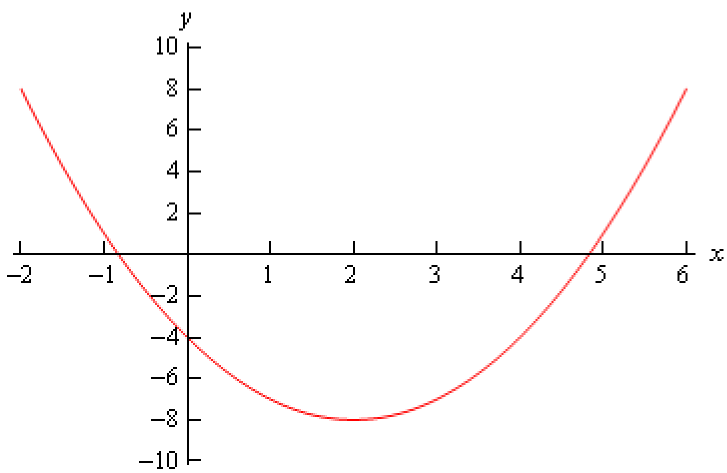
We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y\left(x\right) = \ln\left(x^2 - 4x - 4\right)$$

Finding the interval of validity is the last step that we need to take. Recall that we can't plug negative values or zero into a logarithm, so we need to solve the following inequality

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at the two points $x = 2 \pm 2\sqrt{2}$. A graph of the quadratic (shown below) shows that there are in fact two intervals in which we will get positive values of the polynomial and hence can be possible intervals of validity.



So, possible intervals of validity are

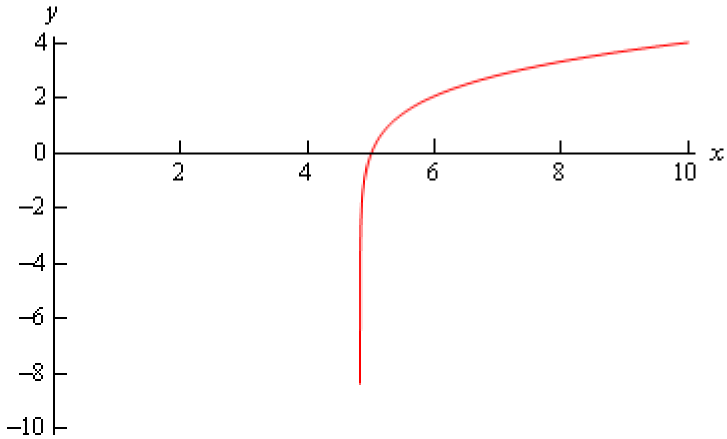
$$-\infty < x < 2 - 2\sqrt{2}$$

$$2 + 2\sqrt{2} < x < \infty$$

From the graph of the quadratic we can see that the second one contains x = 5, the value of the independent variable from the initial condition. Therefore the interval of validity for this solution is.

$$2+2\sqrt{2} < x < \infty$$

Here is a graph of the solution.



Example 5 Solve the following IVP and find the interval of validity for the solution.

$$\frac{dr}{d\theta}=\frac{r^2}{\theta} \qquad r(1)=2$$

Solution

This is actually a fairly simple differential equation to solve. I'm doing this one mostly because of the interval of validity.

So, get things separated out and then integrate.

$$\frac{1}{r^2}\,dr=\frac{1}{\theta}\,d\theta$$

$$\int\frac{1}{r^2}\,dr=\int\frac{1}{\theta}\,d\theta$$

$$-\frac{1}{r}=\ln|\theta|+c$$

Now, apply the initial condition to find *c*.

$$-\frac{1}{2}=\ln(1)+c \qquad c=-\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r}=\ln|\theta|-\frac{1}{2}$$

Solving for *r* gets us our explicit solution.

$$r=\frac{1}{\frac{1}{2}-\ln|\theta|}$$

Now, there are two problems for our solution here. First we need to avoid $\theta = 0$ because of the natural log. Notice that because of the absolute value on the θ we don't need to worry about θ being negative. We will also need to avoid division by zero. In other words, we need to avoid the following points.

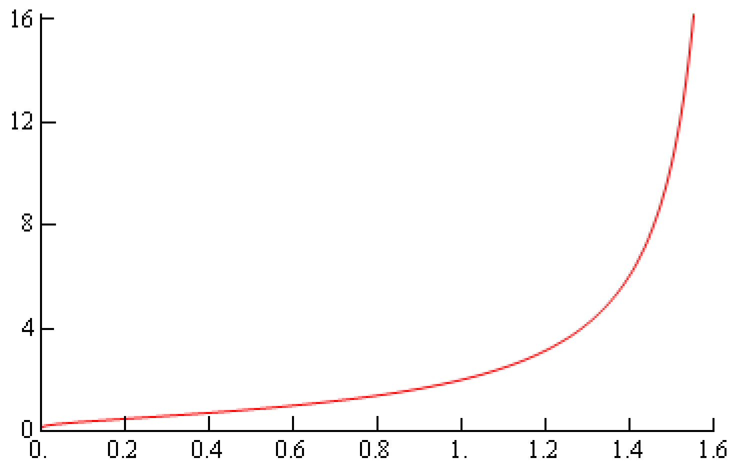
$$\begin{aligned} \frac{1}{2} - \ln|\theta| &= 0 \\ \ln|\theta| &= \frac{1}{2} \qquad \text{exponentiate both sides} \\ |\theta| &= \mathbf{e}^{\frac{1}{2}} \\ \theta &= \pm\sqrt{\mathbf{e}} \end{aligned}$$

So, these three points break the number line up into four portions, each of which could be an interval of validity.

$$\begin{aligned} -\infty < \theta < -\sqrt{\mathbf{e}} \\ -\sqrt{\mathbf{e}} < \theta < 0 \\ 0 < \theta < \sqrt{\mathbf{e}} \\ \sqrt{\mathbf{e}} < \theta < \infty \end{aligned}$$

The interval that will be the actual interval of validity is the one that contains $\theta = 1$. Therefore, the interval of validity is $0 < \theta < \sqrt{\mathbf{e}}$.

Here is a graph of the solution.



Example 6 Solve the following IVP.

$$\frac{dy}{dt} = \mathbf{e}^{y-t}\sec\left(y\right)\left(1+t^2\right) \qquad\qquad y\left(0\right)=0$$

Solution

This problem will require a little work to get it separated and in a form that we can integrate, so let's do that first.

$$\frac{dy}{dt} = \frac{\mathbf{e}^{y}\mathbf{e}^{-t}}{\cos\left(y\right)}\left(1+t^2\right)$$

$$\mathbf{e}^{-y}\cos\left(y\right)dy = \mathbf{e}^{-t}\left(1+t^2\right)dt$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\int \mathbf{e}^{-y}\cos\left(y\right)dy = \int \mathbf{e}^{-t}\left(1+t^2\right)dt$$

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin\left(y\right)-\cos\left(y\right)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+c$$

Applying the initial condition gives.

$$\frac{1}{2}\left(-1\right) = -\left(3\right)+c \qquad\qquad c=\frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin\left(y\right)-\cos\left(y\right)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+\frac{5}{2}$$

It is not possible to find an explicit solution for this problem and so we will have to leave the solution in its implicit form. Finding intervals of validity from implicit solutions can often be very difficult so we will also not bother with that for this problem.

As this last example showed it is not always possible to find explicit solutions so be on the lookout for those cases.

Exact Differential Equations

The next type of first order differential equations that we'll be looking at is exact differential equations. Before we get into the full details behind solving exact differential equations it's probably best to work an example that will help to show us just what an exact differential equation is. It will also show some of the behind the scenes details that we usually don't bother with in the solution process.

The vast majority of the following example will not be done in any of the remaining examples and the work that we will put into the remaining examples will not be shown in this example. The whole point behind this example is to show you just what an exact differential equation is, how we use this fact to arrive at a solution and why the process works as it does. The majority of the actual solution details will be shown in a later example.

Example 1 Solve the following differential equation.

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0$$

Solution

Let's start off by supposing that somewhere out there in the world is a function $\Psi(x,y)$ that we can find. For this example the function that we need is

$$\Psi\left(x, y\right) = y^2 + \left(x^2 + 1\right)y - 3x^3$$

Do not worry at this point about where this function came from and how we found it. Finding the function, $\Psi(x,y)$, that is needed for any particular differential equation is where the vast majority of the work for these problems lies. As stated earlier however, the point of this example is to show you why the solution process works rather than showing you the actual solution process. We will see how to find this function in the next example, so at this point do not worry about how to find it, simply accept that it can be found and that we've done that for this particular differential equation.

Now, take some partial derivatives of the function.

$$\Psi_x = 2xy - 9x^2$$
$$\Psi_y = 2y + x^2 + 1$$

Now, compare these partial derivatives to the differential equation and you'll notice that with these we can now write the differential equation as.

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0 \qquad \textbf{(1)}$$

Now, recall from your multi-variable calculus class (probably Calculus III) that (1) is nothing more than the following derivative (you'll need the multi-variable chain rule for this…).

$$\frac{d}{dx}\left(\Psi\left(x, y\left(x\right)\right)\right)$$

So, the differential equation can now be written as

$$\frac{d}{dx}\left(\Psi\left(x, y\left(x\right)\right)\right)=0$$

Now, if the ordinary (not partial…) derivative of something is zero, that something must have been a constant to start with. In other words, we've got to have $\Psi\left(x, y\right)=c$. Or,

$$y^2+\left(x^2+1\right)y-3x^3=c$$

This then is an implicit solution for our differential equation! If we had an initial condition we could solve for *c*. We could also find an explicit solution if we wanted to, but we'll hold off on that until the next example.

Okay, so what did we learn from the last example? Let's look at things a little more generally. Suppose that we have the following differential equation.

$$M\left(x, y\right)+N\left(x, y\right)\frac{dy}{dx}=0 \qquad \textbf{(2)}$$

Note that it's important that it be in this form! There must be an "= 0" on one side and the sign separating the two terms must be a "+". Now, if there is a function somewhere out there in the world, $\Psi(x,y)$, so that,

$$\Psi_x=M\left(x, y\right) \qquad \text{and} \qquad \Psi_y=N\left(x, y\right)$$

then we call the differential equation **exact**. In these cases we can write the differential equation as

$$\Psi_x+\Psi_y\frac{dy}{dx}=0 \qquad \textbf{(3)}$$

Then using the chain rule from Calculus III we can further reduce the differential equation to the following derivative,

$$\frac{d}{dx}\left(\Psi\left(x, y\left(x\right)\right)\right)=0$$

The (implicit) solution to an exact differential equation is then

$$\Psi\left(x, y\right)=c \qquad \textbf{(4)}$$

Well, it's the solution provided we can find $\Psi(x,y)$ anyway. Therefore, once we have the function we can always just jump straight to (4) to get an implicit solution to our differential equation.

Finding the function $\Psi(x,y)$ is clearly the central task in determining if a differential equation is exact and in finding its solution. As we will see, finding $\Psi(x,y)$ can be a somewhat lengthy process in which there is the chance of mistakes. Therefore, it would be nice if there was some simple test that we could use before even starting to see if a differential equation is exact or not. This will be especially useful if it turns out that the differential equation is not exact, since in this case $\Psi(x,y)$ will not exist. It would be a waste of time to try and find a nonexistent function!

So, let's see if we can find a test for exact differential equations. Let's start with (2) and assume that the differential equation is in fact exact. Since it's exact we know that somewhere out there is a function $\Psi(x,y)$ that satisfies

$$\Psi_x = M$$
$$\Psi_y = N$$

Now, provided $\Psi(x,y)$ is continuous and its first order derivatives are also continuous we know that

$$\Psi_{xy} = \Psi_{yx}$$

However, we also have the following.

$$\Psi_{xy} = \left(\Psi_x\right)_y = \left(M\right)_y = M_y$$
$$\Psi_{yx} = \left(\Psi_y\right)_x = \left(N\right)_x = N_x$$

Therefore, if a differential equation is exact and $\Psi(x,y)$ meets all of its continuity conditions we must have.

$$M_y = N_x \tag{5}$$

Likewise if (5) is not true there is no way for the differential equation to be exact.

Therefore, we will use (5) as a test for exact differential equations. If (5) is true we will assume that the differential equation is exact and that $\Psi(x,y)$ meets all of its continuity conditions and proceed with finding it. Note that for all the examples here the continuity conditions will be met and so this won't be an issue.

Okay, let's go back and rework the first example. This time we will use the example to show how to find $\Psi(x,y)$. We'll also add in an initial condition to the problem.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0, \qquad y(0) = -3$$

Solution

First identify M and N and check that the differential equation is exact.

$$M = 2xy - 9x^2 \qquad M_y = 2x$$
$$N = 2y + x^2 + 1 \qquad N_x = 2x$$

So, the differential equation is exact according to the test. However, we already knew that as we have given you $\Psi(x,y)$. It's not a bad thing to verify it however and to run through the test at least once however.

Now, how do we actually find $\Psi(x,y)$? Well recall that

$$\Psi_x = M$$
$$\Psi_y = N$$

We can use either of these to get a start on finding $\Psi(x,y)$ by integrating as follows.

$$\Psi = \int M\,dx \qquad \text{OR} \qquad \Psi = \int N\,dy$$

However, we will need to be careful as this won't give us the exact function that we need. Often it doesn't matter which one you choose to work with while in other problems one will be significantly easier than the other. In this case it doesn't matter which one we use as either will be just as easy.

So, I'll use the first one.

$$\Psi\left(x,y\right) = \int 2xy - 9x^2\,dx = x^2 y - 3x^3 + h\left(y\right)$$

Note that in this case the "constant" of integration is not really a constant at all, but instead it will be a function of the remaining variable(s), y in this case.

Recall that in integration we are asking what function we differentiated to get the function we are integrating. Since we are working with two variables here and talking about partial differentiation with respect to x, this means that any term that contained only constants or y's would have differentiated away to zero, therefore we need to acknowledge that fact by adding on a function of y instead of the standard c.

Okay, we've got most of $\Psi(x,y)$ we just need to determine $h(y)$ and we'll be done. This is actually easy to do. We used $\Psi_x = M$ to find most of $\Psi(x,y)$ so we'll use $\Psi_y = N$ to find $h(y)$. Differentiate our $\Psi(x,y)$ with respect to y and set this equal to N (since they must be equal after all). Don't forget to "differentiate" $h(y)$! Doing this gives,

$$\Psi_y = x^2 + h'\left(y\right) = 2y + x^2 + 1 = N$$

From this we can see that

$$h'\left(y\right) = 2y + 1$$

Note that at this stage $h(y)$ must be only a function of y and so if there are any x's in the equation at this stage we have made a mistake somewhere and it's time to go look for it.

We can now find $h(y)$ by integrating.

$$h\left(y\right) = \int 2y + 1\,dy = y^2 + y + k$$

You'll note that we included the constant of integration, k, here. It will turn out however that this will end up getting absorbed into another constant so we can drop it in general.

So, we can now write down $\Psi(x,y)$.

$$\Psi\left(x,y\right) = x^2 y - 3x^3 + y^2 + y + k = y^2 + \left(x^2 + 1\right) y - 3x^3 + k$$

With the exception of the k this is identical to the function that we used in the first example. We can now go straight to the implicit solution using (4).

$$y^2 + \left(x^2 + 1\right) y - 3x^3 + k = c$$

We'll now take care of the *k*. Since both *k* and *c* are unknown constants all we need to do is subtract one from both sides and combine and we still have an unknown constant.

$$y^2 + \left(x^2+1\right)y - 3x^3 = c - k$$
$$y^2 + \left(x^2+1\right)y - 3x^3 = c$$

Therefore, we'll not include the *k* in anymore problems.

This is where we left off in the first example. Let's now apply the initial condition to find *c*.

$$\left(-3\right)^2 + \left(0+1\right)\left(-3\right) - 3\left(0\right)^3 = c \qquad \Rightarrow \qquad c = 6$$

The implicit solution is then.

$$y^2 + \left(x^2+1\right)y - 3x^3 - 6 = 0$$

Now, as we saw in the separable differential equation section, this is quadratic in *y* and so we can solve for *y(x)* by using the quadratic formula.

$$\begin{aligned} y\left(x\right) &= \frac{-\left(x^2+1\right) \pm \sqrt{\left(x^2+1\right)^2 - 4\left(1\right)\left(-3x^3-6\right)}}{2\left(1\right)} \\ &= \frac{-\left(x^2+1\right) \pm \sqrt{x^4+12x^3+2x^2+25}}{2} \end{aligned}$$

Now, reapply the initial condition to figure out which of the two signs in the $\pm$ that we need.

$$-3 = y\left(0\right) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3, 2$$

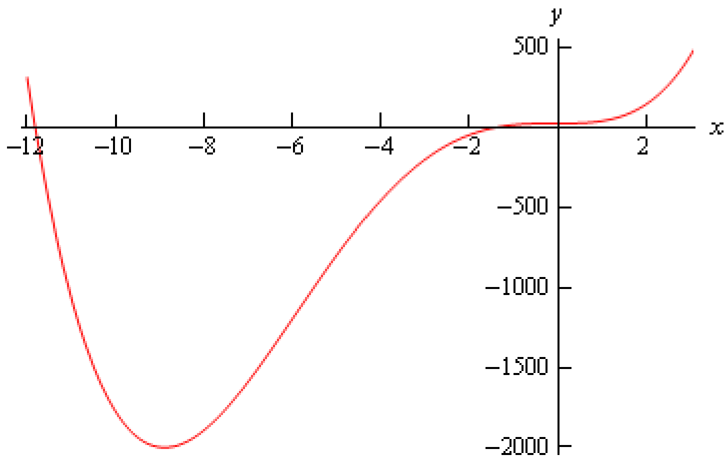
So, it looks like the "-" is the one that we need. The explicit solution is then.

$$y\left(x\right) = \frac{-\left(x^2+1\right) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

Now, for the interval of validity. It looks like we might well have problems with square roots of negative numbers. So, we need to solve

$$x^4+12x^3+2x^2+25 = 0$$

Upon solving this equation is zero at $x = -11.81557624$ and $x = -1.396911133$. Note that you'll need to use some form of computational aid in solving this equation. Here is a graph of the polynomial under the radical.



So, it looks like there are two intervals where the polynomial will be positive.

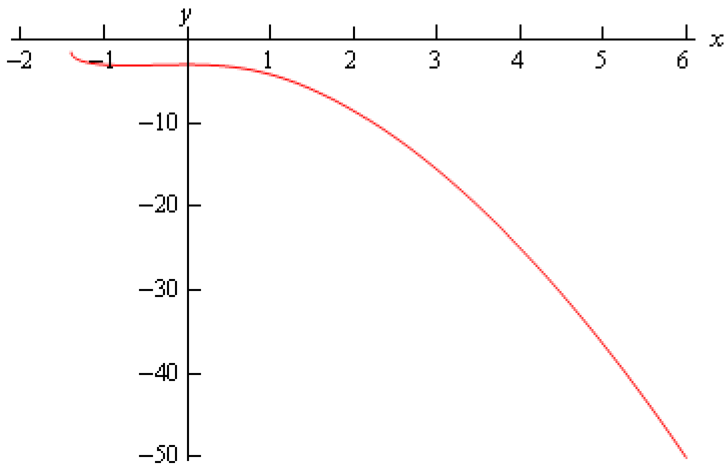
$$-\infty < x \le -11.81557624$$

$$-1.396911133 \le x < \infty$$

However, recall that intervals of validity need to be continuous intervals and contain the value of x that is used in the initial condition. Therefore the interval of validity must be.

$$-1.396911133 \le x < \infty$$

Here is a quick graph of the solution.



That was a long example, but mostly because of the initial explanation of how to find $\Psi(x,y)$. The remaining examples will not be as long.

Example 3 Find the solution and interval of validity for the following IVP.

$$2xy^2 + 4 = 2\left(3 - x^2 y\right) y' \qquad\qquad y\left(-1\right) = 8$$

Solution

Here, we first need to put the differential equation into proper form before proceeding. Recall that it needs to be "= 0" and the sign separating the two terms must be a plus!

$$2xy^2 + 4 - 2\left(3 - x^2 y\right) y' = 0$$

$$2xy^2 + 4 + 2\left(x^2 y - 3\right) y' = 0$$

So we have the following

$$M = 2xy^2 + 4 \qquad\qquad M_y = 4xy$$

$$N = 2x^2 y - 6 \qquad\qquad N_x = 4xy$$

and so the differential equation is exact. We can either integrate M with respect to x or integrate N with respect to y. In this case either would be just as easy so we'll integrate N this time so we can say that we've got an example of both down here.

$$\Psi\left(x, y\right) = \int 2x^2 y - 6\, dy = x^2 y^2 - 6y + h\left(x\right)$$

This time, as opposed to the previous example, our "constant" of integration must be a function of x since we integrated with respect to y. Now differentiate with respect to x and compare this to M.

$$\Psi_x = 2xy^2 + h'\left(x\right) = 2xy^2 + 4 = M$$

So, it looks like

$$h'\left(x\right) = 4 \qquad \Rightarrow \qquad h\left(x\right) = 4x$$

Again, we'll drop the constant of integration that technically should be present in *h(x)* since it will just get absorbed into the constant we pick up in the next step. Also note that, *h(x)* should only involve x's at this point. If there are any y's left at this point a mistake has been made so go back and look for it.

Writing everything down gives us the following for $\Psi(x,y)$.

$$\Psi\left(x, y\right) = x^2 y^2 - 6y + 4x$$

So, the implicit solution to the differential equation is

$$x^2 y^2 - 6y + 4x = c$$

Applying the initial condition gives,

$$64 - 48 - 4 = c \qquad\qquad c = 12$$

The solution is then

$$x^2 y^2 - 6y + 4x - 12 = 0$$

Using the quadratic formula gives us

$$y(x) = \frac{6 \pm \sqrt{36 - 4x^2(4x - 12)}}{2x^2}$$
$$= \frac{6 \pm \sqrt{36 + 48x^2 - 16x^3}}{2x^2}$$
$$= \frac{6 \pm 2\sqrt{9 + 12x^2 - 4x^3}}{2x^2}$$
$$= \frac{3 \pm \sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

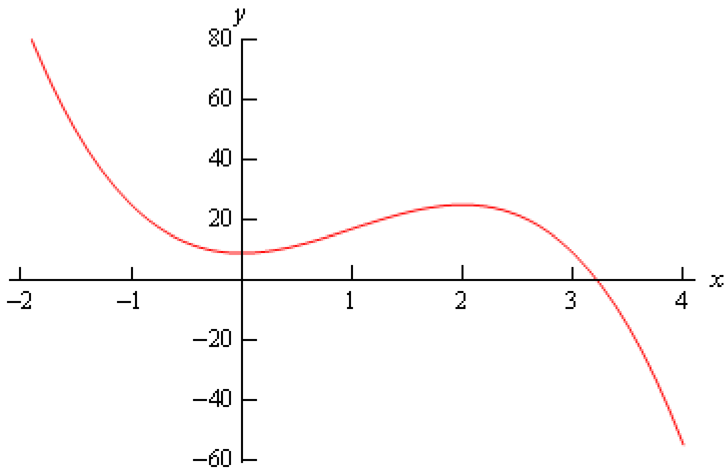
Reapplying the initial condition shows that this time we need the "+" (we'll leave those details to you to check). Therefore, the explicit solution is

$$y(x) = \frac{3 + \sqrt{9 + 12x^2 - 4x^3}}{x^2}$$

Now let's find the interval of validity. We'll need to avoid $x = 0$ so we don't get division by zero. We'll also have to watch out for square roots of negative numbers so solve the following equation.

$$-4x^3 + 12x^2 + 9 = 0$$

The only real solution here is $x = 3.217361577$. Below is a graph of the polynomial.



So, it looks like the polynomial will be positive, and hence okay under the square root on

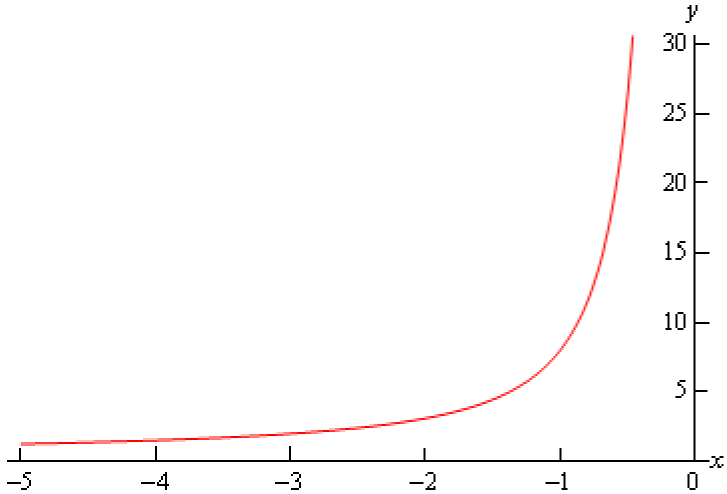
$$-\infty < x < 3.217361577$$

Now, this interval can't be the interval of validity because it contains $x = 0$ and we need to avoid that point. Therefore, this interval actually breaks up into two different possible intervals of validity.

$$-\infty < x < 0$$
$$0 < x < 3.217361577$$

The first one contains $x = -1$, the x value from the initial condition. Therefore, the interval of validity for this problem is $-\infty < x < 0$.

Here is a graph of the solution.



Example 4 Find the solution and interval of validity to the following IVP.

$$\frac{2ty}{t^2+1} - 2t - \left(2 - \ln\left(t^2+1\right)\right) y' = 0 \qquad\qquad y(5) = 0$$

Solution

So, first deal with that minus sign separating the two terms.

$$\frac{2ty}{t^2+1} - 2t + \left(\ln\left(t^2+1\right) - 2\right) y' = 0$$

Now, find M and N and check that it's exact.

$$M = \frac{2ty}{t^2+1} - 2t \qquad\qquad M_y = \frac{2t}{t^2+1}$$

$$N = \ln\left(t^2+1\right) - 2 \qquad\qquad N_t = \frac{2t}{t^2+1}$$

So, it's exact. We'll integrate the first one in this case.

$$\Psi(t,y) = \int \frac{2ty}{t^2+1} - 2t \, dt = y\ln\left(t^2+1\right) - t^2 + h(y)$$

Differentiate with respect to y and compare to N.

$$\Psi_y = \ln\left(t^2+1\right) + h'(y) = \ln\left(t^2+1\right) - 2 = N$$

So, it looks like we've got.

$$h'(y) = -2 \qquad \Rightarrow \qquad h(y) = -2y$$

This gives us

$$\Psi(t,y) = y\ln\left(t^2+1\right) - t^2 - 2y$$

The implicit solution is then,

$$y\ln\left(t^2+1\right)-t^2-2y=c$$

Applying the initial condition gives,

$$-25=c$$

The implicit solution is now,

$$y\left(\ln\left(t^2+1\right)-2\right)-t^2=-25$$

This solution is much easier to solve than the previous ones. No quadratic formula is needed this time, all we need to do is solve for *y*. Here's what we get for an explicit solution.

$$y(t)=\frac{t^2-25}{\ln\left(t^2+1\right)-2}$$

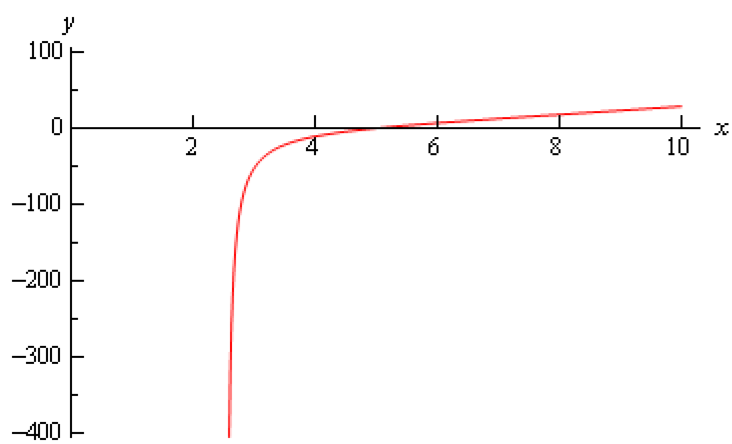
Alright, let's get the interval of validity. The term in the logarithm is always positive so we don't need to worry about negative numbers in that. We do need to worry about division by zero however. We will need to avoid the following point(s).

$$\begin{aligned}\ln\left(t^2+1\right)-2&=0\\ \ln\left(t^2+1\right)&=2\\ t^2+1&=\mathbf{e}^2\\ t&=\pm\sqrt{\mathbf{e}^2-1}\end{aligned}$$

We now have three possible intervals of validity.

$$\begin{aligned}-\infty<t&<-\sqrt{\mathbf{e}^2-1}\\ -\sqrt{\mathbf{e}^2-1}<t&<\sqrt{\mathbf{e}^2-1}\\ \sqrt{\mathbf{e}^2-1}<t&<\infty\end{aligned}$$

The last one contains $t = 5$ and so is the interval of validity for this problem is $\sqrt{\mathbf{e}^2-1}<t<\infty$. Here's a graph of the solution.



Example 5 Find the solution and interval of validity for the following IVP.

$$3y^3\mathbf{e}^{3xy} - 1 + \left(2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy}\right)y' = 0 \qquad\qquad y(0) = 1$$

Solution

Let's identify *M* and *N* and check that it's exact.

$$M = 3y^3\mathbf{e}^{3xy} - 1 \qquad\qquad M_y = 9y^2\mathbf{e}^{3xy} + 9xy^3\mathbf{e}^{3xy}$$

$$N = 2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy} \qquad\qquad N_x = 9y^2\mathbf{e}^{3xy} + 9xy^3\mathbf{e}^{3xy}$$

So, it's exact. With the proper simplification integrating the second one isn't too bad. However, the first is already set up for easy integration so let's do that one.

$$\Psi(x,y) = \int 3y^3\mathbf{e}^{3xy} - 1\,dx = y^2\mathbf{e}^{3xy} - x + h(y)$$

Differentiate with respect to *y* and compare to *N*.

$$\Psi_y = 2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy} + h'(y) = 2y\mathbf{e}^{3xy} + 3xy^2\mathbf{e}^{3xy} = N$$

So, it looks like we've got

$$h'(y) = 0 \qquad \Rightarrow \qquad h(y) = 0$$

Recall that actually *h(y)* = *k*, but we drop the *k* because it will get absorbed in the next step. That gives us *h(y)* = 0. Therefore, we get.

$$\Psi(x,y) = y^2\mathbf{e}^{3xy} - x$$

The implicit solution is then

$$y^2\mathbf{e}^{3xy} - x = c$$

Apply the initial condition.

$$1 = c$$

The implicit solution is then

$$y^2\mathbf{e}^{3xy} - x = 1$$

This is as far as we can go. There is no way to solve this for *y* and get an explicit solution.

Bernoulli Differential Equations

In this section we are going to take a look at differential equations in the form,

$$y' + p(x)\,y = q(x)\,y^n$$

where *p(x)* and *q(x)* are continuous functions on the interval we're working on and *n* is a real number. Differential equations in this form are called **Bernoulli Equations**.

First notice that if $n = 0$ or $n = 1$ then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of *n* other than these two.

In order to solve these we'll first divide the differential equation by y^n to get,

$$y^{-n}\,y' + p(x)\,y^{1-n} = q(x)$$

We are now going to use the substitution $v = y^{1-n}$ to convert this into a differential equation in terms of *v*. As we'll see this will lead to a differential equation that we can solve.

We are going to have to be careful with this however when it comes to dealing with the derivative, y'. We need to determine just what y' is in terms of our substitution. This is easier to do than it might at first look to be. All that we need to do is differentiate both sides of our substitution with respect to *x*. Remember that both *v* and *y* are functions of *x* and so we'll need to use the chain rule on the right side. If you remember your Calculus I you'll recall this is just implicit differentiation. So, taking the derivative gives us,

$$v' = (1-n)\,y^{-n}\,y'$$

Now, plugging this as well as our substitution into the differential equation gives,

$$\frac{1}{1-n}v' + p(x)\,v = q(x)$$

This is a linear differential equation that we can solve for *v* and once we have this in hand we can also get the solution to the original differential equation by plugging *v* back into our substitution and solving for *y*.

Let's take a look at an example.

Example 1 Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{4}{x}y = x^3y^2 \qquad y(2) = -1, \qquad x > 0$$

Solution

So, the first thing that we need to do is get this into the "proper" form and that means dividing everything by y^2. Doing this gives,

$$y^{-2}\,y' + \frac{4}{x}y^{-1} = x^3$$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \qquad\qquad v' = -y^{-2}y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x}v = x^3$$

So, as noted above this is a linear differential equation that we know how to solve. We'll do the details on this one and then for the rest of the examples in this section we'll leave the details for you to fill in. If you need a refresher on solving linear differential equations then go back to that section for a quick review.

Here's the solution to this differential equation.

$$v' - \frac{4}{x}v = -x^3 \qquad \Rightarrow \qquad \mu(x) = \mathbf{e}^{\int -\frac{4}{x}dx} = \mathbf{e}^{-4\ln|x|} = x^{-4}$$

$$\int \left(x^{-4}v\right)' dx = \int -x^{-1}\,dx$$

$$x^{-4}v = -\ln|x| + c \qquad \Rightarrow \qquad v(x) = cx^4 - x^4\ln x$$

Note that we dropped the absolute value bars on the *x* in the logarithm because of the assumption that $x > 0$.

Now we need to determine the constant of integration. This can be done in one of two ways. We can can convert the solution above into a solution in terms of *y* and then use the original initial condition or we can convert the initial condition to an initial condition in terms of *v* and use that. Because we'll need to convert the solution to *y*'s eventually anyway and it won't add that much work in we'll do it that way.

So, to get the solution in terms of *y* all we need to do is plug the substitution back in. Doing this gives,

$$y^{-1} = x^4\left(c - \ln x\right)$$

At this point we can solve for *y* and then apply the initial condition or apply the initial condition and then solve for *y*. We'll generally do this with the later approach so let's apply the initial condition to get,

$$(-1)^{-1} = c2^4 - 2^4\ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for *c* and solving for *y* gives,

$$y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)} = \frac{-16}{x^4\left(1 + 16\ln x - 16\ln 2\right)} = \frac{-16}{x^4\left(1 + 16\ln\frac{x}{2}\right)}$$

Note that we did a little simplification in the solution. This will help with finding the interval of validity.

Before finding the interval of validity however, we mentioned above that we could convert the original initial condition into an initial condition for *v*. Let's briefly talk about how to do that. To do that all we need to do is plug $x = 2$ into the substitution and then use the original initial condition. Doing this gives,

$$v(2) = y^{-1}(2) = (-1)^{-1} = -1$$

So, in this case we got the same value for *v* that we had for *y*. Don't expect that to happen in general if you chose to do the problems in this manner.

Okay, let's now find the interval of validity for the solution. First we already know that $x > 0$ and that means we'll avoid the problems of having logarithms of negative numbers and division by zero at $x = 0$. So, all that we need to worry about then is division by zero in the second term and this will happen where,

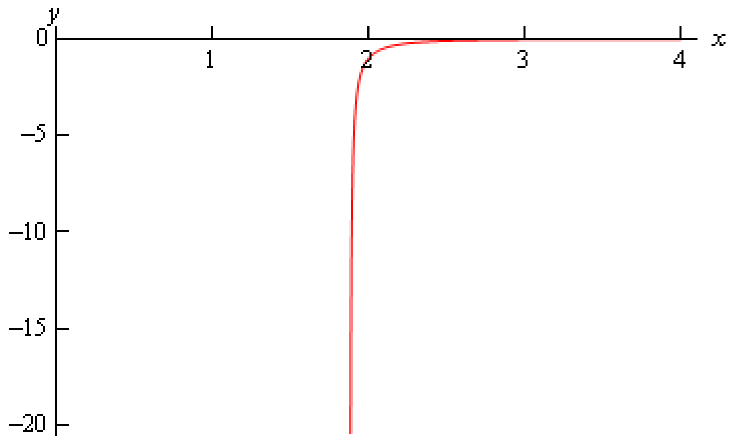
$$\begin{aligned} 1 + 16 \ln \tfrac{x}{2} &= 0 \\ \ln \tfrac{x}{2} &= -\tfrac{1}{16} \\ \tfrac{x}{2} &= \mathbf{e}^{-\frac{1}{16}} \qquad \Rightarrow \qquad x = 2\mathbf{e}^{-\frac{1}{16}} \approx 1.8788 \end{aligned}$$

The two possible intervals of validity are then,

$$0 < x < 2\mathbf{e}^{-\frac{1}{16}} \qquad\qquad 2\mathbf{e}^{-\frac{1}{16}} < x < \infty$$

and since the second one contains the initial condition we know that the interval of validity is then $2\mathbf{e}^{-\frac{1}{16}} < x < \infty$.

Here is a graph of the solution.



Let's do a couple more examples and as noted above we're going to leave it to you to solve the linear differential equation when we get to that stage.

Example 2 Solve the following IVP and find the interval of validity for the solution.

$$y' = 5y + \mathbf{e}^{-2x} y^{-2} \qquad\qquad y(0) = 2$$

Solution

The first thing we'll need to do here is multiply through by y^2 and we'll also do a little rearranging to get things into the form we'll need for the linear differential equation. This gives,

$$y^2 y' - 5y^3 = \mathbf{e}^{-2x}$$

The substitution here and its derivative is,

$$v = y^3 \qquad\qquad v' = 3y^2 y'$$

Plugging the substitution into the differential equation gives,

$$\tfrac{1}{3} v' - 5v = \mathbf{e}^{-2x} \qquad \Rightarrow \qquad v' - 15v = 3\mathbf{e}^{-2x} \qquad\qquad \mu(x) = \mathbf{e}^{-15x}$$

We rearranged a little and gave the integrating factor for the linear differential equation solution. Upon solving we get,

$$v(x) = c\mathbf{e}^{15x} - \tfrac{3}{17}\mathbf{e}^{-2x}$$

Now go back to *y*'s.

$$y^3 = c\mathbf{e}^{15x} - \tfrac{3}{17}\mathbf{e}^{-2x}$$

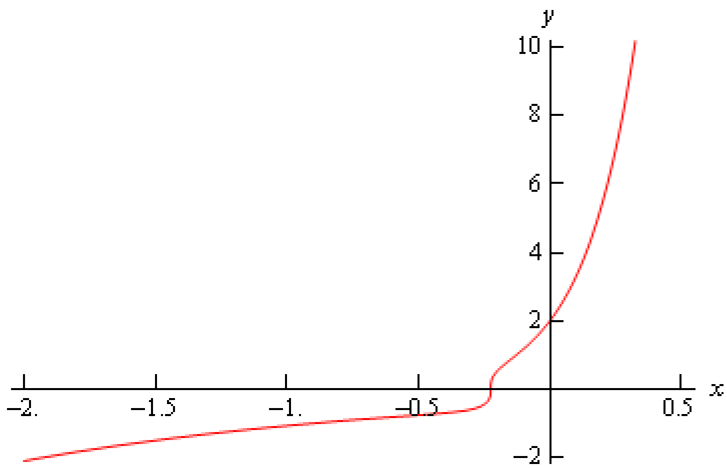
Applying the initial condition and solving for *c* gives,

$$8 = c - \tfrac{3}{17} \qquad \Rightarrow \qquad c = \tfrac{139}{17}$$

Plugging in *c* and solving for *y* gives,

$$y(x) = \left(\frac{139\mathbf{e}^{15x} - 3\mathbf{e}^{-2x}}{17}\right)^{\frac{1}{3}}$$

There are no problem values of *x* for this solution and so the interval of validity is all real numbers. Here's a graph of the solution.



Example 3 Solve the following IVP and find the interval of validity for the solution.

$$6y' - 2y = x\,y^4 \qquad y(0) = -2$$

Solution

First get the differential equation in the proper form and then write down the substitution.

$$6y^{-4}y' - 2y^{-3} = x \qquad \Rightarrow \qquad v = y^{-3} \qquad v' = -3y^{-4}y'$$

Plugging the substitution into the differential equation gives,

$$-2v' - 2v = x \qquad \Rightarrow \qquad v' + v = -\tfrac{1}{2}x \qquad \mu(x) = \mathbf{e}^{x}$$

Again, we've rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

$$v(x) = -\tfrac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now back substitute to get back into y's.

$$y^{-3} = -\tfrac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now we need to apply the initial condition and solve for c.

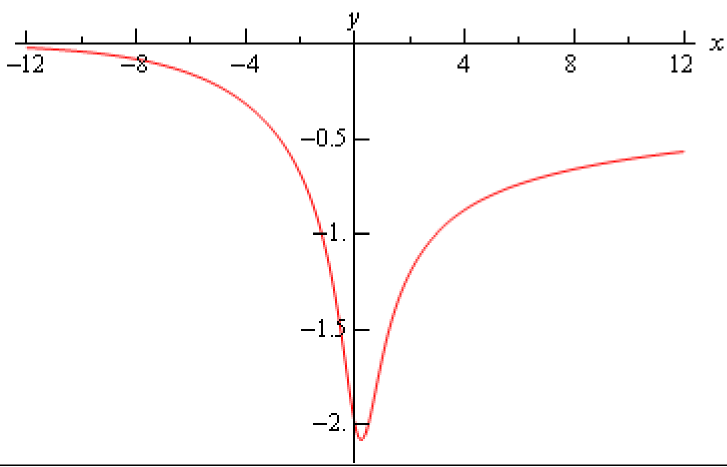
$$-\tfrac{1}{8} = \tfrac{1}{2} + c \qquad \Rightarrow \qquad c = -\tfrac{5}{8}$$

Plugging in c and solving for y gives,

$$y(x) = -\frac{2}{\left(4x - 4 + 5\mathbf{e}^{-x}\right)^{\frac{1}{3}}}$$

Next, we need to think about the interval of validity. In this case all we need to worry about it is division by zero issues and using some form of computational aid (such as Maple or Mathematica) we will see that the denominator of our solution is never zero and so this solution will be valid for all real numbers.

Here is a graph of the solution.



To this point we've only worked examples in which n was an integer (positive and negative) and so we should work a quick example where n is not an integer.

Example 4 Solve the following IVP and find the interval of validity for the solution.

$$y' + \frac{y}{x} - \sqrt{y} = 0 \qquad\qquad y(1) = 0$$

Solution

Let's first get the differential equation into proper form.

$$y' + \frac{1}{x}y = y^{\frac{1}{2}} \qquad \Rightarrow \qquad y^{-\frac{1}{2}}y' + \frac{1}{x}y^{\frac{1}{2}} = 1$$

The substitution is then,

$$v = y^{\frac{1}{2}} \qquad\qquad v' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

Now plug the substitution into the differential equation to get,

$$2v' + \frac{1}{x}v = 1 \qquad \Rightarrow \qquad v' + \frac{1}{2x}v = \frac{1}{2} \qquad\qquad \mu(x) = x^{\frac{1}{2}}$$

As we've done with the previous examples we've done some rearranging and given the integrating factor needed for solving the linear differential equation. Solving this gives us,

$$v(x) = \tfrac{1}{3}x + cx^{-\frac{1}{2}}$$

In terms of *y* this is,

$$y^{\frac{1}{2}} = \tfrac{1}{3}x + cx^{-\frac{1}{2}}$$

Applying the initial condition and solving for *c* gives,

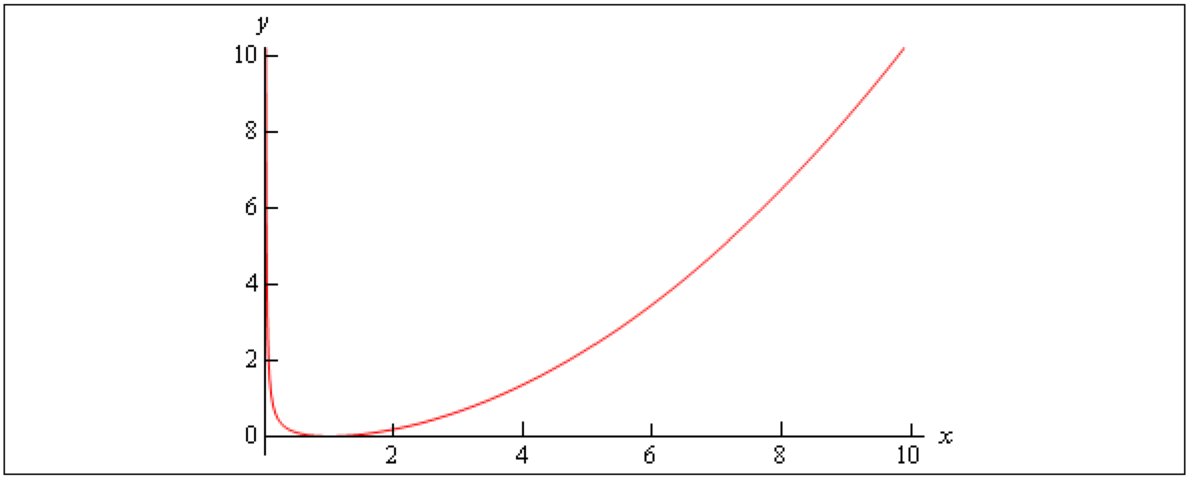
$$0 = \tfrac{1}{3} + c \qquad \Rightarrow \qquad c = -\tfrac{1}{3}$$

Plugging in for *c* and solving for *y* gives us the solution.

$$y(x) = \left(\tfrac{1}{3}x - \tfrac{1}{3}x^{-\frac{1}{2}}\right)^2 = \frac{x^3 - 2x^{\frac{3}{2}} + 1}{9x}$$

Note that we multiplied everything out and converted all the negative exponents to positive exponents to make the interval of validity clear here. Because of the root (in the second term in the numerator) and the *x* in the denominator we can see that we need to require $x > 0$ in order for the solution to exist and it will exist for all positive *x*'s and so this is also the interval of validity.

Here is the graph of the solution.


y
10
8
6
4
2
0
2
4
6
8
10
x

Substitutions

In the previous section we looked at Bernoulli Equations and saw that in order to solve them we needed to use the substitution $v = y^{1-n}$. Upon using this substitution we were able to convert the differential equation into a form that we could deal with (linear in this case). In this section we want to take a look at a couple of other substitutions that can be used to reduce some differential equations down to a solvable form.

The first substitution we'll take a look at will require the differential equation to be in the form,

$$y' = F\left(\frac{y}{x}\right)$$

First order differential equations that can be written in this form are called **homogeneous** differential equations. Note that we will usually have to do some rewriting in order to put the differential equation into the proper form.

Once we have verified that the differential equation is a homogeneous differential equation and we've gotten it written in the proper form we will use the following substitution.

$$v(x) = \frac{y}{x}$$

We can then rewrite this as,

$$y = xv$$

and then remembering that both *y* and *v* are functions of *x* we can use the product rule (recall that is implicit differentiation from Calculus I) to compute,

$$y' = v + xv'$$

Under this substitution the differential equation is then,

$$v + xv' = F(v)$$

$$xv' = F(v) - v \qquad \Rightarrow \qquad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

As we can see with a small rewrite of the new differential equation we will have a separable differential equation after the substitution.

Let's take a quick look at a couple of examples of this kind of substitution.

Example 1 Solve the following IVP and find the interval of validity for the solution.

$$x\,y\,y' + 4x^2 + y^2 = 0 \qquad\qquad y(2) = -7, \qquad\qquad x > 0$$

Solution

Let's first divide both sides by x^2 to rewrite the differential equation as follows,

$$\frac{y}{x}y' = -4 - \frac{y^2}{x^2} = -4 - \left(\frac{y}{x}\right)^2$$

Now, this is not in the officially proper form as we have listed above, but we can see that everywhere the variables are listed they show up as the ratio, *y*/*x* and so this is really as far as we need to go. So, let's plug the substitution into this form of the differential equation to get,

$$v\left(v + xv'\right) = -4 - v^2$$

Next, rewrite the differential equation to get everything separated out.

$$\begin{aligned} v\,x\,v' &= -4 - 2v^2 \\ x\,v' &= -\frac{4+2v^2}{v} \\ \frac{v}{4+2v^2}\,dv &= -\frac{1}{x}\,dx \end{aligned}$$

Integrating both sides gives,

$$\tfrac{1}{4}\ln\left(4+2v^2\right) = -\ln\left(x\right) + c$$

We need to do a little rewriting using basic logarithm properties in order to be able to easily solve this for *v*.

$$\ln\left(4+2v^2\right)^{\frac{1}{4}} = \ln\left(x\right)^{-1} + c$$

Now exponentiate both sides and do a little rewriting

$$\left(4+2v^2\right)^{\frac{1}{4}} = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^{c}\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x}$$

Note that because c is an unknown constant then so is $\mathbf{e}^{c}$ and so we may as well just call this c as we did above.

Finally, let's solve for v and then plug the substitution back in and we'll play a little fast and loose with constants again.

$$\begin{aligned} 4+2v^2 &= \frac{c^4}{x^4} = \frac{c}{x^4} \\ v^2 &= \tfrac{1}{2}\left(\frac{c}{x^4} - 4\right) \\ \frac{y^2}{x^2} &= \tfrac{1}{2}\left(\frac{c-4x^4}{x^4}\right) \\ y^2 &= \tfrac{1}{2}x^2\left(\frac{c-4x^4}{x^4}\right) = \frac{c-4x^4}{2x^2} \end{aligned}$$

At this point it would probably be best to go ahead and apply the initial condition. Doing that gives,

$$49 = \frac{c-4\left(16\right)}{2\left(4\right)} \qquad \Rightarrow \qquad c = 456$$

Note that we could have also converted the original initial condition into one in terms of v and then applied it upon solving the separable differential equation. In this case however, it was probably a little easier to do it in terms of y given all the logarithms in the solution to the separable differential equation.

Finally, plug in c and solve for y to get,

$$y^2 = \frac{228 - 2x^4}{x^2} \qquad \Rightarrow \qquad y(x) = \pm\sqrt{\frac{228 - 2x^4}{x^2}}$$

The initial condition tells us that the "–" must be the correct sign and so the actual solution is,

$$y(x) = -\sqrt{\frac{228 - 2x^4}{x^2}}$$

For the interval of validity we can see that we need to avoid $x = 0$ and because we can't allow negative numbers under the square root we also need to require that,

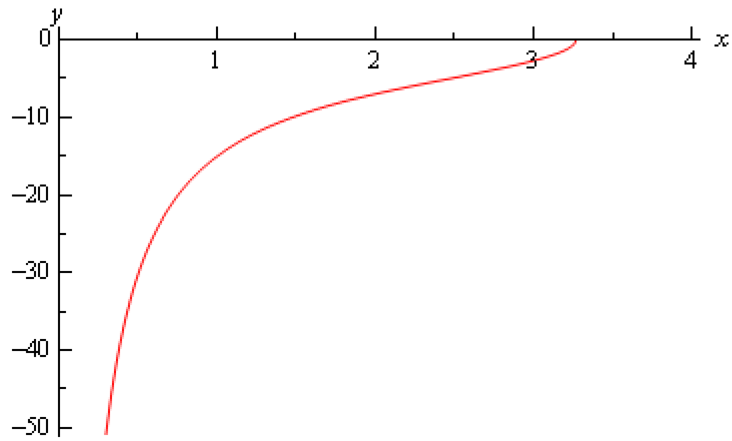
$$\begin{aligned} 228 - 2x^4 &\geq 0 \\ x^4 &\leq 114 \qquad \Rightarrow \qquad -3.2676 \leq x \leq 3.2676 \end{aligned}$$

So, we have two possible intervals of validity,

$$-3.2676 \leq x < 0 \qquad\qquad 0 < x \leq 3.2676$$

and the initial condition tells us that it must be $0 < x \leq 3.2676$.

The graph of the solution is,



Example 2 Solve the following IVP and find the interval of validity for the solution.

$$x\,y' = y\left(\ln x - \ln y\right) \qquad y(1) = 4, \qquad x > 0$$

Solution
On the surface this differential equation looks like it won't be homogeneous. However, with a quick logarithm property we can rewrite this as,

$$y' = \frac{y}{x}\ln\left(\frac{x}{y}\right)$$

In this form the differential equation is clearly homogeneous. Applying the substitution and separating gives,

$$\begin{aligned} v + xv' &= v\ln\left(\frac{1}{v}\right) \\ xv' &= v\left(\ln\left(\frac{1}{v}\right) - 1\right) \\ \frac{dv}{v\left(\ln\left(\frac{1}{v}\right)-1\right)} &= \frac{dx}{x} \end{aligned}$$

Integrate both sides and do a little rewrite to get,

$$\begin{aligned} -\ln\left(\ln\left(\tfrac{1}{v}\right)-1\right) &= \ln x + c \\ \ln\left(\ln\left(\tfrac{1}{v}\right)-1\right) &= c - \ln x \end{aligned}$$

You were able to do the integral on the left right? It used the substitution $u = \ln\left(\frac{1}{v}\right) - 1$.

Now, solve for v and note that we'll need to exponentiate both sides a couple of times and play fast and loose with constants again.

$$\ln\left(\tfrac{1}{v}\right) - 1 = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^{c}\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x}$$

$$\ln\left(\tfrac{1}{v}\right) = \frac{c}{x} + 1$$

$$\frac{1}{v} = \mathbf{e}^{\frac{c}{x}+1} \qquad \Rightarrow \qquad v = \mathbf{e}^{-\frac{c}{x}-1}$$

Plugging the substitution back in and solving for y gives,

$$\frac{y}{x} = \mathbf{e}^{-\frac{c}{x}-1} \qquad \Rightarrow \qquad y(x) = x\mathbf{e}^{-\frac{c}{x}-1}$$

Applying the initial condition and solving for c gives,

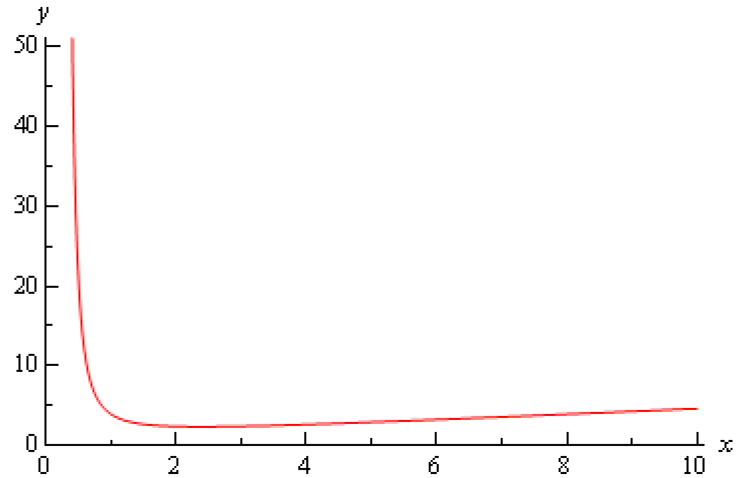
$$4 = \mathbf{e}^{-c-1} \qquad \Rightarrow \qquad c = -(1+\ln 4)$$

The solution is then,

$$y(x) = x\mathbf{e}^{\frac{1+\ln 4}{x}-1}$$

We clearly need to avoid $x = 0$ to avoid division by zero and so with the initial condition we can see that the interval of validity is $x > 0$.

The graph of the solution is,



For the next substitution we'll take a look at we'll need the differential equation in the form,

$$y' = G(ax + by)$$

In these cases we'll use the substitution,

$$v = ax + by \qquad \Rightarrow \qquad v' = a + by'$$

Plugging this into the differential equation gives,

$$\tfrac{1}{b}(v' - a) = G(v)$$

$$v' = a + bG(v) \qquad \Rightarrow \qquad \frac{dv}{a + bG(v)} = dx$$

So, with this substitution we'll be able to rewrite the original differential equation as a new separable differential equation that we can solve.

Let's take a look at a couple of examples.

Example 3 Solve the following IVP and find the interval of validity for the solution.

$$y' - (4x - y + 1)^2 = 0 \qquad y(0) = 2$$

Solution
In this case we'll use the substitution.

$$v = 4x - y \qquad v' = 4 - y'$$

Note that we didn't include the "+1" in our substitution. Usually only the *ax* + *by* part gets included in the substitution. There are times where including the extra constant may change the difficulty of the solution process, either easier or harder, however in this case it doesn't really make much difference so we won't include it in our substitution.

So, plugging this into the differential equation gives,

$$4 - v' - (v+1)^2 = 0$$

$$v' = 4 - (v+1)^2$$

$$\frac{dv}{(v+1)^2 - 4} = -dx$$

As we've shown above we definitely have a separable differential equation. Also note that to help with the solution process we left a minus sign on the right side. We'll need to integrate both sides and in order to do the integral on the left we'll need to use partial fractions. We'll leave it to you to fill in the missing details and given that we'll be doing quite a bit of partial fraction work in a few chapters you should really make sure that you can do the missing details.

$$\int \frac{dv}{v^2 + 2v - 3} = \int \frac{dv}{(v+3)(v-1)} = \int -dx$$

$$\frac{1}{4}\int \frac{1}{v-1} - \frac{1}{v+3}\,dv = \int -dx$$

$$\tfrac{1}{4}\left(\ln(v-1) - \ln(v+3)\right) = -x + c$$

$$\ln\left(\frac{v-1}{v+3}\right) = c - 4x$$

Note that we played a little fast and loose with constants above. The next step is fairly messy but needs to be done and that is to solve for *v* and note that we'll be playing fast and loose with constants again where we can get away with it and we'll be skipping a few steps that you shouldn't have any problem verifying.

$$\frac{v-1}{v+3} = \mathbf{e}^{c-4x} = c\,\mathbf{e}^{-4x}$$

$$v - 1 = c\,\mathbf{e}^{-4x}(v+3)$$

$$v\left(1 - c\,\mathbf{e}^{-4x}\right) = 1 + 3c\,\mathbf{e}^{-4x}$$

At this stage we should back away a bit and note that we can't play fast and loose with constants anymore. We were able to do that in first step because the *c* appeared only once in the equation. At this point however the *c* appears twice and so we've got to keep them around. If we "absorbed" the 3 into the *c* on the right the "new" *c* would be different from the *c* on the left because the *c* on the left didn't have the 3 as well.

So, let's solve for *v* and then go ahead and go back into terms of *y*.

$$v = \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

$$4x - y = \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

$$y(x) = 4x - \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

The last step is to then apply the initial condition and solve for c.

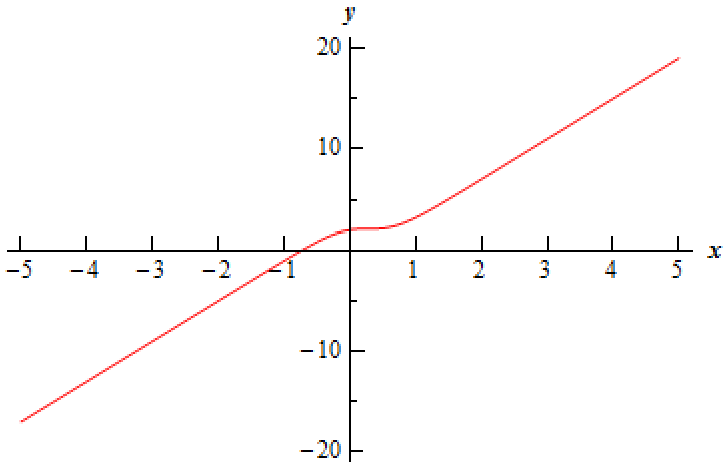
$$2 = y(0) = -\frac{1+3c}{1-c} \qquad \Rightarrow \qquad c = -3$$

The solution is then,

$$y(x) = 4x - \frac{1-9\mathbf{e}^{-4x}}{1+3\mathbf{e}^{-4x}}$$

Note that because exponentials exist everywhere and the denominator of the second term is always positive (because exponentials are always positive and adding a positive one onto that won't change the fact that it's positive) the interval of validity for this solution will be all real numbers.

Here is a graph of the solution.



Example 4 Solve the following IVP and find the interval of validity for the solution.

$$y' = \mathbf{e}^{9y-x} \qquad y(0) = 0$$

Solution

Here is the substitution that we'll need for this example.

$$v = 9y - x \qquad v' = 9y' - 1$$

Plugging this into our differential equation gives,

$$\tfrac{1}{9}(v'+1) = \mathbf{e}^{v}$$

$$v' = 9\mathbf{e}^{v} - 1$$

$$\frac{dv}{9\mathbf{e}^{v}-1} = dx \qquad \Rightarrow \qquad \frac{\mathbf{e}^{-v}\,dv}{9-\mathbf{e}^{-v}} = dx$$

Note that we did a little rewrite on the separated portion to make the integrals go a little easier. By multiplying the numerator and denominator by $\mathbf{e}^{-v}$ we can turn this into a fairly simply substitution integration problem. So, upon integrating both sides we get,

$$\ln\left(9-\mathbf{e}^{-v}\right)=x+c$$

Solving for v gives,

$$\begin{aligned} 9-\mathbf{e}^{-v} &= \mathbf{e}^{c}\mathbf{e}^{x}=c\mathbf{e}^{x} \\ \mathbf{e}^{-v} &= 9-c\mathbf{e}^{x} \\ v &= -\ln\left(9-c\mathbf{e}^{x}\right) \end{aligned}$$

Plugging the substitution back in and solving for y gives us,

$$y(x)=\tfrac{1}{9}\left(x-\ln\left(9-c\mathbf{e}^{x}\right)\right)$$

Next, apply the initial condition and solve for c.

$$0=y(0)=-\tfrac{1}{9}\ln(9-c) \qquad \Rightarrow \qquad c=8$$

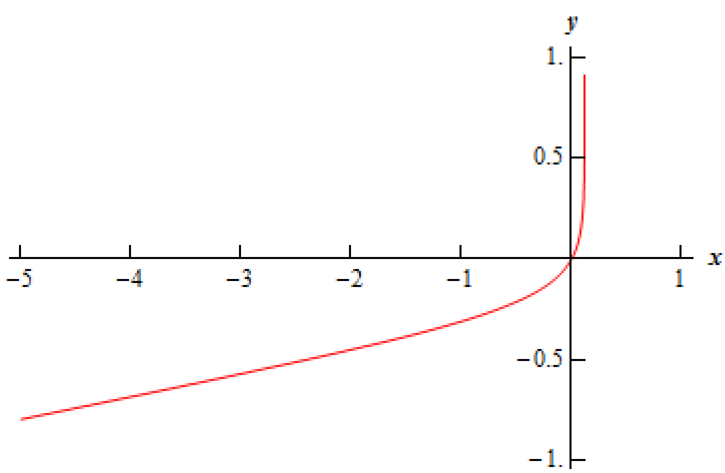
The solution is then,

$$y(x)=\tfrac{1}{9}\left(x-\ln\left(9-8\mathbf{e}^{x}\right)\right)$$

Now, for the interval of validity we need to make sure that we only take logarithms of positive numbers as we'll need to require that,

$$9-8\mathbf{e}^{x}>0 \qquad \Rightarrow \qquad \mathbf{e}^{x}<\tfrac{9}{8} \qquad \Rightarrow \qquad x<\ln\tfrac{9}{8}=0.1178$$

Here is a graph of the solution.



In both this section and the previous section we've seen that sometimes a substitution will take a differential equation that we can't solve and turn it into one that we can solve. This idea of substitutions is an important idea and should not be forgotten. Not every differential equation can be made easier with a substitution and there is no way to show every possible substitution but remembering that a substitution may work is a good thing to do. If you get stuck on a differential equation you may try to see if a substitution if some kind will work for you.

Intervals of Validity

I've called this section Intervals of Validity because all of the examples will involve them. However, there is a lot more to this section. We will see a couple of theorems that will tell us when we can solve a differential equation. We will also see some of the differences between linear and nonlinear differential equations.

First let's take a look at a theorem about linear first order differential equations. This is a very important theorem although we're not going to really use it for its most important aspect.

Theorem 1

Consider the following IVP.

$$y' + p(t)\,y = g(t) \qquad\qquad y(t_0) = y_0$$

If *p(t)* and *g(t)* are continuous functions on an open interval $\alpha < t < \beta$ and the interval contains t_o, then there is a unique solution to the IVP on that interval.

So, just what does this theorem tell us? First, it tells us that for nice enough linear first order differential equations solutions are guaranteed to exist and more importantly the solution will be unique. We may not be able to find the solution, but do know that it exists and that there will only be one of them. This is the very important aspect of this theorem. Knowing that a differential equation has a unique solution is sometimes more important than actually having the solution itself!

Next, if the interval in the theorem is the largest possible interval on which *p(t)* and *g(t)* are continuous then the interval is the interval of validity for the solution. This means, that for linear first order differential equations, we won't need to actually solve the differential equation in order to find the interval of validity. Notice as well that the interval of validity will depend only partially on the initial condition. The interval must contain t_o, but the value of y_o, has no effect on the interval of validity.

Let's take a look at an example.

Example 1 Without solving, determine the interval of validity for the following initial value problem.

$$\left(t^2 - 9\right) y' + 2y = \ln\left|20 - 4t\right| \qquad\qquad y(4) = -3$$

Solution

First, in order to use the theorem to find the interval of validity we must write the differential equation in the proper form given in the theorem. So we will need to divide out by the coefficient of the derivative.

$$y' + \frac{2}{t^2 - 9}\,y = \frac{\ln\left|20 - 4t\right|}{t^2 - 9}$$

Next, we need to identify where the two functions are not continuous. This will allow us to find all possible intervals of validity for the differential equation. So, *p(t)* will be discontinuous at $t = \pm 3$ since these points will give a division by zero. Likewise, *g(t)* will also be discontinuous at $t = \pm 3$ as well as $t = 5$ since at this point we will have the natural logarithm of zero. Note that in this case we won't have to worry about natural log of negative numbers because of the absolute values.

Now, with these points in hand we can break up the real number line into four intervals where both $p(t)$ and $g(t)$ will be continuous. These four intervals are,

$$-\infty < t < -3 \qquad -3 < t < 3 \qquad 3 < t < 5 \qquad 5 < t < \infty$$

The endpoints of each of the intervals are points where at least one of the two functions is discontinuous. This will guarantee that both functions are continuous everywhere in each interval.

Finally, let's identify the actual interval of validity for the initial value problem. The actual interval of validity is the interval that will contain $t_o = 4$. So, the interval of validity for the initial value problem is.

$$3 < t < 5$$

In this last example we need to be careful to not jump to the conclusion that the other three intervals cannot be intervals of validity. By changing the initial condition, in particular the value of t_o, we can make any of the four intervals the interval of validity.

The first theorem required a linear differential equation. There is a similar theorem for non-linear first order differential equations. This theorem is not as useful for finding intervals of validity as the first theorem was so we won't be doing all that much with it.

Here is the theorem.

Theorem 2

Consider the following IVP.

$$y' = f(t,y) \qquad\qquad y(t_0) = y_0$$

If $f(t,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous functions in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_o, y_o) then there is a unique solution to the IVP in some interval $t_o - h < t < t_o + h$ that is contained in $\alpha < t < \beta$.

That's it. Unlike the first theorem, this one cannot really be used to find an interval of validity. So, we will know that a unique solution exists if the conditions of the theorem are met, but we will actually need the solution in order to determine its interval of validity. Note as well that for non-linear differential equations it appears that the value of y_0 may affect the interval of validity.

Here is an example of the problems that can arise when the conditions of this theorem aren't met.

Example 2 Determine all possible solutions to the following IVP.

$$y' = y^{\frac{1}{3}} \qquad y(0) = 0$$

Solution

First, notice that this differential equation does NOT satisfy the conditions of the theorem.

$$f(y) = y^{\frac{1}{3}} \qquad\qquad \frac{df}{dy} = \frac{1}{3y^{\frac{2}{3}}}$$

So, the function is continuous on any interval, but the derivative is not continuous at $y = 0$ and so will not be continuous at any interval containing $y = 0$. In order to use the theorem both must be continuous on an interval that contains $y_o = 0$ and this is problem for us since we do have $y_o = 0$.

Now, let's actually work the problem. This differential equation is separable and is fairly simple to solve.

$$\int y^{-\frac{1}{3}} dy = \int dt$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

Applying the initial condition gives $c = 0$ and so the solution is.

$$\frac{3}{2} y^{\frac{2}{3}} = t$$

$$y^{\frac{2}{3}} = \frac{2}{3} t$$

$$y^2 = \left(\frac{2}{3} t \right)^3$$

$$y(t) = \pm \left(\frac{2}{3} t \right)^{\frac{3}{2}}$$

So, we've got two possible solutions here, both of which satisfy the differential equation and the initial condition. There is also a third solution to the IVP. *$y(t) = 0$* is also a solution to the differential equation and satisfies the initial condition.

In this last example we had a very simple IVP and it only violated one of the conditions of the theorem, yet it had three different solutions. All the examples we've worked in the previous sections satisfied the conditions of this theorem and had a single unique solution to the IVP. This example is a useful reminder of the fact that, in the field of differential equations, things don't always behave nicely. It's easy to forget this as most of the problems that are worked in a differential equations class are nice and behave in a nice, predictable manner.

Let's work one final example that will illustrate one of the differences between linear and non-linear differential equations.

Example 3 Determine the interval of validity for the initial value problem below and give its dependence on the value of y_o

$$y' = y^2 \qquad y(0) = y_0$$

Solution

Before proceeding in this problem, we should note that the differential equation is non-linear and meets both conditions of the Theorem 2 and so there will be a unique solution to the IVP for each possible value of y_o.

Also, note that the problem asks for any dependence of the interval of validity on the value of y_o. This immediately illustrates a difference between linear and non-linear differential equations.

Intervals of validity for linear differential equations do not depend on the value of y_o. Intervals of validity for non-linear differential can depend on the value of y_o as we pointed out after the second theorem.

So, let's solve the IVP and get some intervals of validity.

First note that if $y_o = 0$ then $y(t) = 0$ is the solution and this has an interval of validity of

$$-\infty < t < \infty$$

So for the rest of the problem let's assume that $y_0 \neq 0$. Now, the differential equation is separable so let's solve it and get a general solution.

$$\int y^{-2} dy = \int dt$$

$$-\frac{1}{y} = t + c$$

Applying the initial condition gives

$$c = -\frac{1}{y_0}$$

The solution is then.

$$-\frac{1}{y} = t - \frac{1}{y_0}$$

$$y(t) = \frac{1}{\frac{1}{y_0} - t}$$

$$y(t) = \frac{y_0}{1 - y_0 t}$$

Now that we have a solution to the initial value problem we can start finding intervals of validity. From the solution we can see that the only problem that we'll have is division by zero at

$$t = \frac{1}{y_0}$$

This leads to two possible intervals of validity.

$$-\infty < t < \frac{1}{y_0}$$

$$\frac{1}{y_0} < t < \infty$$

That actual interval of validity will be the interval that contains $t_o = 0$. This however, depends on the value of y_o. If $y_o < 0$ then $\frac{1}{y_0} < 0$ and so the second interval will contain $t_o = 0$. Likewise if $y_o > 0$ then $\frac{1}{y_0} > 0$ and in this case the first interval will contain $t_o = 0$.

This leads to the following possible intervals of validity, depending on the value of y_o.

If $y_0 > 0$ $\qquad -\infty < t < \frac{1}{y_0}$ is the interval of validity.

If $y_0 = 0$ $\qquad -\infty < t < \infty$ is the interval of validity.

If $y_0 < 0$ $\qquad \frac{1}{y_0} < t < \infty$ is the interval of validity.

On a side note, notice that the solution, in its final form, will also work if $y_o = 0$.

So what did this example show us about the difference between linear and non-linear differential equations?

First, as pointed out in the solution to the example, intervals of validity for non-linear differential equations can depend on the value of y_o, whereas intervals of validity for linear differential equations don't.

Second, intervals of validity for linear differential equations can be found from the differential equation with no knowledge of the solution. This is definitely not the case with non-linear differential equations. It would be very difficult to see how any of these intervals in the last example could be found from the differential equation. Knowledge of the solution was required in order for us to find the interval of validity.

Modeling with First Order Differential Equations

We now move into one of the main applications of differential equations both in this class and in general. Modeling is the process of writing a differential equation to describe a physical situation. Almost all of the differential equations that you will use in your job (for the engineers out there in the audience) are there because somebody, at some time, modeled a situation to come up with the differential equation that you are using.

This section is not intended to completely teach you how to go about modeling all physical situations. A whole course could be devoted to the subject of modeling and still not cover everything! This section is designed to introduce you to the process of modeling and show you what is involved in modeling. We will look at three different situations in this section : Mixing Problems, Population Problems, and Falling Bodies.

In all of these situations we will be forced to make assumptions that do not accurately depict reality in most cases, but without them the problems would be very difficult and beyond the scope of this discussion (and the course in most cases to be honest).

So let's get started.

Mixing Problems
In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it. If $Q(t)$ gives the amount of the substance dissolved in the liquid in the tank at any time t we want to develop a differential equation that, when solved, will give us an expression for $Q(t)$. Note as well that in many situations we can think of air as a liquid for the purposes of these kinds of discussions and so we don't actually need to have an actual liquid, but could instead use air as the "liquid".

The main assumption that we'll be using here is that the concentration of the substance in the liquid is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course.

The main "equation" that we'll be using to model this situation is :

$$\begin{array}{c}\text{Rate of}\\ \text{change of}\\ Q(t)\end{array} = \begin{array}{c}\text{Rate at}\\ \text{which } Q(t)\\ \text{enters the}\\ \text{tank}\end{array} - \begin{array}{c}\text{Rate at}\\ \text{which } Q(t)\\ \text{exits the}\\ \text{tank}\end{array}$$

where,

Rate of change of $Q(t)$ = $\dfrac{dQ}{dt} = Q'\left(t\right)$

Rate at which $Q(t)$ enters the tank = (flow rate of liquid entering) x (concentration of substance in liquid entering)

Rate at which $Q(t)$ exits the tank = (flow rate of liquid exiting) x (concentration of substance in liquid exiting)

Let's take a look at the first problem.

Example 1 A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}\left(1+\cos\left(t\right)\right)$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

Solution

First off, let's address the "well mixed solution" bit. This is the assumption that was mentioned earlier. We are going to assume that the instant the water enters the tank it somehow instantly disperses evenly throughout the tank to give a uniform concentration of salt in the tank at every point. Again, this will clearly not be the case in reality, but it will allow us to do the problem.

Now, to set up the IVP that we'll need to solve to get *Q(t)* we'll need the flow rate of the water entering (we've got that), the concentration of the salt in the water entering (we've got that), the flow rate of the water leaving (we've got that) and the concentration of the salt in the water exiting (we don't have this yet).

So, we first need to determine the concentration of the salt in the water exiting the tank. Since we are assuming a uniform concentration of salt in the tank the concentration at any point in the tank and hence in the water exiting is given by,

$$\text{Concentration} = \frac{\text{Amount of salt in the tank at any time, } t}{\text{Volume of water in the tank at any time, } t}$$

The amount at any time t is easy it's just *Q(t)*. The volume is also pretty easy. We start with 600 gallons and every hour 9 gallons enters and 6 gallons leave. So, if we use t in hours, every hour 3 gallons enters the tank, or at any time t there is 600 + 3t gallons of water in the tank.

So, the IVP for this situation is,

$$Q'\left(t\right) = \left(9\right)\left(\frac{1}{5}\left(1+\cos\left(t\right)\right)\right) - \left(6\right)\left(\frac{Q\left(t\right)}{600+3t}\right) \qquad Q\left(0\right)=5$$

$$Q'\left(t\right) = \frac{9}{5}\left(1+\cos\left(t\right)\right) - \frac{2Q\left(t\right)}{200+t} \qquad Q\left(0\right)=5$$

This is a linear differential equation and it isn't too difficult to solve (hopefully). We will show most of the details, but leave the description of the solution process out. If you need a refresher on solving linear first order differential equations go back and take a look at that section.

$$Q'\left(t\right) + \frac{2Q\left(t\right)}{200+t} = \frac{9}{5}\left(1+\cos\left(t\right)\right)$$

$$\mu\left(t\right) = \mathbf{e}^{\int \frac{2}{200+t}\,dt} = \mathbf{e}^{2\ln\left(200+t\right)} = \left(200+t\right)^2$$

$$\int \left(\left(200+t\right)^2 Q\left(t\right)\right)' dt = \int \frac{9}{5}\left(200+t\right)^2\left(1+\cos\left(t\right)\right) dt$$

$$(200+t)^2 Q(t)=\frac{9}{5}\left(\frac{1}{3}(200+t)^3+(200+t)^2\sin(t)+2(200+t)\cos(t)-2\sin(t)\right)+c$$

$$Q(t)=\frac{9}{5}\left(\frac{1}{3}(200+t)+\sin(t)+\frac{2\cos(t)}{200+t}-\frac{2\sin(t)}{(200+t)^2}\right)+\frac{c}{(200+t)^2}$$

So, here's the general solution. Now, apply the initial condition to get the value of the constant, c.

$$5=Q(0)=\frac{9}{5}\left(\frac{1}{3}(200)+\frac{2}{200}\right)+\frac{c}{(200)^2} \qquad\qquad c=-4600720$$

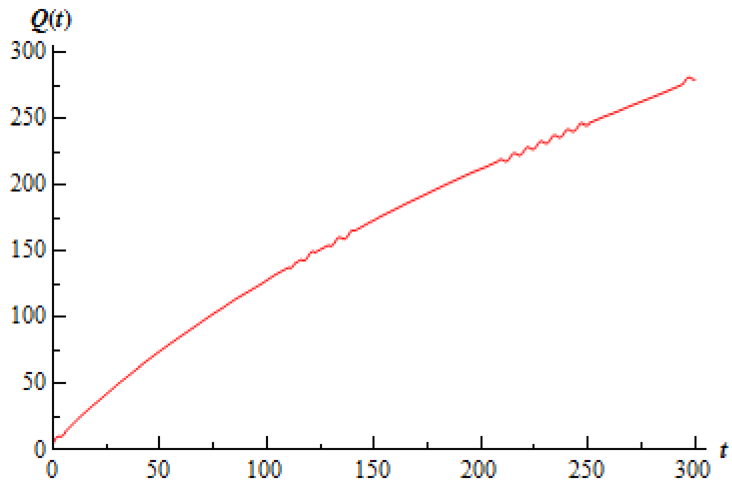
So, the amount of salt in the tank at any time t is.

$$Q(t)=\frac{9}{5}\left(\frac{1}{3}(200+t)+\sin(t)+\frac{2\cos(t)}{200+t}-\frac{2\sin(t)}{(200+t)^2}\right)-\frac{4600720}{(200+t)^2}$$

Now, the tank will overflow at t = 300 hrs. The amount of salt in the tank at that time is.

$$Q(300)=279.797 \text{ lbs}$$

Here's a graph of the salt in the tank before it overflows.



Note that the whole graph should have small oscillations in it as you can see in the range from 200 to 250. The scale of the oscillations however was small enough that the program used to generate the image had trouble showing all of them.

The work was a little messy with that one, but they will often be that way so don't get excited about it. This first example also assumed that nothing would change throughout the life of the process. That, of course will usually not be the case. Let's take a look at an example where something changes in the process.

Example 2 A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gallons while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time *t*.

Solution

Okay, so clearly the pollution in the tank will increase as time passes. If the amount of pollution ever reaches the maximum allowed there will be a change in the situation. This will necessitate a change in the differential equation describing the process as well. In other words, we'll need two IVP's for this problem. One will describe the initial situation when polluted runoff is entering the tank and one for after the maximum allowed pollution is reached and fresh water is entering the tank.

Here are the two IVP's for this problem.

$$Q_1'(t) = (3)(5) - (3)\left(\frac{Q_1(t)}{800}\right) \qquad Q_1(0) = 2 \qquad 0 \le t \le t_m$$

$$Q_2'(t) = (2)(0) - (4)\left(\frac{Q_2(t)}{800 - 2(t - t_m)}\right) \qquad Q_2(t_m) = 500 \quad t_m \le t \le t_e$$

The first one is fairly straight forward and will be valid until the maximum amount of pollution is reached. We'll call that time t_m. Also, the volume in the tank remains constant during this time so we don't need to do anything fancy with that this time in the second term as we did in the previous example.

We'll need a little explanation for the second one. First notice that we don't "start over" at $t = 0$. We start this one at t_m, the time at which the new process starts. Next, fresh water is flowing into the tank and so the concentration of pollution in the incoming water is zero. This will drop out the first term, and that's okay so don't worry about that.

Now, notice that the volume at any time looks a little funny. During this time frame we are losing two gallons of water every hour of the process so we need the "-2" in there to account for that. However, we can't just use *t* as we did in the previous example. When this new process starts up there needs to be 800 gallons of water in the tank and if we just use *t* there we won't have the required 800 gallons that we need in the equation. So, to make sure that we have the proper volume we need to put in the difference in times. In this way once we are one hour into the new process (*i.e* $t - t_m = 1$) we will have 798 gallons in the tank as required.

Finally, the second process can't continue forever as eventually the tank will empty. This is denoted in the time restrictions as t_e. We can also note that $t_e = t_m + 400$ since the tank will empty 400 hours after this new process starts up. Well, it will end provided something doesn't come along and start changing the situation again.

Okay, now that we've got all the explanations taken care of here's the simplified version of the IVP's that we'll be solving.

$$Q_1'(t) = 15 - \frac{3Q_1(t)}{800} \qquad Q_1(0) = 2 \qquad 0 \le t \le t_m$$
$$Q_2'(t) = -\frac{2Q_2(t)}{400-(t-t_m)} \qquad Q_2(t_m) = 500 \quad t_m \le t \le t_e$$

The first IVP is a fairly simple linear differential equation so we'll leave the details of the solution to you to check. Upon solving you get.

$$Q_1(t) = 4000 - 3998\mathbf{e}^{-\frac{3t}{800}}$$

Now, we need to find t_m. This isn't too bad all we need to do is determine when the amount of pollution reaches 500. So we need to solve.

$$Q_1(t) = 4000 - 3998\mathbf{e}^{-\frac{3t}{800}} = 500 \qquad \Rightarrow \qquad t_m = 35.475$$

So, the second process will pick up at 35.475 hours. For completeness sake here is the IVP with this information inserted.

$$Q_2'(t) = -\frac{2Q_2(t)}{435.475-t} \qquad Q_2(35.475) = 500 \qquad 35.475 \le t \le 435.475$$

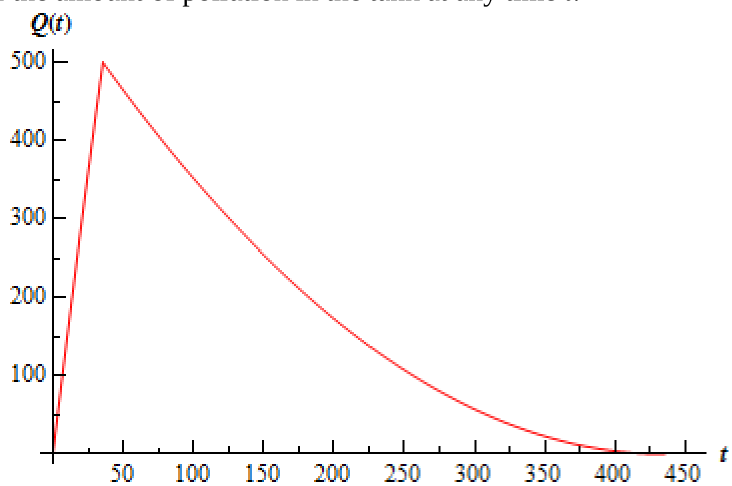
This differential equation is both linear and separable and again isn't terribly difficult to solve so I'll leave the details to you again to check that we should get.

$$Q_2(t) = \frac{(435.476-t)^2}{320}$$

So, a solution that encompasses the complete running time of the process is

$$Q(t) = \begin{cases} 4000 - 3998\mathbf{e}^{-\frac{3t}{800}} & 0 \le t \le 35.475 \\ \dfrac{(435.476-t)^2}{320} & 35.475 \le t \le 435.4758 \end{cases}$$

Here is a graph of the amount of pollution in the tank at any time t.



As you can surely see, these problems can get quite complicated if you want them to. Take the last example. A more realistic situation would be that once the pollution dropped below some predetermined point the polluted runoff would, in all likelihood, be allowed to flow back in and then the whole process would repeat itself. So, realistically, there should be at least one more IVP in the process.

Let's move on to another type of problem now.

Population

These are somewhat easier than the mixing problems although, in some ways, they are very similar to mixing problems. So, if *P(t)* represents a population in a given region at any time *t* the basic equation that we'll use is identical to the one that we used for mixing. Namely,

Rate of change of *P(t)*	=	Rate at which *P(t)* enters the region	-	Rate at which *P(t)* exits the region

Here the rate of change of *P(t)* is still the derivative. What's different this time is the rate at which the population enters and exits the region. For population problems all the ways for a population to enter the region are included in the entering rate. Birth rate and migration into the region are examples of terms that would go into the rate at which the population enters the region. Likewise, all the ways for a population to leave an area will be included in the exiting rate. Therefore things like death rate, migration out and predation are examples of terms that would go into the rate at which the population exits the area.

Here's an example.

Example 3 A population of insects in a region will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. On any given day there is a net migration into the area of 15 insects and 16 are eaten by the local bird population and 7 die of natural causes. If there are initially 100 insects in the area will the population survive? If not, when do they die out?

Solution

Let's start out by looking at the birth rate. We are told that the insects will be born at a rate that is proportional to the current population. This means that the birth rate can be written as

$$rP$$

where r is a positive constant that will need to be determined. Now, let's take everything into account and get the IVP for this problem.

$$\begin{aligned} P' &= \left(rP + 15\right) - \left(16 + 7\right) & \qquad P\left(0\right) &= 100 \\ P' &= rP - 8 & \qquad P\left(0\right) &= 100 \end{aligned}$$

Note that in the first line we used parenthesis to note which terms went into which part of the differential equation. Also note that we don't make use of the fact that the population will triple in two weeks time in the absence of outside factors here. In the absence of outside factors means that the ONLY thing that we can consider is birth rate. Nothing else can enter into the picture and clearly we have other influences in the differential equation.

So, just how does this tripling come into play? Well, we should also note that without knowing r we will have a difficult time solving the IVP completely. We will use the fact that the population triples in two week time to help us find r. In the absence of outside factors the differential equation would become.

$$P' = rP \qquad P(0) = 100 \qquad P(14) = 300$$

Note that since we used days as the time frame in the actual IVP I needed to convert the two weeks to 14 days. We could have just as easily converted the original IVP to weeks as the time frame, in which case there would have been a net change of –56 per week instead of the –8 per day that we are currently using in the original differential equation.

Okay back to the differential equation that ignores all the outside factors. This differential equation is separable and linear and is a simple differential equation to solve. I'll leave the detail to you to get the general solution.

$$P(t) = c\mathbf{e}^{rt}$$

Applying the initial condition gives $c = 100$. Now apply the second condition.

$$300 = P(14) = 100\mathbf{e}^{14r} \qquad 300 = 100\mathbf{e}^{14r}$$

We need to solve this for r. First divide both sides by 100, then take the natural log of both sides.

$$\begin{aligned} 3 &= \mathbf{e}^{14r} \\ \ln 3 &= \ln \mathbf{e}^{14r} \\ \ln 3 &= 14r \\ r &= \frac{\ln 3}{14} \end{aligned}$$

We made use of the fact that $\ln \mathbf{e}^{g(x)} = g(x)$ here to simplify the problem. Now, that we have r we can go back and solve the original differential equation. We'll rewrite it a little for the solution process.

$$P' - \frac{\ln 3}{14} P = -8 \qquad P(0) = 100$$

This is a fairly simple linear differential equation, but that coefficient of P always get people bent out of shape, so we'll go through at least some of the details here.

$$\mu(t) = \mathbf{e}^{\int -\frac{\ln 3}{14} dt} = \mathbf{e}^{-\frac{\ln 3}{14} t}$$

Now, don't get excited about the integrating factor here. It's just like $\mathbf{e}^{2t}$ only this time the constant is a little more complicated than just a 2, but it is a constant! Now, solve the differential equation.

$$\int \left(P\mathbf{e}^{-\frac{\ln 3}{14}t}\right)' dt = \int -8\mathbf{e}^{-\frac{\ln 3}{14}t}\, dt$$

$$P\mathbf{e}^{-\frac{\ln 3}{14}t} = -8\left(-\frac{14}{\ln 3}\right)\mathbf{e}^{-\frac{\ln 3}{14}t} + c$$

$$P(t) = \frac{112}{\ln 3} + c\mathbf{e}^{\frac{\ln 3}{14}t}$$

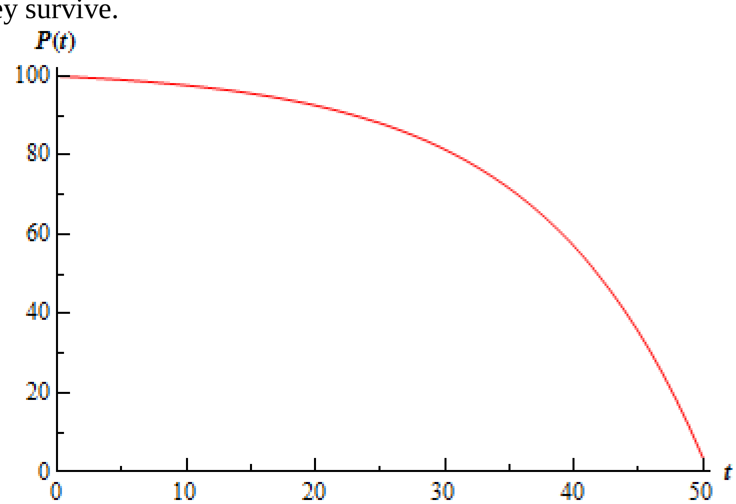
Again, do not get excited about doing the right hand integral, it's just like integrating $\mathbf{e}^{2t}$! Applying the initial condition gives the following.

$$P(t) = \frac{112}{\ln 3} + \left(100 - \frac{112}{\ln 3}\right)\mathbf{e}^{\frac{\ln 3}{14}t} = \frac{112}{\ln 3} - 1.94679\mathbf{e}^{\frac{\ln 3}{14}t}$$

Now, the exponential has a positive exponent and so will go to plus infinity as *t* increases. Its coefficient, however, is negative and so the whole population will go negative eventually. Clearly, population can't be negative, but in order for the population to go negative it must pass through zero. In other words, eventually all the insects must die. So, they don't survive and we can solve the following to determine when they die out.

$$0 = \frac{112}{\ln 3} - 1.94679\mathbf{e}^{\frac{\ln 3}{14}t} \qquad \Rightarrow \qquad t = 50.4415 \text{ days}$$

So, the insects will survive for around 7.2 weeks. Here is a graph of the population during the time in which they survive.



As with the mixing problems, we could make the population problems more complicated by changing the circumstances at some point in time. For instance, if at some point in time the local bird population saw a decrease due to disease they wouldn't eat as much after that point and a second differential equation to govern the time after this point.

Let's now take a look at the final type of problem that we'll be modeling in this section.

Falling Body
This will not be the first time that we've looked into falling bodies. If you recall, we looked at one of these when we were looking at Direction Fields. In that section we saw that the basic equation that we'll use is Newton's Second Law of Motion.

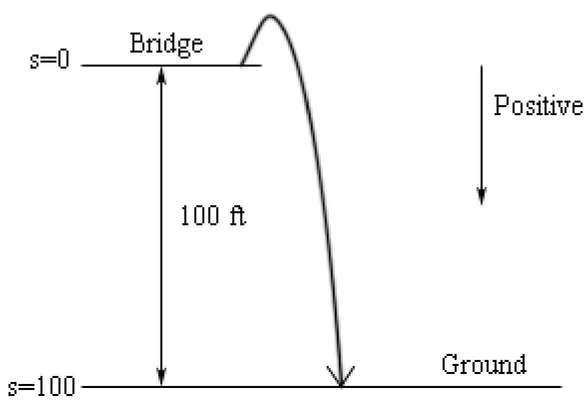
$$mv' = F(t,v)$$

The two forces that we'll be looking at here are gravity and air resistance. The main issue with these problems is to correctly define conventions and then remember to keep those conventions. By this we mean define which direction will be termed the positive direction and then make sure that all your forces match that convention. This is especially important for air resistance as this is usually dependent on the velocity and so the "sign" of the velocity can and does affect the "sign" of the air resistance force.

Let's take a look at an example.

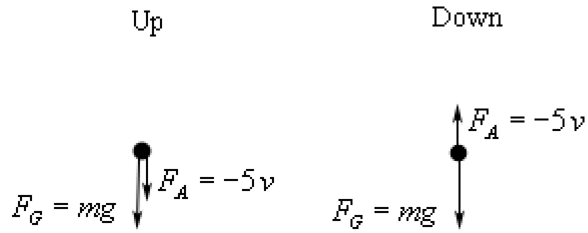
Example 4 A 50 kg mass is shot from a cannon straight up with an initial velocity of 10m/s off a bridge that is 100 meters above the ground. If air resistance is given by 5*v* determine the velocity of the mass when it hits the ground.

Solution
First, notice that when we say straight up, we really mean straight up, but in such a way that it will miss the bridge on the way back down. Here is a sketch of the situation.



Notice the conventions that we set up for this problem. Since the vast majority of the motion will be in the downward direction we decided to assume that everything acting in the downward direction should be positive. Note that we also defined the "zero position" as the bridge, which makes the ground have a "position" of 100.

Okay, if you think about it we actually have two situations here. The initial phase in which the mass is rising in the air and the second phase when the mass is on its way down. We will need to examine both situations and set up an IVP for each. We will do this simultaneously. Here are the forces that are acting on the object on the way up and on the way down.



Notice that the air resistance force needs a negative in both cases in order to get the correct "sign" or direction on the force. When the mass is moving upwards the velocity (and hence v) is negative, yet the force must be acting in a downward direction. Therefore, the "-" must be part of the force to make sure that, overall, the force is positive and hence acting in the downward direction. Likewise, when the mass is moving downward the velocity (and so v) is positive. Therefore, the air resistance must also have a "-" in order to make sure that it's negative and hence acting in the upward direction.

So, the IVP for each of these situations are.

Up	Down
$mv' = mg - 5v$	$mv' = mg - 5v$
$v(0) = -10$	$v(t_0) = 0$

In the second IVP, the t_0 is the time when the object is at the highest point and is ready to start on the way down. Note that at this time the velocity would be zero. Also note that the initial condition of the first differential equation will have to be negative since the initial velocity is upward.

In this case, the differential equation for both of the situations is identical. This won't always happen, but in those cases where it does, we can ignore the second IVP and just let the first govern the whole process.

So, let's actually plug in for the mass and gravity (we'll be using $g = 9.8$ m/s^2 here). We'll go ahead and divide out the mass while we're at it since we'll need to do that eventually anyway.

$$v' = 9.8 - \frac{5v}{50} = 9.8 - \frac{v}{10} \qquad\qquad v(0) = -10$$

This is a simple linear differential equation to solve so we'll leave the details to you. Upon solving we arrive at the following equation for the velocity of the object at any time t.

$$v(t) = 98 - 108\mathbf{e}^{-\frac{t}{10}}$$

Okay, we want the velocity of the ball when it hits the ground. Of course we need to know *when* it hits the ground before we can ask this. In order to find this we will need to find the position function. This is easy enough to do.

$$s(t) = \int v(t)\,dt = \int 98 - 108\mathbf{e}^{-\frac{t}{10}}\,dt = 98t + 1080\mathbf{e}^{-\frac{t}{10}} + c$$

We can now use the fact that I took the convention that $s(0) = 0$ to find that $c = -1080$. The position at any time is then.

$$s(t) = 98t + 1080\mathbf{e}^{-\frac{t}{10}} - 1080$$

To determine when the mass hits the ground we just need to solve.

$$100 = 98t + 1080\mathbf{e}^{-\frac{t}{10}} - 1080 \qquad\qquad t = -3.32203,\ 5.98147$$

We've got two solutions here, but since we are starting things at $t = 0$, the negative is clearly the incorrect value. Therefore the mass hits the ground at $t = 5.98147$. The velocity of the object upon hitting the ground is then.

$$v(5.98147) = 38.61841$$

This last example gave us an example of a situation where the two differential equations needed for the problem ended up being identical and so we didn't need the second one after all. Be careful however to not always expect this. We could very easily change this problem so that it required two different differential equations. For instance we could have had a parachute on the mass open at the top of its arc changing its air resistance. This would have completely changed the second differential equation and forced us to use it as well. Or, we could have put a river under the bridge so that before it actually hit the ground it would have first had to go through some water which would have a different "air" resistance for that phase necessitating a new differential equation for that portion.

Or, we could be really crazy and have both the parachute and the river which would then require three IVP's to be solved before we determined the velocity of the mass before it actually hits the solid ground.

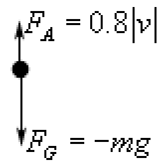
Before leaving this section let's work a couple examples illustrating the importance of remembering the conventions that you set up for the positive direction in these problems.

Awhile back I gave my students a problem in which a sky diver jumps out of a plane. Most of my students are engineering majors and following the standard convention from most of their engineering classes they defined the positive direction as upward, despite the fact that all the motion in the problem was downward. There is nothing wrong with this assumption, however, because they forgot the convention that up was positive they did not correctly deal with the air resistance which caused them to get the incorrect answer.

So, let's take a look at the problem and set up the IVP that will give the sky diver's velocity at any time t.

Example 5 Set up the IVP that will give the velocity of a 60 kg sky diver that jumps out of a plane with no initial velocity and an air resistance of $0.8|v|$. For this example assume that the positive direction is upward.

Solution
Here are the forces that are acting on the sky diver



Because of the conventions the force due to gravity is negative and the force due to air resistance is positive. As set up, these forces have the correct sign and so the IVP is

$$mv' = -mg + 0.8|v| \qquad v(0) = 0$$

The problem arises when you go to remove the absolute value bars. In order to do the problem they do need to be removed. This is where most of the students made their mistake. Because they had forgotten about the convention and the direction of motion they just dropped the absolute value bars to get.

$$mv' = -mg + 0.8v \qquad v(0) = 0 \qquad (\text{incorrect IVP!!})$$

So, why is this incorrect? Well remember that the convention is that positive is upward. However in this case the object is moving downward and so v is negative! Upon dropping the absolute value bars the air resistance became a negative force and hence was acting in the downward direction!

To get the correct IVP recall that because v is negative then $|v| = -v$. Using this, the air resistance becomes $F_A = -0.8v$ and despite appearances this is a positive force since the "-" cancels out against the velocity (which is negative) to get a positive force.

The correct IVP is then

$$mv' = -mg - 0.8v \qquad v(0) = 0$$

Plugging in the mass gives

$$v' = -9.8 - \frac{v}{75} \qquad v(0) = 0$$

For the sake of completeness the velocity of the sky diver, at least until the parachute opens, which I didn't include in this problem is.

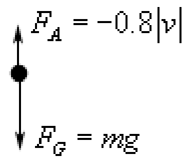
$$v(t) = -735 + 735\mathbf{e}^{-\frac{t}{75}}$$

This mistake was made in part because the students were in a hurry and weren't paying attention, but also because they simply forgot about their convention and the direction of motion! Don't fall into this mistake. Always pay attention to your conventions and what is happening in the problems.

Just to show you the difference here is the problem worked by assuming that down is positive.

Example 6 Set up the IVP that will give the velocity of a 60 kg sky diver that jumps out of a plane with no initial velocity and an air resistance of $0.8|v|$. For this example assume that the positive direction is downward.

Solution
Here are the forces that are acting on the sky diver



In this case the force due to gravity is positive since it's a downward force and air resistance is an upward force and so needs to be negative. In this case since the motion is downward the velocity is positive so $|v| = v$. The air resistance is then $F_A = -0.8v$. The IVP for this case is

$$mv' = mg - 0.8v \qquad\qquad v(0) = 0$$

Plugging in the mass gives

$$v' = 9.8 - \frac{v}{75} \qquad\qquad v(0) = 0$$

Solving this gives

$$v(t) = 735 - 735\mathbf{e}^{-\frac{t}{75}}$$

This is the same solution as the previous example, except that it's got the opposite sign. This is to be expected since the conventions have been switched between the two examples.

Equilibrium Solutions

In the previous section we modeled a population based on the assumption that the growth rate would be a constant. However, in reality this doesn't make much sense. Clearly a population cannot be allowed to grow forever at the same rate. The growth rate of a population needs to depend on the population itself. Once a population reaches a certain point the growth rate will start reduce, often drastically. A much more realistic model of a population growth is given by the **logistic growth equation**. Here is the logistic growth equation.

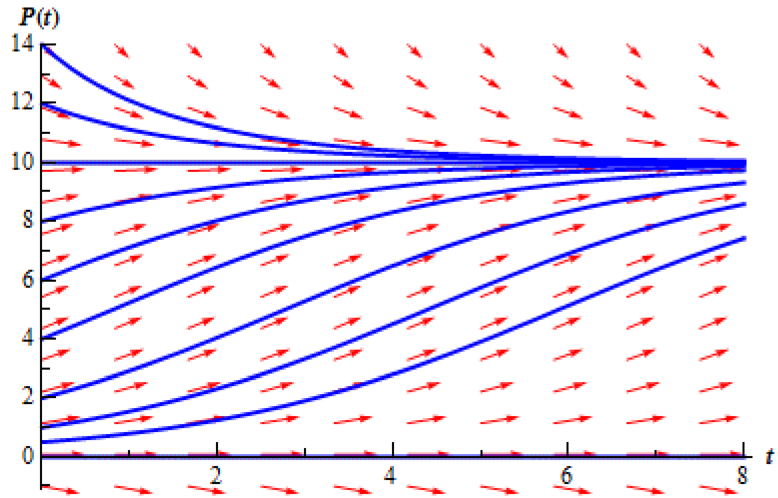
$$P' = r\left(1 - \frac{P}{K}\right)P$$

In the logistic growth equation r is the **intrinsic growth rate** and is the same r as in the last section. In other words, it is the growth rate that will occur in the absence of any limiting factors. K is called either the **saturation level** or the **carrying capacity**.

Now, we claimed that this was a more realistic model for a population. Let's see if that in fact is correct. To allow us to sketch a direction field let's pick a couple of numbers for r and K. We'll use $r = \frac{1}{2}$ and $K = 10$. For these values the logistics equation is.

$$P' = \frac{1}{2}\left(1 - \frac{P}{10}\right)P$$

If you need a refresher on sketching direction fields go back and take a look at that section. First notice that the derivative will be zero at $P = 0$ and $P = 10$. Also notice that these are in fact solutions to the differential equation. These two values are called **equilibrium solutions** since they are constant solutions to the differential equation. We'll leave the rest of the details on sketching the direction field to you. Here is the direction field as well as a couple of solutions sketched in as well.



Note, that we included a small portion of negative P's in here even though they really don't make any sense for a population problem. The reason for this will be apparent down the road. Also, notice that a population of say 8 doesn't make all that much sense so let's assume that population is in thousands or millions so that 8 actually represents 8,000 or 8,000,000 individuals in a population.

Notice that if we start with a population of zero, there is no growth and the population stays at zero. So, the logistic equation will correctly figure out that. Next, notice that if we start with a population in the range $0 < P(0) < 10$ then the population will grow, but start to level off once we get close to a population of 10. If we start with a population of 10, the population will stay at 10. Finally if we start with a population that is greater than 10, then the population will actually die off until we start nearing a population of 10, at which point the population decline will start to slow down.

Now, from a realistic standpoint this should make some sense. Populations can't just grow forever without bound. Eventually the population will reach such a size that the resources of an area are no longer able to sustain the population and the population growth will start to slow as it comes closer to this threshold. Also, if you start off with a population greater than what an area can sustain there will actually be a die off until we get near to this threshold.

In this case that threshold appears to be 10, which is also the value of K for our problem. That should explain the name that we gave K initially. The carrying capacity or saturation level of an area is the maximum sustainable population for that area.

So, the logistics equation, while still quite simplistic, does a much better job of modeling what will happen to a population.

Now, let's move on to the point of this section. The logistics equation is an example of an **autonomous differential equation**. Autonomous differential equations are differential equations that are of the form.

$$\frac{dy}{dt} = f\left(y\right)$$

The only place that the independent variable, t in this case, appears is in the derivative.

Notice that if $f\left(y_0\right) = 0$ for some value $y = y_0$ then this will also be a solution to the differential equation. These values are called **equilibrium solutions** or **equilibrium points**. What we would like to do is classify these solutions. By classify we mean the following. If solutions start "near" an equilibrium solution will they move away from the equilibrium solution or towards the equilibrium solution? Upon classifying the equilibrium solutions we can then know what all the other solutions to the differential equation will do in the long term simply by looking at which equilibrium solutions they start near.

So, just what do I mean by "near"? Go back to our logistics equation.

$$P' = \frac{1}{2}\left(1 - \frac{P}{10}\right)P$$

As we pointed out there are two equilibrium solutions to this equation $P = 0$ and $P = 10$. If we ignore the fact that we're dealing with population these points break up the P number line into three distinct regions.

$$-\infty < P < 0 \qquad\qquad 0 < P < 10 \qquad\qquad 10 < P < \infty$$

We will say that a solution starts "near" an equilibrium solution if it starts in a region that is on either side of that equilibrium solution. So solutions that start "near" the equilibrium solution $P = 10$ will start in either

$$0 < P < 10 \qquad \text{OR} \qquad 10 < P < \infty$$

and solutions that start "near" $P = 0$ will start in either

$$-\infty < P < 0 \qquad \text{OR} \qquad 0 < P < 10$$

For regions that lie between two equilibrium solutions we can think of any solutions starting in that region as starting "near" either of the two equilibrium solutions as we need to.

Now, solutions that start "near" $P = 0$ all move away from the solution as t increases. Note that moving away does not necessarily mean that they grow without bound as they move away. It only means that they move away. Solutions that start out greater than $P = 0$ move away, but do stay bounded as t grows. In fact, they move in towards $P = 10$.

Equilibrium solutions in which solutions that start "near" them move away from the equilibrium solution are called **unstable equilibrium points** or **unstable equilibrium solutions**. So, for our logistics equation, $P = 0$ is an unstable equilibrium solution.

Next, solutions that start "near" $P = 10$ all move in toward $P = 10$ as t increases. Equilibrium solutions in which solutions that start "near" them move toward the equilibrium solution are called **asymptotically stable equilibrium points** or **asymptotically stable equilibrium solutions**. So, $P = 10$ is an asymptotically stable equilibrium solution.

There is one more classification, but I'll wait until we get an example in which this occurs to introduce it. So, let's take a look at a couple of examples.

Example 1 Find and classify all the equilibrium solutions to the following differential equation.

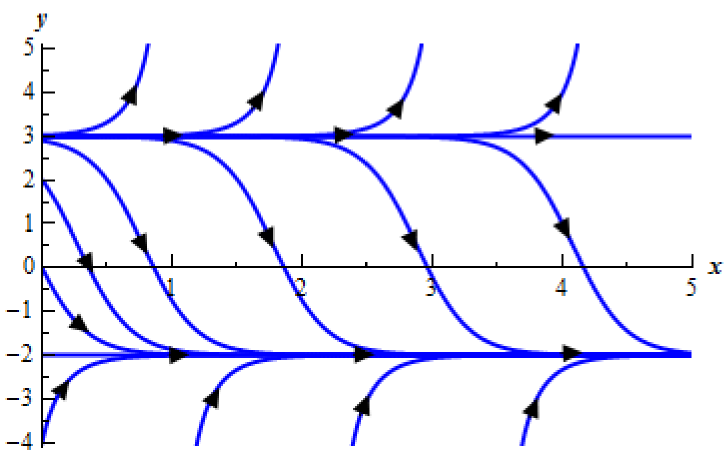
$$y' = y^2 - y - 6$$

Solution

First, find the equilibrium solutions. This is generally easy enough to do.

$$y^2 - y - 6 = (y-3)(y+2) = 0$$

So, it looks like we've got two equilibrium solutions. Both $y = -2$ and $y = 3$ are equilibrium solutions. Below is the sketch of some integral curves for this differential equation. A sketch of the integral curves or direction fields can simplify the process of classifying the equilibrium solutions.



From this sketch it appears that solutions that start "near" $y = -2$ all move towards it as t increases and so $y = -2$ is an asymptotically stable equilibrium solution and solutions that start "near" $y = 3$

all move away from it as t increases and so $y = 3$ is an unstable equilibrium solution.

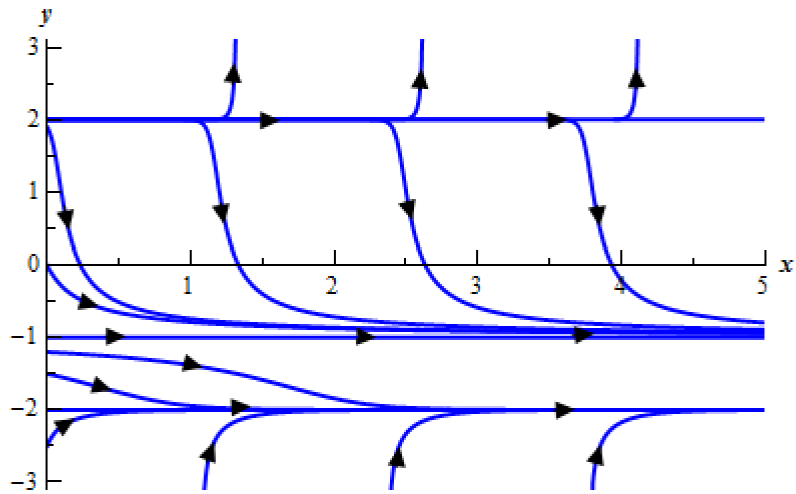
This next example will introduce the third classification that we can give to equilibrium solutions.

Example 2 Find and classify the equilibrium solutions of the following differential equation.

$$y' = \left(y^2 - 4\right)\left(y+1\right)^2$$

Solution

The equilibrium solutions are to this differential equation are $y = -2$, $y = 2$, and $y = -1$. Below is the sketch of the integral curves.



From this it is clear (hopefully) that $y = 2$ is an unstable equilibrium solution and $y = -2$ is an asymptotically stable equilibrium solution. However, $y = -1$ behaves differently from either of these two. Solutions that start above it move towards $y = -1$ while solutions that start below $y = -1$ move away as t increases.

In cases where solutions on one side of an equilibrium solution move towards the equilibrium solution and on the other side of the equilibrium solution move away from it we call the equilibrium solution **semi-stable**.

So, $y = -1$ is a semi-stable equilibrium solution.

Euler's Method

Up to this point practically every differential equation that we've been presented with could be solved. The problem with this is that these are the exceptions rather than the rule. The vast majority of first order differential equations can't be solved.

In order to teach you something about solving first order differential equations we've had to restrict ourselves down to the fairly restrictive cases of linear, separable, or exact differential equations or differential equations that could be solved with a set of very specific substitutions. Most first order differential equations however fall into none of these categories. In fact even those that are separable or exact cannot always be solved for an explicit solution. Without explicit solutions to these it would be hard to get any information about the solution.

So what do we do when faced with a differential equation that we can't solve? The answer depends on what you are looking for. If you are only looking for long term behavior of a solution you can always sketch a direction field. This can be done without too much difficulty for some fairly complex differential equations that we can't solve to get exact solutions.

The problem with this approach is that it's only really good for getting general trends in solutions and for long term behavior of solutions. There are times when we will need something more. For instance, maybe we need to determine how a specific solution behaves, including some values that the solution will take. There are also a fairly large set of differential equations that are not easy to sketch good direction fields for.

In these cases we resort to numerical methods that will allow us to approximate solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation and in fact whole classes can be taught just dealing with the various methods. We are going to look at one of the oldest and easiest to use here. This method was originally devised by Euler and is called, oddly enough, Euler's Method.

Let's start with a general first order IVP

$$\frac{dy}{dt} = f\left(t, y\right) \qquad\qquad y\left(t_0\right) = y_0 \tag{1}$$

where $f(t,y)$ is a known function and the values in the initial condition are also known numbers. From the second theorem in the Intervals of Validity section we know that if f and f_y are continuous functions then there is a unique solution to the IVP in some interval surrounding $t = t_0$. So, let's assume that everything is nice and continuous so that we know that a solution will in fact exist.

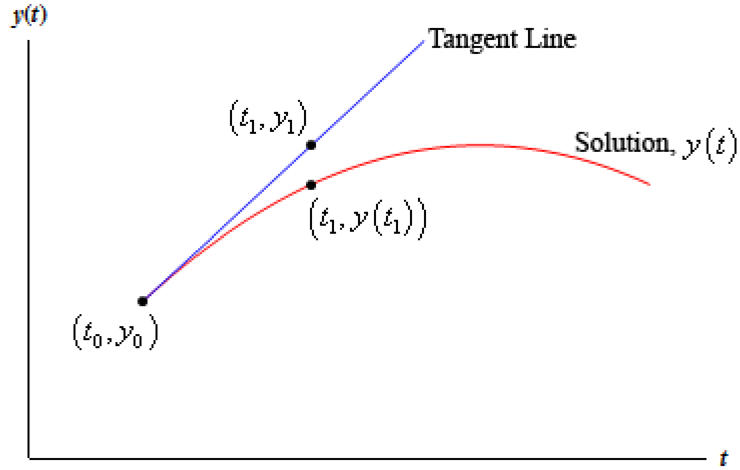
We want to approximate the solution to (1) near $t = t_0$. We'll start with the two pieces of information that we do know about the solution. First, we know the value of the solution at $t = t_0$ from the initial condition. Second, we also know the value of the derivative at $t = t_0$. We can get this by plugging the initial condition into $f(t,y)$ into the differential equation itself. So, the derivative at this point is.

$$\left.\frac{dy}{dt}\right|_{t=t_0} = f\left(t_0, y_0\right)$$

Now, recall from your Calculus I class that these two pieces of information are enough for us to write down the equation of the tangent line to the solution at $t = t_0$. The tangent line is

$$y = y_0 + f\left(t_0, y_0\right)\left(t - t_0\right)$$

Take a look at the figure below



If t_1 is close enough to t_0 then the point y_1 on the tangent line should be fairly close to the actual value of the solution at t_1, or $y(t_1)$. Finding y_1 is easy enough. All we need to do is plug t_1 in the equation for the tangent line.

$$y_1 = y_0 + f\left(t_0, y_0\right)\left(t_1 - t_0\right)$$

Now, we would like to proceed in a similar manner, but we don't have the value of the solution at t_1 and so we won't know the slope of the tangent line to the solution at this point. This is a problem. We can partially solve it however, by recalling that y_1 is an approximation to the solution at t_1. If y_1 is a very good approximation to the actual value of the solution then we can use that to estimate the slope of the tangent line at t_1.

So, let's hope that y_1 is a good approximation to the solution and construct a line through the point (t_1, y_1) that has slope $f(t_1, y_1)$. This gives

$$y = y_1 + f\left(t_1, y_1\right)\left(t - t_1\right)$$

Now, to get an approximation to the solution at $t = t_2$ we will hope that this new line will be fairly close to the actual solution at t_2 and use the value of the line at t_2 as an approximation to the actual solution. This gives.

$$y_2 = y_1 + f\left(t_1, y_1\right)\left(t_2 - t_1\right)$$

We can continue in this fashion. Use the previously computed approximation to get the next approximation. So,

$$y_3 = y_2 + f\left(t_2, y_2\right)\left(t_3 - t_2\right)$$
$$y_4 = y_3 + f\left(t_3, y_3\right)\left(t_4 - t_3\right)$$
$$\text{etc.}$$

In general, if we have t_n and the approximation to the solution at this point, y_n, and we want to find the approximation at t_{n+1} all we need to do is use the following.

$$y_{n+1} = y_n + f\left(t_n, y_n\right)\cdot\left(t_{n+1} - t_n\right)$$

If we define $f_n = f\left(t_n, y_n\right)$ we can simplify the formula to

$$y_{n+1} = y_n + f_n \cdot\left(t_{n+1} - t_n\right) \tag{2}$$

Often, we will assume that the step sizes between the points t_0 , t_1 , t_2 , … are of a uniform size of h. In other words, we will often assume that

$$t_{n+1} - t_n = h$$

This doesn't have to be done and there are times when it's best that we not do this. However, if we do the formula for the next approximation becomes.

$$y_{n+1} = y_n + h\, f_n \tag{3}$$

So, how do we use Euler's Method? It's fairly simple. We start with (1) and then decide if we want to use a uniform step size or not. Then starting with (t_0, y_0) we repeatedly evaluate (2) or (3) depending on whether we chose to use a uniform set size or not. We continue until we've gone the desired number of steps or reached the desired time. This will give us a sequence of numbers y_1 , y_2 , y_3 , … y_n that will approximate the value of the actual solution at t_1 , t_2 , t_3 , … t_n.

What do we do if we want a value of the solution at some other point than those used here? One possibility is to go back and redefine our set of points to a new set that will include the points we are after and redo Euler's Method using this new set of points. However this is cumbersome and could take a lot of time especially if we had to make changes to the set of points more than once.

Another possibility is to remember how we arrived at the approximations in the first place. Recall that we used the tangent line

$$y = y_0 + f\left(t_0, y_0\right)\left(t - t_0\right)$$

to get the value of y_1. We could use this tangent line as an approximation for the solution on the interval [t_0, t_1]. Likewise, we used the tangent line

$$y = y_1 + f\left(t_1, y_1\right)\left(t - t_1\right)$$

to get the value of y_2. We could use this tangent line as an approximation for the solution on the interval [t_1, t_2]. Continuing in this manner we would get a set of lines that, when strung together, should be an approximation to the solution as a whole.

In practice you would need to write a computer program to do these computations for you. In most cases the function $f(t,y)$ would be too large and/or complicated to use by hand and in most serious uses of Euler's Method you would want to use hundreds of steps which would make doing this by hand prohibitive. So, here is a bit of *pseudo-code* that you can use to write a program for Euler's Method that uses a uniform step size, h.

1. **define** $f(t, y)$.
2. **input** t_0 and y_0.
3. **input** step size, h and the number of steps, n.
4. **for** j from 1 to n **do**
    a. $m = f(t_0, y_0)$
    b. $y_1 = y_0 + h*m$
    c. $t_1 = t_0 + h$
    d. Print t_1 and y_1
    e. $t_0 = t_1$
    f. $y_0 = y_1$
5. **end**

The *pseudo-code* for a non-uniform step size would be a little more complicated, but it would essentially be the same.

So, let's take a look at a couple of examples. We'll use Euler's Method to approximate solutions to a couple of first order differential equations. The differential equations that we'll be using are linear first order differential equations that can be easily solved for an exact solution. Of course, in practice we wouldn't use Euler's Method on these kinds of differential equations, but by using easily solvable differential equations we will be able to check the accuracy of the method. Knowing the accuracy of any approximation method is a good thing. It is important to know if the method is liable to give a good approximation or not.

Example 1 For the IVP

$$y' + 2y = 2 - \mathbf{e}^{-4t} \qquad y(0) = 1$$

Use Euler's Method with a step size of $h = 0.1$ to find approximate values of the solution at t = 0.1, 0.2, 0.3, 0.4, and 0.5. Compare them to the exact values of the solution as these points.

Solution

This is a fairly simple linear differential equation so we'll leave it to you to check that the solution is

$$y(t) = 1 + \frac{1}{2}\mathbf{e}^{-4t} - \frac{1}{2}\mathbf{e}^{-2t}$$

In order to use Euler's Method we first need to rewrite the differential equation into the form given in (1).

$$y' = 2 - \mathbf{e}^{-4t} - 2y$$

From this we can see that $f(t, y) = 2 - \mathbf{e}^{-4t} - 2y$. Also note that $t_0 = 0$ and $y_0 = 1$. We can now start doing some computations.

$$f_0 = f(0,1) = 2 - \mathbf{e}^{-4(0)} - 2(1) = -1$$
$$y_1 = y_0 + h\, f_0 = 1 + (0.1)(-1) = 0.9$$

So, the approximation to the solution at $t_1 = 0.1$ is $y_1 = 0.9$.

At the next step we have

$$f_1 = f\left(0.1, 0.9\right) = 2 - \mathbf{e}^{-4(0.1)} - 2\left(0.9\right) = -0.470320046$$
$$y_2 = y_1 + h\, f_1 = 0.9 + \left(0.1\right)\left(-0.470320046\right) = 0.852967995$$

Therefore, the approximation to the solution at $t_2 = 0.2$ is $y_2 = 0.852967995$.

I'll leave it to you to check the remainder of these computations.

$$f_2 = -0.155264954 \qquad y_3 = 0.837441500$$
$$f_3 = 0.023922788 \qquad y_4 = 0.839833779$$
$$f_4 = 0.1184359245 \qquad y_5 = 0.851677371$$

Here's a quick table that gives the approximations as well as the exact value of the solutions at the given points.

Time, t_n	**Approximation**	**Exact**	**Error**
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

We've also included the error as a percentage. It's often easier to see how well an approximation does if you look at percentages. The formula for this is,

$$percent\ error = \frac{\left|exact - approximate\right|}{exact} \times 100$$

We used absolute value in the numerator because we really don't care at this point if the approximation is larger or smaller than the exact. We're only interested in how close the two are.

The maximum error in the approximations from the last example was 4.42%, which isn't too bad, but also isn't all that great of an approximation. So, provided we aren't after very accurate approximations this didn't do too badly. This kind of error is generally unacceptable in almost all real applications however. So, how can we get better approximations?

Recall that we are getting the approximations by using a tangent line to approximate the value of the solution and that we are moving forward in time by steps of h. So, if we want a more accurate approximation, then it seems like one way to get a better approximation is to not move forward as much with each step. In other words, take smaller h's.

Example 2 Repeat the previous example only this time give the approximations at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation. We'll leave the computational details to you to check.

Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$t = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$t = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$t = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$t = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

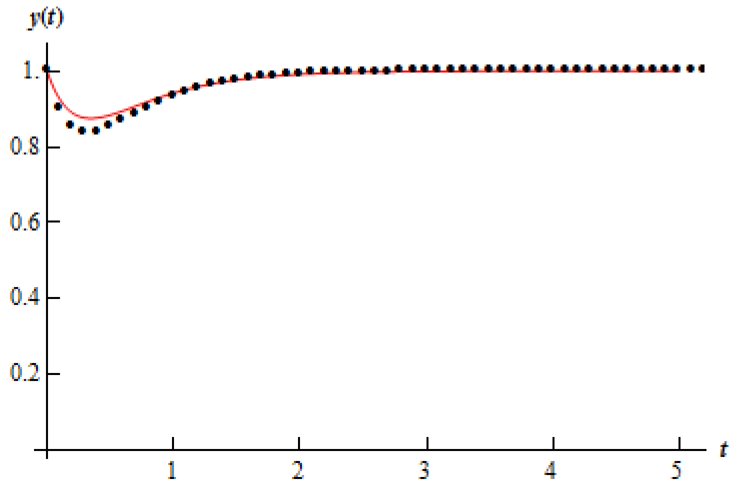
Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

We can see from these tables that decreasing h does in fact improve the accuracy of the approximation as we expected.

There are a couple of other interesting things to note from the data. First, notice that in general, decreasing the step size, h, by a factor of 10 also decreased the error by about a factor of 10 as well.

Also, notice that as t increases the approximation actually tends to get better. This isn't the case completely as we can see that in all but the first case the $t = 3$ error is worse than the error at $t = 2$, but after that point, it only gets better. This should not be expected in general. In this case this is more a function of the shape of the solution. Below is a graph of the solution (the line) as well as the approximations (the dots) for $h = 0.1$.



Notice that the approximation is worst where the function is changing rapidly. This should not be too surprising. Recall that we're using tangent lines to get the approximations and so the value of the tangent line at a given t will often be significantly different than the function due to the rapidly changing function at that point.

Also, in this case, because the function ends up fairly flat as t increases, the tangents start looking like the function itself and so the approximations are very accurate. This won't always be the case of course.

Let's take a look at one more example.

Example 3 For the IVP

$$y' - y = -\frac{1}{2}\mathbf{e}^{\frac{t}{2}}\sin(5t) + 5\mathbf{e}^{\frac{t}{2}}\cos(5t) \qquad\qquad y(0) = 0$$

Use Euler's Method to find the approximation to the solution at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. Use $h = 0.1$, $h = 0.05$, $h = 0.01$, $h = 0.005$, and $h = 0.001$ for the approximations.

Solution

I'll leave it to you to check the details of the solution process. The solution to this linear first order differential equation is.

$$y(t) = \mathbf{e}^{\frac{t}{2}}\sin(5t)$$

Here are two tables giving the approximations and the percentage error for each approximation.

Approximations

Time	**Exact**	***h* = 0.1**	***h* = 0.05**	***h* = 0.01**	***h* = 0.005**	***h* = 0.001**
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-2.18657	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

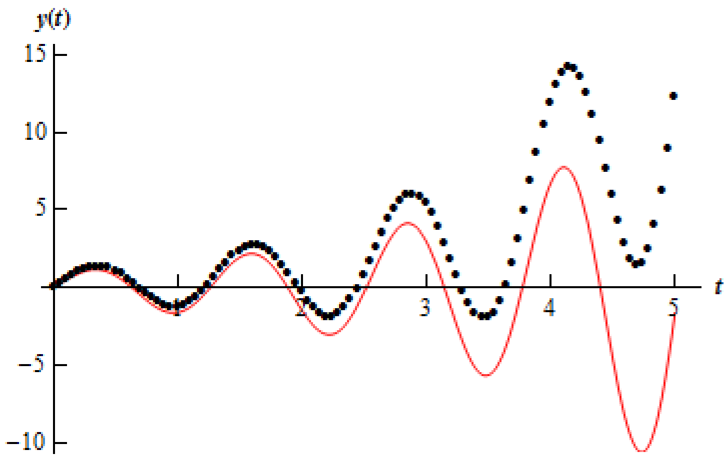
Percentage Errors

Time	***h* = 0.1**	***h* = 0.05**	***h* = 0.01**	***h* = 0.005**	***h* = 0.001**
$t = 1$	38.54 %	19.98 %	4.12 %	2.07 %	0.42 %
$t = 2$	144.14 %	76.79 %	16.21 %	8.16 %	1.64 %
$t = 3$	150.55 %	83.46 %	18.20 %	9.20 %	1.86 %
$t = 4$	130.68 %	75.64 %	17.08 %	8.67 %	1.76 %
$t = 5$	1461.63 %	859.14 %	196.79 %	100.12 %	20.30 %

So, with this example Euler's Method does not do nearly as well as it did on the first IVP. Some of the observations we made in Example 2 are still true however. Decreasing the size of h decreases the error as we saw with the last example and would expect to happen. Also, as we saw in the last example, decreasing h by a factor of 10 also decreases the error by about a factor of 10.

However, unlike the last example increasing t sees an increasing error. This behavior is fairly common in the approximations. We shouldn't expect the error to decrease as t increases as we saw in the last example. Each successive approximation is found using a previous approximation. Therefore, at each step we introduce error and so approximations should, in general, get worse as t increases.

Below is a graph of the solution (the line) as well as the approximations (the dots) for $h = 0.05$.



As we can see the approximations do follow the general shape of the solution, however, the error is clearly getting much worse as t increases.

So, Euler's method is a nice method for approximating fairly nice solutions that don't change rapidly. However, not all solutions will be this nicely behaved. There are other approximation methods that do a much better job of approximating solutions. These are not the focus of this course however, so I'll leave it to you to look further into this field if you are interested.

Also notice that we don't generally have the actual solution around to check the accuracy of the approximation. We generally try to find bounds on the error for each method that will tell us how well an approximation should do. These error bounds are again not really the focus of this course, so I'll leave these to you as well if you're interested in looking into them.

Second Order Differential Equations

Introduction

In the previous chapter we looked at first order differential equations. In this chapter we will move on to second order differential equations. Just as we did in the last chapter we will look at some special cases of second order differential equations that we can solve. Unlike the previous chapter however, we are going to have to be even more restrictive as to the kinds of differential equations that we'll look at. This will be required in order for us to actually be able to solve them.

Here is a list of topics that will be covered in this chapter.

Basic Concepts – Some of the basic concepts and ideas that are involved in solving second order differential equations.

Real Roots – Solving differential equations whose characteristic equation has real roots.

Complex Roots – Solving differential equations whose characteristic equation complex real roots.

Repeated Roots – Solving differential equations whose characteristic equation has repeated roots.

Reduction of Order – A brief look at the topic of reduction of order. This will be one of the few times in this chapter that non-constant coefficient differential equation will be looked at.

Fundamental Sets of Solutions – A look at some of the theory behind the solution to second order differential equations, including looks at the Wronskian and fundamental sets of solutions.

More on the Wronskian – An application of the Wronskian and an alternate method for finding it.

Nonhomogeneous Differential Equations – A quick look into how to solve nonhomogeneous differential equations in general.

Undetermined Coefficients – The first method for solving nonhomogeneous differential equations that we'll be looking at in this section.

Variation of Parameters – Another method for solving nonhomogeneous differential equations.

Mechanical Vibrations – An application of second order differential equations. This section focuses on mechanical vibrations, yet a simple change of notation can move this into almost any other engineering field.

Basic Concepts

In this chapter we will be looking exclusively at linear second order differential equations. The most general linear second order differential equation is in the form.

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = g(t) \qquad \textbf{(1)}$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$ay'' + by' + cy = g(t) \qquad \textbf{(2)}$$

Where possible we will use (1) just to make the point that certain facts, theorems, properties, and/or techniques can be used with the non-constant form. However, most of the time we will be using (2) as it can be fairly difficult to solve second order non-constant coefficient differential equations.

Initially we will make our life easier by looking at differential equations with $g(t) = 0$. When $g(t) = 0$ we call the differential equation **homogeneous** and when $g(t) \neq 0$ we call the differential equation **nonhomogeneous**.

So, let's start thinking about how to go about solving a constant coefficient, homogeneous, linear, second order differential equation. Here is the general constant coefficient, homogeneous, linear, second order differential equation.

$$ay'' + by' + cy = 0$$

It's probably best to start off with an example. This example will lead us to a very important fact that we will use in every problem from this point on. The example will also give us clues into how to go about solving these in general.

Example 1 Determine some solutions to

$$y'' - 9y = 0$$

Solution

We can get some solutions here simply by inspection. We need functions whose second derivative is 9 times the original function. One of the first functions that I can think of that comes back to itself after two derivatives is an exponential function and with proper exponents the 9 will get taken care of as well.

So, it looks like the following two functions are solutions.

$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$

We'll leave it to you to verify that these are in fact solutions.

These two functions are not the only solutions to the differential equation however. Any of the following are also solutions to the differential equation.

$$y(t) = -9\mathbf{e}^{3t} \qquad y(t) = 123\mathbf{e}^{3t}$$

$$y(t) = 56\mathbf{e}^{-3t} \qquad y(t) = \frac{14}{9}\mathbf{e}^{-3t}$$

$$y(t) = 7\mathbf{e}^{3t} - 6\mathbf{e}^{-3t} \qquad y(t) = -92\mathbf{e}^{3t} - 16\mathbf{e}^{-3t}$$

In fact if you think about it any function that is in the form

$$y(t) = c_1\mathbf{e}^{3t} + c_2\mathbf{e}^{-3t}$$

will be a solution to the differential equation.

This example leads us to a very important fact that we will use in practically every problem in this chapter.

Principle of Superposition

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, homogeneous differential equation then so is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \qquad \textbf{(3)}$$

Note that we didn't include the restriction of constant coefficient or second order in this. This will work for any linear homogeneous differential equation.

If we further assume second order and one other condition (which we'll give in a second) we can go a step further.

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, second order homogeneous differential equation and they are "nice enough" then the general solution to the linear, second order differential equation is given by (3).

So, just what do we mean by "nice enough"? We'll hold off on that until a later section. At this point you'll hopefully believe it when we say that specific functions are "nice enough".

So, if we now make the assumption that we are dealing with a linear, second order differential equation, we now know that (3) will be its general solution. The next question that we can ask is how to find the constants c_1 and c_2. Since we have two constants it makes sense, hopefully, that we will need two equations, or conditions, to find them.

One way to do this is to specify the value of the solution at two distinct points, or,

$$y(t_0) = y_0 \qquad y(t_1) = y_1$$

These are typically called boundary values and are not really the focus of this course so we won't be working with them.

Another way to find the constants would be to specify the value of the solution and its derivative at a particular point. Or,

$$y(t_0) = y_0 \qquad y'(t_0) = y'_0$$

These are the two conditions that we'll be using here. As with the first order differential equations these will be called initial conditions.

Example 2 Solve the following IVP.

$$y'' - 9y = 0 \qquad y(0) = 2 \qquad y'(0) = -1$$

Solution

First, the two functions

$$y(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y(t) = \mathbf{e}^{-3t}$$

are "nice enough" for us to form the general solution to the differential equation. At this point, please just believe this. You will be able to verify this for yourself in a couple of sections.

The general solution to our differential equation is then

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{3t}$$

Now all we need to do is apply the initial conditions. This means that we need the derivative of the solution.

$$y'(t) = -3c_1\mathbf{e}^{-3t} + 3c_2\mathbf{e}^{3t}$$

Plug in the initial conditions

$$2 = y(0) = c_1 + c_2$$
$$-1 = y'(0) = -3c_1 + 3c_2$$

This gives us a system of two equations and two unknowns that can be solved. Doing this yields

$$c_1 = \frac{7}{6} \qquad c_2 = \frac{5}{6}$$

The solution to the IVP is then,

$$y(t) = \frac{7}{6}\mathbf{e}^{-3t} + \frac{5}{6}\mathbf{e}^{3t}$$

Up to this point we've only looked at a single differential equation and we got its solution by inspection. For a rare few differential equations we can do this. However, for the vast majority of the second order differential equations out there we will be unable to do this.

So, we would like a method for arriving at the two solutions we will need in order to form a general solution that will work for any linear, constant coefficient, second order differential equation. This is easier than it might initially look.

We will use the solutions we found in the first example as a guide. All of the solutions in this example were in the form

$$y(t) = \mathbf{e}^{rt}$$

Note, that we didn't include a constant in front of it since we can literally include any constant that we want and still get a solution. The important idea here is to get the exponential function. Once we have that we can add on constants to our hearts content.

So, let's assume that all solutions to

$$ay'' + by' + cy = 0 \qquad \textbf{(4)}$$

will be of the form

$$y(t) = \mathbf{e}^{rt} \qquad \textbf{(5)}$$

To see if we are correct all we need to do is plug this into the differential equation and see what happens. So, let's get some derivatives and then plug in.

$$y'(t) = r\mathbf{e}^{rt} \qquad\qquad y''(t) = r^2\mathbf{e}^{rt}$$

$$a\left(r^2\mathbf{e}^{rt}\right) + b\left(r\mathbf{e}^{rt}\right) + c\left(\mathbf{e}^{rt}\right) = 0$$

$$\mathbf{e}^{rt}\left(ar^2 + br + c\right) = 0$$

So, if (5) is to be a solution to (4) then the following must be true

$$\mathbf{e}^{rt}\left(ar^2 + br + c\right) = 0$$

This can be reduced further by noting that exponentials are never zero. Therefore, (5) will be a solution to (4) provided r is a solution to

$$ar^2 + br + c = 0 \qquad \textbf{(6)}$$

This equation is typically called the **characteristic equation** for (4).

Okay, so how do we use this to find solutions to a linear, constant coefficient, second order differential equation? First write down the characteristic equation, (6), for the differential equation, (4). This will be a quadratic equation and so we should expect two roots, r_1 and r_2. Once we have these two roots we have two solutions to the differential equation.

$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t} \qquad \textbf{(7)}$$

Let's take a look at a quick example.

Example 3 Find two solutions to

$$y'' - 9y = 0$$

Solution

This is the same differential equation that we looked at in the first example. This time however, let's not just guess. Let's go through the process as outlined above to see the functions that we guess above are the same as the functions the process gives us.

First write down the characteristic equation for this differential equation and solve it.

$$r^2 - 9 = 0 \qquad \Rightarrow \qquad r = \pm 3$$

The two roots are 3 and -3. Therefore, two solutions are

$$y_1(t) = \mathbf{e}^{3t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{-3t}$$

These match up with the first guesses that we made in the first example.

You'll notice that we neglected to mention whether or not the two solutions listed in (7) are in fact "nice enough" to form the general solution to (4). This was intentional. We have three cases that we need to look at and this will be addressed differently in each of these cases.

So, what are the cases? As we previously noted the characteristic equation is quadratic and so will have two roots, r_1 and r_2. The roots will have three possible forms. These are

1. Real, distinct roots, $r_1 \neq r_2$.
2. Complex root, $r_{1,2} = \lambda \pm \mu i$.
3. Double roots, $r_1 = r_2 = r$.

The next three sections will look at each of these in some more depth, including giving forms for the solution that will be "nice enough" to get a general solution.

Real, Distinct Roots

It's time to start solving constant coefficient, homogeneous, linear, second order differential equations. So, let's recap how we do this from the last section. We start with the differential equation.

$$ay'' + by' + cy = 0$$

Write down the characteristic equation.

$$ar^2 + br + c = 0$$

Solve the characteristic equation for the two roots, r_1 and r_2. This gives the two solutions

$$y_1(t) = \mathbf{e}^{r_1 t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{r_2 t}$$

Now, if the two roots are real and distinct (*i.e.* $r_1 \neq r_2$) it will turn out that these two solutions are "nice enough" to form the general solution

$$y(t) = c_1 \mathbf{e}^{r_1 t} + c_2 \mathbf{e}^{r_2 t}$$

As with the last section, we'll ask that you believe us when we say that these are "nice enough". You will be able to prove this easily enough once we reach a later section.

With real, distinct roots there really isn't a whole lot to do other than work a couple of examples so let's do that.

Example 1 Solve the following IVP.

$$y'' + 11y' + 24y = 0 \qquad y(0) = 0 \qquad y'(0) = -7$$

Solution

The characteristic equation is

$$r^2 + 11r + 24 = 0$$
$$(r+8)(r+3) = 0$$

Its roots are r_1 = - 8 and r_2 = -3 and so the general solution and its derivative is.

$$y(t) = c_1 \mathbf{e}^{-8t} + c_2 \mathbf{e}^{-3t}$$
$$y'(t) = -8c_1 \mathbf{e}^{-8t} - 3c_2 \mathbf{e}^{-3t}$$

Now, plug in the initial conditions to get the following system of equations.

$$0 = y(0) = c_1 + c_2$$
$$-7 = y'(0) = -8c_1 - 3c_2$$

Solving this system gives $c_1 = \frac{7}{5}$ and $c_2 = -\frac{7}{5}$. The actual solution to the differential equation is then

$$y(t) = \frac{7}{5}\mathbf{e}^{-8t} - \frac{7}{5}\mathbf{e}^{-3t}$$

Example 2 Solve the following IVP

$$y'' + 3y' - 10y = 0 \qquad y(0) = 4 \qquad y'(0) = -2$$

Solution

The characteristic equation is

$$r^2 + 3r - 10 = 0$$
$$(r+5)(r-2) = 0$$

Its roots are r_1 = - 5 and r_2 = 2 and so the general solution and its derivative is.

$$y(t) = c_1\mathbf{e}^{-5t} + c_2\mathbf{e}^{2t}$$
$$y'(t) = -5c_1\mathbf{e}^{-5t} + 2c_2\mathbf{e}^{2t}$$

Now, plug in the initial conditions to get the following system of equations.

$$4 = y(0) = c_1 + c_2$$
$$-2 = y'(0) = -5c_1 + 2c_2$$

Solving this system gives $c_1 = \frac{10}{7}$ and $c_2 = \frac{18}{7}$. The actual solution to the differential equation is then

$$y(t) = \frac{10}{7}\mathbf{e}^{-5t} + \frac{18}{7}\mathbf{e}^{2t}$$

Example 3 Solve the following IVP.

$$3y'' + 2y' - 8y = 0 \qquad y(0) = -6 \qquad y'(0) = -18$$

Solution

The characteristic equation is

$$3r^2 + 2r - 8 = 0$$
$$(3r-4)(r+2) = 0$$

Its roots are $r_1 = \frac{4}{3}$ and r_2 = -2 and so the general solution and its derivative is.

$$y(t) = c_1\mathbf{e}^{\frac{4t}{3}} + c_2\mathbf{e}^{-2t}$$
$$y'(t) = \frac{4}{3}c_1\mathbf{e}^{\frac{4t}{3}} - 2c_2\mathbf{e}^{-2t}$$

Now, plug in the initial conditions to get the following system of equations.

$$-6 = y(0) = c_1 + c_2$$
$$-18 = y'(0) = \frac{4}{3}c_1 - 2c_2$$

Solving this system gives c_1 = -9 and c_2 = 3. The actual solution to the differential equation is then.

$$y(t) = -9\mathbf{e}^{\frac{4t}{3}} + 3\mathbf{e}^{-2t}$$

Example 4 Solve the following IVP

$$4y'' - 5y' = 0 \qquad y(-2) = 0 \qquad y'(-2) = 7$$

Solution

The characteristic equation is

$$\begin{aligned} 4r^2 - 5r &= 0 \\ r(4r-5) &= 0 \end{aligned}$$

The roots of this equation are $r_1 = 0$ and $r_2 = \frac{5}{4}$. Here is the general solution as well as its derivative.

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^0 + c_2 \mathbf{e}^{\frac{5t}{4}} = c_1 + c_2 \mathbf{e}^{\frac{5t}{4}} \\ y'(t) &= \frac{5}{4} c_2 \mathbf{e}^{\frac{5t}{4}} \end{aligned}$$

Up to this point all of the initial conditions have been at $t = 0$ and this one isn't. Don't get too locked into initial conditions always being at $t = 0$ and you just automatically use that instead of the actual value for a given problem.

So, plugging in the initial conditions gives the following system of equations to solve.

$$\begin{aligned} 0 &= y(-2) = c_1 + c_2 \mathbf{e}^{-\frac{5}{2}} \\ 7 &= y'(-2) = \frac{5}{4} c_2 \mathbf{e}^{-\frac{5}{2}} \end{aligned}$$

Solving this gives.

$$c_1 = -\frac{28}{5} \qquad c_2 = \frac{28}{5} \mathbf{e}^{\frac{5}{2}}$$

The solution to the differential equation is then.

$$y(t) = -\frac{28}{5} + \frac{28}{5} \mathbf{e}^{\frac{5}{2}} \mathbf{e}^{\frac{5t}{4}} = -\frac{28}{5} + \frac{28}{5} \mathbf{e}^{\frac{5t}{4} + \frac{5}{2}}$$

In a differential equations class most instructors (including me….) tend to use initial conditions at $t = 0$ because it makes the work a little easier for the students as they are trying to learn the subject. However, there is no reason to always expect that this will be the case, so do not start to always expect initial conditions at $t = 0$!

Let's do one final example to make another point that you need to be made aware of.

Example 5 Find the general solution to the following differential equation.

$$y'' - 6y' - 2y = 0$$

Solution

The characteristic equation is.

$$r^2 - 6r - 2 = 0$$

The roots of this equation are.

$$r_{1,2} = 3 \pm \sqrt{11}$$

Now, do NOT get excited about these roots they are just two real numbers.

$$r_1 = 3 + \sqrt{11} \qquad \text{and} \qquad r_1 = 3 - \sqrt{11}$$

Admittedly they are not as nice looking as we may be used to, but they are just real numbers. Therefore, the general solution is

$$y(t) = c_1 \mathbf{e}^{\left(3+\sqrt{11}\right)t} + c_2 \mathbf{e}^{\left(3-\sqrt{11}\right)t}$$

If we had initial conditions we could proceed as we did in the previous two examples although the work would be somewhat messy and so we aren't going to do that for this example.

The point of the last example is make sure that you don't get to used to "nice", simple roots. In practice roots of the characteristic equation will generally not be nice, simple integers or fractions so don't get too used to them!

Complex Roots

In this section we will be looking at solutions to the differential equation

$$ay'' + by' + cy = 0$$

in which roots of the characteristic equation,

$$ar^2 + br + c = 0$$

are complex roots in the form $r_{1,2} = \lambda \pm \mu i$.

Now, recall that we arrived at the characteristic equation by assuming that all solutions to the differential equation will be of the form

$$y(t) = \mathbf{e}^{rt}$$

Plugging our two roots into the general form of the solution gives the following solutions to the differential equation.

$$y_1(t) = \mathbf{e}^{(\lambda+\mu i)t} \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{(\lambda-\mu i)t}$$

Now, these two functions are "nice enough" (there's those words again… we'll get around to defining them eventually) to form the general solution. We do have a problem however. Since we started with only real numbers in our differential equation we would like our solution to only involve real numbers. The two solutions above are complex and so we would like to get our hands on a couple of solutions ("nice enough" of course…) that are real.

To do this we'll need Euler's Formula.

$$\mathbf{e}^{i\theta} = \cos\theta + i\sin\theta$$

A nice variant of Euler's Formula that we'll need is.

$$\mathbf{e}^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Now, split up our two solutions into exponentials that only have real exponents and exponentials that only have imaginary exponents. Then use Euler's formula, or its variant, to rewrite the second exponential.

$$y_1(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)$$
$$y_2(t) = \mathbf{e}^{\lambda t}\mathbf{e}^{-i\mu t} = \mathbf{e}^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right)$$

This doesn't eliminate the complex nature of the solutions, but it does put the two solutions into a form that we can eliminate the complex parts.

Recall from the basics section that if two solutions are "nice enough" then any solution can be written as a combination of the two solutions. In other words,

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

will also be a solution.

Using this let's notice that if we add the two solutions together we will arrive at.

$$y_1(t) + y_2(t) = 2\mathbf{e}^{\lambda t}\cos(\mu t)$$

This is a real solution and just to eliminate the extraneous 2 let's divide everything by a 2. This gives the first real solution that we're after.

$$u(t) = \frac{1}{2} y_1(t) + \frac{1}{2} y_2(t) = \mathbf{e}^{\lambda t} \cos(\mu t)$$

Note that this is just equivalent to taking

$$c_1 = c_2 = \frac{1}{2}$$

Now, we can arrive at a second solution in a similar manner. This time let's subtract the two original solutions to arrive at.

$$y_1(t) - y_2(t) = 2i\,\mathbf{e}^{\lambda t} \sin(\mu t)$$

On the surface this doesn't appear to fix the problem as the solution is still complex. However, upon learning that the two constants, c_1 and c_2 can be complex numbers we can arrive at a real solution by dividing this by $2i$. This is equivalent to taking

$$c_1 = \frac{1}{2i} \qquad \text{and} \qquad c_2 = -\frac{1}{2i}$$

Our second solution will then be

$$v(t) = \frac{1}{2i} y_1(t) - \frac{1}{2i} y_2(t) = \mathbf{e}^{\lambda t} \sin(\mu t)$$

We now have two solutions (we'll leave it to you to check that they are in fact solutions) to the differential equation.

$$u(t) = \mathbf{e}^{\lambda t} \cos(\mu t) \qquad \text{and} \qquad v(t) = \mathbf{e}^{\lambda t} \sin(\mu t)$$

It also turns out that these two solutions are "nice enough" to form a general solution.

So, if the roots of the characteristic equation happen to be $r_{1,2} = \lambda \pm \mu i$ the general solution to the differential equation is.

$$y(t) = c_1 \mathbf{e}^{\lambda t} \cos(\mu t) + c_2 \mathbf{e}^{\lambda t} \sin(\mu t)$$

Let's take a look at a couple of examples now.

Example 1 Solve the following IVP.

$$y'' - 4y' + 9y = 0 \qquad y(0) = 0 \qquad y'(0) = -8$$

Solution

The characteristic equation for this differential equation is.

$$r^2 - 4r + 9 = 0$$

The roots of this equation are $r_{1,2} = 2 \pm \sqrt{5}\, i$. The general solution to the differential equation is then.

$$y(t) = c_1 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right) + c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Now, you'll note that we didn't differentiate this right away as we did in the last section. The reason for this is simple. While the differentiation is not terribly difficult, it can get a little messy. So, first looking at the initial conditions we can see from the first one that if we just applied it we would get the following.

$$0 = y(0) = c_1$$

In other words, the first term will drop out in order to meet the first condition. This makes the solution, along with its derivative

$$y(t) = c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$
$$y'(t) = 2c_2 \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right) + \sqrt{5}c_2 \mathbf{e}^{2t} \cos\left(\sqrt{5}t\right)$$

A much nicer derivative than if we'd done the original solution. Now, apply the second initial condition to the derivative to get.

$$-8 = y'(0) = \sqrt{5}c_2 \qquad \Rightarrow \qquad c_2 = -\frac{8}{\sqrt{5}}$$

The actual solution is then.

$$y(t) = -\frac{8}{\sqrt{5}} \mathbf{e}^{2t} \sin\left(\sqrt{5}t\right)$$

Example 2 Solve the following IVP.

$$y'' - 8y' + 17y = 0 \qquad y(0) = -4 \qquad y'(0) = -1$$

Solution

The characteristic equation this time is.

$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$y(t) = c_1 \mathbf{e}^{4t} \cos(t) + c_2 \mathbf{e}^{4t} \sin(t)$$
$$y'(t) = 4c_1 \mathbf{e}^{4t} \cos(t) - c_1 \mathbf{e}^{4t} \sin(t) + 4c_2 \mathbf{e}^{4t} \sin(t) + c_2 \mathbf{e}^{4t} \cos(t)$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$-4 = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + c_2$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.

$$y(t) = -4\mathbf{e}^{4t} \cos(t) + 15\mathbf{e}^{4t} \sin(t)$$

Example 3 Solve the following IVP.

$$4y'' + 24y' + 37y = 0 \qquad y(\pi) = 1 \qquad y'(\pi) = 0$$

Solution

The characteristic equation this time is.

$$4r^2 + 24r + 37 = 0$$

The roots of this are $r_{1,2} = -3 \pm \frac{1}{2} i$. The general solution as well as its derivative is

$$y(t) = c_1 \mathbf{e}^{-3t} \cos\left(\frac{t}{2}\right) + c_2 \mathbf{e}^{-3t} \sin\left(\frac{t}{2}\right)$$

$$y'(t) = -3c_1 \mathbf{e}^{-3t} \cos\left(\frac{t}{2}\right) - \frac{c_1}{2}\mathbf{e}^{-3t} \sin\left(\frac{t}{2}\right) - 3c_2 \mathbf{e}^{-3t} \sin\left(\frac{t}{2}\right) + \frac{c_2}{2}\mathbf{e}^{-3t} \cos\left(\frac{t}{2}\right)$$

Applying the initial conditions gives the following system.

$$1 = y(\pi) = c_1 \mathbf{e}^{-3\pi} \cos\left(\frac{\pi}{2}\right) + c_2 \mathbf{e}^{-3\pi} \sin\left(\frac{\pi}{2}\right) = c_2 \mathbf{e}^{-3\pi}$$

$$0 = y'(\pi) = -\frac{c_1}{2}\mathbf{e}^{-3\pi} - 3c_2 \mathbf{e}^{-3\pi}$$

Do not forget to plug the $t = \pi$ into the exponential! This is one of the more common mistakes that students make on these problems. Also, make sure that you evaluate the trig functions as much as possible in these cases. It will only make your life simpler. Solving this system gives

$$c_1 = -6\mathbf{e}^{3\pi} \qquad c_2 = \mathbf{e}^{3\pi}$$

The actual solution to the IVP is then.

$$y(t) = -6\mathbf{e}^{3\pi}\mathbf{e}^{-3t} \cos\left(\frac{t}{2}\right) + \mathbf{e}^{3\pi}\mathbf{e}^{-3t} \sin\left(\frac{t}{2}\right)$$

$$y(t) = -6\mathbf{e}^{-3(t-\pi)} \cos\left(\frac{t}{2}\right) + \mathbf{e}^{-3(t-\pi)} \sin\left(\frac{t}{2}\right)$$

Let's do one final example before moving on to the next topic.

Example 4 Solve the following IVP.

$$y'' + 16y = 0 \qquad y\left(\frac{\pi}{2}\right) = -10 \qquad y'\left(\frac{\pi}{2}\right) = 3$$

Solution

The characteristic equation for this differential equation and its roots are.

$$r^2 + 16 = 0 \qquad \Rightarrow \qquad r = \pm 4i$$

The general solution to this differential equation and its derivative is.

$$y(t) = c_1 \cos(4t) + c_2 \sin(4t)$$

$$y'(t) = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

Plugging in the initial conditions gives the following system.

$$-10 = y\left(\frac{\pi}{2}\right) = c_1 \qquad\qquad c_1 = -10$$

$$3 = y'\left(\frac{\pi}{2}\right) = 4c_2 \qquad\qquad c_2 = \frac{3}{4}$$

So, the constants drop right out with this system and the actual solution is.

$$y(t) = -10\cos(4t) + \frac{3}{4}\sin(4t)$$

Repeated Roots

In this section we will be looking at the last case for the constant coefficient, linear, homogeneous second order differential equations. In this case we want solutions to

$$ay'' + by' + cy = 0$$

where solutions to the characteristic equation

$$ar^2 + br + c = 0$$

are double roots $r_1 = r_2 = r$.

This leads to a problem however. Recall that the solutions are

$$y_1(t) = \mathbf{e}^{r_1 t} = \mathbf{e}^{r\,t} \qquad\qquad y_2(t) = \mathbf{e}^{r_2 t} = \mathbf{e}^{r\,t}$$

These are the same solution and will NOT be "nice enough" to form a general solution. I do promise that I'll define "nice enough" eventually! So, we can use the first solution, but we're going to need a second solution.

Before finding this second solution let's take a little side trip. The reason for the side trip will be clear eventually. From the quadratic formula we know that the roots to the characteristic equation are,

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, since we have double roots we must have

$$b^2 - 4ac = 0$$

This is the only way that we can get double roots and in this case the roots will be

$$r_{1,2} = \frac{-b}{2a}$$

So, the one solution that we've got is

$$y_1(t) = \mathbf{e}^{-\frac{bt}{2a}}$$

To find a second solution we will use the fact that a constant times a solution to a linear homogeneous differential equation is also a solution. If this is true then *maybe* we'll get lucky and the following will also be a solution

$$y_2(t) = v(t)\,y_1(t) = v(t)\,\mathbf{e}^{-\frac{bt}{2a}} \qquad \textbf{(1)}$$

with a proper choice of $v(t)$. To determine if this in fact can be done, let's plug this back into the differential equation and see what we get. We'll first need a couple of derivatives.

$$\begin{aligned} y_2'(t) &= v'\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v\,\mathbf{e}^{-\frac{bt}{2a}} \\ y_2''(t) &= v''\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v'\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{2a} v'\,\mathbf{e}^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v\,\mathbf{e}^{-\frac{bt}{2a}} \\ &= v''\,\mathbf{e}^{-\frac{bt}{2a}} - \frac{b}{a} v'\,\mathbf{e}^{-\frac{bt}{2a}} + \frac{b^2}{4a^2} v\,\mathbf{e}^{-\frac{bt}{2a}} \end{aligned}$$

We dropped the *(t)* part on the *v* to simplify things a little for the writing out of the derivatives. Now, plug these into the differential equation.

$$a\left(v''\mathbf{e}^{-\frac{bt}{2a}}-\frac{b}{a}v'\mathbf{e}^{-\frac{bt}{2a}}+\frac{b^2}{4a^2}v\,\mathbf{e}^{-\frac{bt}{2a}}\right)+b\left(v'\mathbf{e}^{-\frac{bt}{2a}}-\frac{b}{2a}v\,\mathbf{e}^{-\frac{bt}{2a}}\right)+c\left(v\,\mathbf{e}^{-\frac{bt}{2a}}\right)=0$$

We can factor an exponential out of all the terms so let's do that. We'll also collect all the coefficients of *v* and its derivatives.

$$\begin{aligned}\mathbf{e}^{-\frac{bt}{2a}}\left(av''+\left(-b+b\right)v'+\left(\frac{b^2}{4a}-\frac{b^2}{2a}+c\right)v\right)&=0\\ \mathbf{e}^{-\frac{bt}{2a}}\left(av''+\left(-\frac{b^2}{4a}+c\right)v\right)&=0\\ \mathbf{e}^{-\frac{bt}{2a}}\left(av''-\frac{1}{4a}\left(b^2-4ac\right)v\right)&=0\end{aligned}$$

Now, because we are working with a double root we know that that the second term will be zero. Also exponentials are never zero. Therefore, (1) will be a solution to the differential equation provided *v(t)* is a function that satisfies the following differential equation.

$$av''=0 \qquad \text{OR} \qquad v''=0$$

We can drop the *a* because we know that it can't be zero. If it were we wouldn't have a second order differential equation! So, we can now determine the most general possible form that is allowable for *v(t)*.

$$v'=\int v''\,dt=c \qquad\qquad v\left(t\right)=\int v'\,dt=ct+k$$

This is actually more complicated than we need and in fact we can drop both of the constants from this. To see why this is let's go ahead and use this to get the second solution. The two solutions are then

$$y_1\left(t\right)=\mathbf{e}^{-\frac{bt}{2a}} \qquad\qquad y_2\left(t\right)=\left(ct+k\right)\mathbf{e}^{-\frac{bt}{2a}}$$

Eventually you will be able to show that these two solutions are "nice enough" to form a general solution. The general solution would then be the following.

$$\begin{aligned}y\left(t\right)&=c_1\mathbf{e}^{-\frac{bt}{2a}}+c_2\left(ct+k\right)\mathbf{e}^{-\frac{bt}{2a}}\\ &=c_1\mathbf{e}^{-\frac{bt}{2a}}+\left(c_2ct+c_2k\right)\mathbf{e}^{-\frac{bt}{2a}}\\ &=\left(c_1+c_2k\right)\mathbf{e}^{-\frac{bt}{2a}}+c_2c\,t\,\mathbf{e}^{-\frac{bt}{2a}}\end{aligned}$$

Notice that we rearranged things a little. Now, c, k, c_1, and c_2 are all unknown constants so any combination of them will also be unknown constants. In particular, $c_1+c_2\,k$ and $c_2\,c$ are unknown constants so we'll just rewrite them as follows.

$$y\left(t\right)=c_1\mathbf{e}^{-\frac{bt}{2a}}+c_2\,t\,\mathbf{e}^{-\frac{bt}{2a}}$$

So, if we go back to the most general form for *v(t)* we can take *c=1* and *k=0* and we will arrive at the same general solution.

Let's recap. If the roots of the characteristic equation are $r_1 = r_2 = r$, then the general solution is then

$$y(t) = c_1\mathbf{e}^{rt} + c_2 t\mathbf{e}^{rt}$$

Now, let's work a couple of examples.

Example 1 Solve the following IVP.

$$y'' - 4y' + 4y = 0 \qquad y(0) = 12 \qquad y'(0) = -3$$

Solution

The characteristic equation and its roots are.

$$r^2 - 4r + 4 = (r-2)^2 = 0 \qquad\qquad r_{1,2} = 2$$

The general solution and its derivative are

$$y(t) = c_1\mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t}$$
$$y'(t) = 2c_1\mathbf{e}^{2t} + c_2\mathbf{e}^{2t} + 2c_2 t\mathbf{e}^{2t}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$12 = y(0) = c_1$$
$$-3 = y'(0) = 2c_1 + c_2$$

This system is easily solve to get $c_1 = 12$ and $c_2 = -27$. The actual solution to the IVP is then.

$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

Example 2 Solve the following IVP.

$$16y'' - 40y' + 25y = 0 \qquad y(0) = 3 \qquad y'(0) = -\frac{9}{4}$$

Solution

The characteristic equation and its roots are.

$$16r^2 - 40r + 25 = (4r-5)^2 = 0 \qquad r_{1,2} = \frac{5}{4}$$

The general solution and its derivative are

$$y(t) = c_1\mathbf{e}^{\frac{5t}{4}} + c_2 t\mathbf{e}^{\frac{5t}{4}}$$
$$y'(t) = \frac{5}{4}c_1\mathbf{e}^{\frac{5t}{4}} + c_2\mathbf{e}^{\frac{5t}{4}} + \frac{5}{4}c_2 t\mathbf{e}^{\frac{5t}{4}}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$3 = y(0) = c_1$$
$$-\frac{9}{4} = y'(0) = \frac{5}{4}c_1 + c_2$$

This system is easily solve to get $c_1 = 3$ and $c_2 = -6$. The actual solution to the IVP is then.

$$y(t) = 3\mathbf{e}^{\frac{5t}{4}} - 6t\mathbf{e}^{\frac{5t}{4}}$$

Example 3 Solve the following IVP

$$y'' + 14y' + 49y = 0 \qquad y(-4) = -1 \qquad y'(-4) = 5$$

Solution

The characteristic equation and its roots are.

$$r^2 + 14r + 49 = (r+7)^2 = 0 \qquad\qquad r_{1,2} = -7$$

The general solution and its derivative are

$$y(t) = c_1\mathbf{e}^{-7t} + c_2t\mathbf{e}^{-7t}$$
$$y'(t) = -7c_1\mathbf{e}^{-7t} + c_2\mathbf{e}^{-7t} - 7c_2t\mathbf{e}^{-7t}$$

Plugging in the initial conditions gives the following system of equations.

$$-1 = y(-4) = c_1\mathbf{e}^{28} - 4c_2\mathbf{e}^{28}$$
$$5 = y'(-4) = -7c_1\mathbf{e}^{28} + c_2\mathbf{e}^{28} + 28c_2\mathbf{e}^{28} = -7c_1\mathbf{e}^{28} + 29c_2\mathbf{e}^{28}$$

Solving this system gives the following constants.

$$c_1 = -9\mathbf{e}^{-28} \qquad\qquad c_2 = -2\mathbf{e}^{-28}$$

The actual solution to the IVP is then.

$$y(t) = -9\mathbf{e}^{-28}\mathbf{e}^{-7t} - 2t\mathbf{e}^{-28}\mathbf{e}^{-7t}$$
$$y(t) = -9\mathbf{e}^{-7(t+4)} - 2t\mathbf{e}^{-7(t+4)}$$

Reduction of Order

We're now going to take a brief detour and look at solutions to non-constant coefficient, second order differential equations of the form.

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = 0$$

In general, finding solutions to these kinds of differential equations can be much more difficult than finding solutions to constant coefficient differential equations. However, if we already know one solution to the differential equation we can use the method that we used in the last section to find a second solution. This method is called **reduction of order**.

Let's take a quick look at an example to see how this is done.

Example 1 Find the general solution to

$$2t^2 y'' + ty' - 3y = 0$$

given that $y_1(t) = t^{-1}$ is a solution.

Solution

Reduction of order requires that a solution already be known. Without this known solution we won't be able to do reduction of order.

Once we have this first solution we will then assume that a second solution will have the form

$$y_2(t) = v(t)\,y_1(t) \tag{1}$$

for a proper choice of $v(t)$. To determine the proper choice, we plug the guess into the differential equation and get a new differential equation that can be solved for $v(t)$.

So, let's do that for this problem. Here is the form of the second solution as well as the derivatives that we'll need.

$$y_2(t) = t^{-1}v \qquad\qquad y_2'(t) = -t^{-2}v + t^{-1}v' \qquad\qquad y_2''(t) = 2t^{-3}v - 2t^{-2}v' + t^{-1}v''$$

Plugging these into the differential equation gives

$$2t^2\left(2t^{-3}v - 2t^{-2}v' + t^{-1}v''\right) + t\left(-t^{-2}v + t^{-1}v'\right) - 3\left(t^{-1}v\right) = 0$$

Rearranging and simplifying gives

$$\begin{aligned} 2tv'' + (-4+1)v' + \left(4t^{-1} - t^{-1} - 3t^{-1}\right)v &= 0 \\ 2tv'' - 3v' &= 0 \end{aligned}$$

Note that upon simplifying the only terms remaining are those involving the derivatives of v. The term involving v drops out. If you've done all of your work correctly this should always happen. Sometimes, as in the repeated roots case, the first derivative term will also drop out.

So, in order for (1) to be a solution then v must satisfy

$$2tv'' - 3v' = 0 \tag{2}$$

This appears to be a problem. In order to find a solution to a second order non-constant coefficient differential equation we need to solve a different second order non-constant coefficient differential equation.

However, this isn't the problem that it appears to be. Because the term involving the *v* drops out we can actually solve (2) and we can do it with the knowledge that we already have at this point. We will solve this by making the following **change of variable**.

$$w = v' \qquad \Rightarrow \qquad w' = v''$$

With this change of variable (2) becomes

$$2tw' - 3w = 0$$

and this is a linear, first order differential equation that we can solve. This also explains the name of this method. We've managed to reduce a second order differential equation down to a first order differential equation.

This is a fairly simple first order differential equation so I'll leave the details of the solving to you. If you need a refresher on solving linear, first order differential equations go back to the second chapter and check out that section. The solution to this differential equation is

$$w(t) = c\,t^{\frac{3}{2}}$$

Now, this is not quite what we were after. We are after a solution to (2). However, we can now find this. Recall our change of variable.

$$v' = w$$

With this we can easily solve for *v(t)*.

$$v(t) = \int w\,dt = \int ct^{\frac{3}{2}}\,dt = \frac{2}{5}ct^{\frac{5}{2}} + k$$

This is the most general possible *v(t)* that we can use to get a second solution. So, just as we did in the repeated roots section, we can choose the constants to be anything we want so choose them to clear out all the extraneous constants. In this case we can use

$$c = \frac{5}{2} \qquad\qquad k = 0$$

Using these gives the following for *v(t)* and for the second solution.

$$v(t) = t^{\frac{5}{2}} \qquad \Rightarrow \qquad y_2(t) = t^{-1}\left(t^{\frac{5}{2}}\right) = t^{\frac{3}{2}}$$

Then general solution will then be,

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$$

If we had been given initial conditions we could then differentiate, apply the initial conditions and solve for the constants.

Reduction of order, the method used in the previous example can be used to find second solutions to differential equations. However, this does require that we already have a solution and often finding that first solution is a very difficult task and often in the process of finding the first solution you will also get the second solution without needing to resort to reduction of order. So, for those cases when we do have a first solution this is a nice method for getting a second solution.

Let's do one more example.

Example 2 Find the general solution to

$$t^2 y'' + 2ty' - 2y = 0$$

given that $y_1(t) = t$ is a solution.

Solution

The form for the second solution as well as its derivatives are,

$$y_2(t) = tv \qquad y_2'(t) = v + tv' \qquad y_2''(t) = 2v' + tv''$$

Plugging these into the differential equation gives,

$$t^2\left(2v' + tv''\right) + 2t\left(v + tv'\right) - 2\left(tv\right) = 0 = 0$$

Rearranging and simplifying gives the differential equation that we'll need to solve in order to determine the correct *v* that we'll need for the second solution.

$$t^3 v'' + 4t^2 v' = 0$$

Next use the variable transformation as we did in the previous example.

$$w = v' \qquad \Rightarrow \qquad w' = v''$$

With this change of variable the differential equation becomes

$$t^3 w' + 4t^2 w = 0$$

and this is a linear, first order differential equation that we can solve. We'll leave the details of the solution process to you.

$$w(t) = c\,t^{-4}$$

Now solve for *v(t)*.

$$v(t) = \int w\,dt = \int ct^{-4}\,dt = -\frac{1}{3}ct^{-3} + k$$

As with the first example we'll drop the constants and use the following *v(t)*

$$v(t) = t^{-3} \qquad \Rightarrow \qquad y_2(t) = t\left(t^{-3}\right) = t^{-2}$$

Then general solution will then be,

$$y(t) = c_1 t + \frac{c_2}{t^2}$$

On a side note, both of the differential equations in this section were of the form,

$$t^2 y'' + \alpha\, t\, y' + \beta y = 0$$

These are called Euler differential equations and are fairly simple to solve directly for both solutions. To see how to solve these directly take a look at the Euler Differential Equation section.

Fundamental Sets of Solutions

The time has finally come to define "nice enough". We've been using this term throughout the last few sections to describe those solutions that could be used to form a general solution and it is now time to officially define it.

First, because everything that we're going to be doing here only requires linear and homogeneous we won't require constant coefficients in our differential equation. So, let's start with the following IVP.

$$\begin{gathered} p(t)\,y'' + q(t)\,y' + r(t)\,y = 0 \\ y(t_0) = y_0 \qquad y'(t_0) = y'_0 \end{gathered} \tag{1}$$

Let's also suppose that we have already found two solutions to this differential equation, $y_1(t)$ and $y_2(t)$. We know from the Principle of Superposition that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{2}$$

will also be a solution to the differential equation. What we want to know is whether or not it will be a general solution. In order for (2) to be considered a general solution it must satisfy the general initial conditions in (1).

$$y(t_0) = y_0 \qquad\qquad y'(t_0) = y'_0$$

This will also imply that any solution to the differential equation can be written in this form.

So, let's see if we can find constants that will satisfy these conditions. First differentiate (2) and plug in the initial conditions.

$$\begin{aligned} y_0 &= y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \\ y'_0 &= y'(t_0) = c_1 y'_1(t_0) + c_2 y'_2(t_0) \end{aligned} \tag{3}$$

Since we are assuming that we've already got the two solutions everything in this system is technically known and so this is a system that can be solved for c_1 and c_2. This can be done in general using Cramer's Rule. Using Cramer's Rule gives the following solution.

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \qquad\qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} \tag{4}$$

where,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

is the determinant of a 2x2 matrix. If you don't know about determinants that is okay, just use the formula that we've provided above.

Now, (4) will give the solution to the system (3). Note that in practice we generally don't use Cramer's Rule to solve systems, we just proceed in a straightforward manner and solve the system using basic algebra techniques. So, why did we use Cramer's Rule here then?

We used Cramer's Rule because we can use (4) to develop a condition that will allow us to determine when we can solve for the constants. All three (yes three, the denominators are the same!) of the quantities in (4) are just numbers and the only thing that will prevent us from actually getting a solution will be when the denominator is zero.

The quantity in the denominator is called the **Wronskian** and is denoted as

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)\,g'(t) - g(t)\,f'(t)$$

When it is clear what the functions and/or t are we often just denote the Wronskian by W.

Let's recall what we were after for here. We wanted to determine when two solutions to (1) would be nice enough to form a general solution. The two solutions will form a general solution to (1) if they satisfy the general initial conditions given in (1) and we can see from Cramer's Rule that they will satisfy the initial conditions provided the Wronskian isn't zero. Or,

$$W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\,y_2'(t_0) - y_2(t_0)\,y_1'(t_0) \neq 0$$

So, suppose that $y_1(t)$ and $y_2(t)$ are two solutions to (1) and that $W(y_1,y_2)(t) \neq 0$. Then the two solutions are called a **fundamental set of solutions** and the general solution to (1) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

We know now what "nice enough" means. Two solutions are "nice enough" if they are a fundamental set of solutions.

So, let's check one of the claims that we made in a previous section. We'll leave the other two to you to check if you'd like to.

Example 1 Back in the complex root section we made the claim that

$$y_1(t) = \mathbf{e}^{\lambda t}\cos(\mu t) \qquad \text{and} \qquad y_2(t) = \mathbf{e}^{\lambda t}\sin(\mu t)$$

were a fundamental set of solutions. Prove that they in fact are.

Solution

So, to prove this we will need to take find the Wronskian for these two solutions and show that it isn't zero.

$$W = \begin{vmatrix} \mathbf{e}^{\lambda t}\cos(\mu t) & \mathbf{e}^{\lambda t}\sin(\mu t) \\ \lambda\mathbf{e}^{\lambda t}\cos(\mu t) - \mu\mathbf{e}^{\lambda t}\sin(\mu t) & \lambda\mathbf{e}^{\lambda t}\sin(\mu t) + \mu\mathbf{e}^{\lambda t}\cos(\mu t) \end{vmatrix}$$
$$= \mathbf{e}^{\lambda t}\cos(\mu t)\left(\lambda\mathbf{e}^{\lambda t}\sin(\mu t) + \mu\mathbf{e}^{\lambda t}\cos(\mu t)\right) - \mathbf{e}^{\lambda t}\sin(\mu t)\left(\lambda\mathbf{e}^{\lambda t}\cos(\mu t) - \mu\mathbf{e}^{\lambda t}\sin(\mu t)\right)$$
$$= \mu\mathbf{e}^{2\lambda t}\cos^2(\mu t) + \mu\mathbf{e}^{2\lambda t}\sin^2(\mu t)$$
$$= \mu\mathbf{e}^{2\lambda t}\left(\cos^2(\mu t) + \sin^2(\mu t)\right)$$
$$= \mu\mathbf{e}^{2\lambda t}$$

Now, the exponential will never be zero and $\mu \neq 0$ (if it were we wouldn't have complex roots!) and so $W \neq 0$. Therefore, these two solutions are in fact a fundamental set of solutions and so the general solution in this case is.

$$y(t) = c_1\mathbf{e}^{\lambda t}\cos(\mu t) + c_2\mathbf{e}^{\lambda t}\sin(\mu t)$$

Example 2 In the first example that we worked in the Reduction of Order section we found a second solution to

$$2t^2 y'' + ty' - 3y = 0$$

Show that this second solution, along with the given solution, form a fundamental set of solutions for the differential equation.

Solution

The two solutions from that example are

$$y_1(t) = t^{-1} \qquad\qquad y_2(t) = t^{\frac{3}{2}}$$

Let's compute the Wronskian of these two solutions.

$$W = \begin{vmatrix} t^{-1} & t^{\frac{3}{2}} \\ -t^{-2} & \frac{3}{2}t^{\frac{1}{2}} \end{vmatrix} = \frac{3}{2}t^{-\frac{1}{2}} - \left(-t^{-\frac{1}{2}}\right) = \frac{5}{2}t^{-\frac{1}{2}} = \frac{5}{2\sqrt{t}}$$

So, the Wronskian will never be zero. Note that we can't plug $t = 0$ into the Wronskian. This would be a problem in finding the constants in the general solution, except that we also can't plug $t = 0$ into the solution either and so this isn't the problem that it might appear to be.

So, since the Wronskian isn't zero for any t the two solutions form a fundamental set of solutions and the general solution is

$$y(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$$

as we claimed in that example.

To this point we've found a set of solutions then we've claimed that they are in fact a fundamental set of solutions. Of course, you can now verify all those claims that we've made, however this does bring up a question. How do we know that for a given differential equation a set of fundamental solutions will exist? The following theorem answers this question.

Theorem

Consider the differential equation

$$y'' + p(t)\,y' + q(t)\,y = 0$$

where *p(t)* and *q(t)* are continuous functions on some interval I. Choose t_0 to be any point in the interval I. Let $y_1(t)$ be a solution to the differential equation that satisfies the initial conditions.

$$y(t_0) = 1 \qquad y'(t_0) = 0$$

Let $y_2(t)$ be a solution to the differential equation that satisfies the initial conditions.

$$y(t_0) = 0 \qquad y'(t_0) = 1$$

Then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the differential equation.

It is easy enough to show that these two solutions form a fundamental set of solutions. Just compute the Wronskian.

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

So, fundamental sets of solutions will exist provided we can solve the two IVP's given in the theorem.

Example 3 Use the theorem to find a fundamental set of solutions for

$$y'' + 4y' + 3y = 0$$

using $t_0 = 0$.

Solution

Using the techniques from the first part of this chapter we can find the two solutions that we've been using to this point.

$$y(t) = \mathbf{e}^{-3t} \qquad\qquad y(t) = \mathbf{e}^{-t}$$

These do form a fundamental set of solutions as we can easily verify. However, they are NOT the set that will be given by the theorem. Neither of these solutions will satisfy either of the two sets of initial conditions given in the theorem. We will have to use these to find the fundamental set of solutions that is given by the theorem.

We know that the following is also solution to the differential equation.

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t}$$

So, let's apply the first set of initial conditions and see if we can find constants that will work.

$$y(0) = 1 \qquad y'(0) = 0$$

We'll leave it to you to verify that we get the following solution upon doing this.

$$y_1(t) = -\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t}$$

Likewise, if we apply the second set of initial conditions,

$$y(0) = 0 \qquad y'(0) = 1$$

we will get

$$y_2(t) = -\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t}$$

According to the theorem these should form a fundament set of solutions. This is easy enough to check.

$$\begin{aligned} W &= \begin{vmatrix} -\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t} & -\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t} \\ \frac{3}{2}\mathbf{e}^{-3t} - \frac{3}{2}\mathbf{e}^{-t} & \frac{3}{2}\mathbf{e}^{-3t} - \frac{1}{2}\mathbf{e}^{-t} \end{vmatrix} \\ &= \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t}\right)\left(\frac{3}{2}\mathbf{e}^{-3t} - \frac{1}{2}\mathbf{e}^{-t}\right) - \left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t}\right)\left(\frac{3}{2}\mathbf{e}^{-3t} - \frac{3}{2}\mathbf{e}^{-t}\right) \\ &= \mathbf{e}^{-4t} \neq 0 \end{aligned}$$

So, we got a completely different set of fundamental solutions from the theorem than what we've been using up to this point. This is not a problem. There are an infinite number of pairs of functions that we could use as a fundamental set of solutions for this problem.

So, which set of fundamental solutions should we use? Well, if we use the ones that we originally found, the general solution would be,

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t}$$

Whereas, if we used the set from the theorem the general solution would be,

$$y(t) = c_1\left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{3}{2}\mathbf{e}^{-t}\right) + c_2\left(-\frac{1}{2}\mathbf{e}^{-3t} + \frac{1}{2}\mathbf{e}^{-t}\right)$$

This would not be very fun to work with when it came to determining the coefficients to satisfy a general set of initial conditions.

So, which set of fundamental solutions should we use? We should always try to use the set that is the most convenient to use for a given problem.

More on the Wronskian

In the previous section we introduced the Wronskian to help us determine whether two solutions were a fundamental set of solutions. In this section we will look at another application of the Wronskian as well as an alternate method of computing the Wronskian.

Let's start with the application. We need to introduce a couple of new concepts first.

Given two non-zero functions *f(x)* and *g(x)* write down the following equation.

$$c\, f\left(x\right) + k\, g\left(x\right) = 0 \tag{1}$$

Notice that $c = 0$ and $k = 0$ will make (1) true for all x regardless of the functions that we use.

Now, if we can find non-zero constants c and k for which (1) will also be true for all x then we call the two functions **linearly dependent**. On the other hand if the only two constants for which (1) is true are $c = 0$ and $k = 0$ then we call the functions **linearly independent.**

Example 1 Determine if the following sets of functions are linearly dependent or linearly independent.

(a) $f\left(x\right) = 9\cos\left(2x\right) \qquad g\left(x\right) = 2\cos^2\left(x\right) - 2\sin^2\left(x\right)$ [Solution]

(b) $f\left(t\right) = 2t^2 \qquad g\left(t\right) = t^4$ [Solution]

Solution

(a) $f\left(x\right) = 9\cos\left(2x\right) \qquad g\left(x\right) = 2\cos^2\left(x\right) - 2\sin^2\left(x\right)$

We'll start by writing down (1) for these two functions.

$$c\left(9\cos\left(2x\right)\right) + k\left(2\cos^2\left(x\right) - 2\sin^2\left(x\right)\right) = 0$$

We need to determine if we can find non-zero constants c and k that will make this true for all x or if $c = 0$ and $k = 0$ are the only constants that will make this true for all x. This is often a fairly difficult process. The process can be simplified with a good intuition for this kind of thing, but that's hard to come by, especially if you haven't done many of these kinds of problems.

In this case the problem can be simplified by recalling

$$\cos^2\left(x\right) - \sin^2\left(x\right) = \cos\left(2x\right)$$

Using this fact our equation becomes.

$$\begin{aligned} 9c\cos\left(2x\right) + 2k\cos\left(2x\right) &= 0 \\ \left(9c + 2k\right)\cos\left(2x\right) &= 0 \end{aligned}$$

With this simplification we can see that this will be zero for any pair of constants c and k that satisfy

$$9c + 2k = 0$$

Among the possible pairs on constants that we could use are the following pairs.

$$
\begin{array}{ll}
c = 1, & k = -\frac{9}{2} \\
c = \frac{2}{9}, & k = -1 \\
c = -2 & k = 9 \\
c = -\frac{7}{6} & k = \frac{21}{4} \\
etc. &
\end{array}
$$

As I'm sure you can see there are literally thousands of possible pairs and they can be made as "simple" or as "complicated" as you want them to be.

So, we've managed to find a pair of non-zero constants that will make the equation true for all x and so the two functions are linearly dependent.

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(b) $f(t) = 2t^2 \qquad g(t) = t^4$

As with the last part, we'll start by writing down (1) for these functions.

$$2ct^2 + kt^4 = 0$$

In this case there isn't any quick and simple formula to write one of the functions in terms of the other as we did in the first part. So, we're just going to have to see if we can find constants. We'll start by noticing that if the original equation is true, then if we differentiate everything we get a new equation that must also be true. In other words, we've got the following system of two equations in two unknowns.

$$
\begin{aligned}
2ct^2 + kt^4 &= 0 \\
4ct + 4kt^3 &= 0
\end{aligned}
$$

We can solve this system for c and k and see what we get. We'll start by solving the second equation for c.

$$c = -kt^2$$

Now, plug this into the first equation.

$$
\begin{aligned}
2\left(-kt^2\right)t^2 + kt^4 &= 0 \\
-kt^4 &= 0
\end{aligned}
$$

Recall that we are after constants that will make this true for all t. The only way that this will ever be zero for all t is if $k = 0$! So, if $k = 0$ we must also have $c = 0$.

Therefore, we've shown that the only way that

$$2ct^2 + kt^4 = 0$$

will be true for all t is to require that $c = 0$ and $k = 0$. The two functions therefore, are linearly independent.

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As we saw in the previous examples determining whether two functions are linearly independent or linearly dependent can be a fairly involved process. This is where the Wronskian can help.

Fact

Given two functions *f(x)* and *g(x)* that are differentiable on some interval I.

(1) If $W\left(f,g\right)\left(x_0\right) \neq 0$ for some x_0 in I, then *f(x)* and *g(x)* are linearly independent on the interval I.

(2) If *f(x)* and *g(x)* are linearly dependent on I then $W(f,g)(x) = 0$ for all x in the interval I.

Be very careful with this fact. It DOES NOT say that if $W(f,g)(x) = 0$ then *f(x)* and *g(x)* are linearly dependent! In fact it is possible for two linearly independent functions to have a zero Wronskian!

This fact is used to quickly identify linearly independent functions and functions that are liable to be linearly dependent.

Example 2 Verify the fact using the functions from the previous example.

Solution

(a) $f\left(x\right) = 9\cos\left(2x\right) \qquad g\left(x\right) = 2\cos^2\left(x\right) - 2\sin^2\left(x\right)$

In this case if we compute the Wronskian of the two functions we should get zero since we have already determined that these functions are linearly dependent.

$$\begin{aligned} W &= \begin{vmatrix} 9\cos\left(2x\right) & 2\cos^2\left(x\right) - 2\sin^2\left(x\right) \\ -18\sin\left(2x\right) & -4\cos\left(x\right)\sin\left(x\right) - 4\sin\left(x\right)\cos\left(x\right) \end{vmatrix} \\ &= \begin{vmatrix} 9\cos\left(2x\right) & 2\cos\left(2x\right) \\ -18\sin\left(2x\right) & -2\sin\left(2x\right) - 2\sin\left(2x\right) \end{vmatrix} \\ &= \begin{vmatrix} 9\cos\left(2x\right) & 2\cos\left(2x\right) \\ -18\sin\left(2x\right) & -4\sin\left(2x\right) \end{vmatrix} \\ &= -36\cos\left(2x\right)\sin\left(2x\right) - \left(-36\cos\left(2x\right)\sin\left(2x\right)\right) = 0 \end{aligned}$$

So, we get zero as we should have. Notice the heavy use of trig formulas to simplify the work!

(b) $f\left(t\right) = 2t^2 \qquad g\left(t\right) = t^4$

Here we know that the two functions are linearly independent and so we should get a non-zero Wronskian.

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} = 8t^5 - 4t^5 = 4t^5$$

The Wronskian is non-zero as we expected provided $t \neq 0$. This is not a problem. As long as the Wronskian is not identically zero for all t we are okay.

Example 3 Determine if the following functions are linearly dependent or linearly independent.

(a) $f(t) = \cos t$ $\qquad g(t) = \sin t$ [Solution]

(b) $f(x) = 6^x$ $\qquad g(x) = 6^{x+2}$ [Solution]

Solution

(a) Now that we have the Wronskian to use here let's first check that. If its non-zero then we will know that the two functions are linearly independent and if its zero then we can be pretty sure that they are linearly dependent.

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

So, by the fact these two functions are linearly independent. Much easier this time around!

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(b) We'll do the same thing here as we did in the first part. Recall that

$$\left(a^x\right)' = a^x \ln a$$

Now compute the Wronskian.

$$W = \begin{vmatrix} 6^x & 6^{x+2} \\ 6^x \ln 6 & 6^{x+2} \ln 6 \end{vmatrix} = 6^x 6^{x+2} \ln 6 - 6^{x+2} 6^x \ln 6 = 0$$

Now, this does not say that the two functions are linearly dependent! However, we can guess that they probably are linearly dependent. To prove that they are in fact linearly dependent we'll need to write down (1) and see if we can find non-zero *c* and *k* that will make it true for all *x*.

$$\begin{aligned} c\,6^x + k\,6^{x+2} &= 0 \\ c\,6^x + k\,6^x 6^2 &= 0 \\ c\,6^x + 36k\,6^x &= 0 \\ \left(c + 36k\right) 6^x &= 0 \end{aligned}$$

So, it looks like we could use any constants that satisfy

$$c + 36k = 0$$

to make this zero for all *x*. In particular we could use

$$\begin{aligned} &c = 36 && k = -1 \\ &c = -36 && k = 1 \\ &c = 9 && k = -\frac{1}{4} \\ &etc. \end{aligned}$$

We have non-zero constants that will make the equation true for all *x*. Therefore, the functions are linearly dependent.

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Before proceeding to the next topic in this section let's talk a little more about linearly independent and linearly dependent functions. Let's start off by assuming that *f(x)* and *g(x)* are linearly dependent. So, that means there are non-zero constants *c* and *k* so that

$$c\, f(x) + k\, g(x) = 0$$

is true for all *x*.

Now, we can solve this in either of the following two ways.

$$f(x) = -\frac{k}{c}\, g(x) \qquad \text{OR} \qquad g(x) = -\frac{c}{k}\, f(x)$$

Note that this can be done because we know that *c* and *k* are non-zero and hence the divisions can be done without worrying about division by zero.

So, this means that two linearly dependent functions can be written in such a way that one is nothing more than a constants time the other. Go back and look at both of the sets of linearly dependent functions that we wrote down and you will see that this is true for both of them.

Two functions that are linearly independent can't be written in this manner and so we can't get from one to the other simply by multiplying by a constant.

Next, we don't want to leave you with the impression that linear independence and linear dependence is only for two functions. We can easily extend the idea to as many functions as we'd like.

Let's suppose that we have *n* non-zero functions, $f_1(x), f_2(x), \ldots, f_n(x)$. Write down the following equation.

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \tag{2}$$

If we can find constants $c_1, c_2, \ldots, c_n$ with at least two non-zero so that (2) is true for all *x* then we call the functions linearly dependent. If, on the other hand, the only constants that make (2) true for *x* are $c_1 = 0,\ c_2 = 0, \ldots, c_n = 0$ then we call the functions linearly independent.

Note that unlike the two function case we can have some of the constants be zero and still have the functions be linearly dependent.

In this case just what does it mean for the functions to be linearly dependent? Well, let's suppose that they are. So, this means that we can find constants, with at least two non-zero so that (2) is true for all *x*. For the sake of argument let's suppose that c_1 is one of the non-zero constants. This means that we can do the following.

$$\begin{aligned} c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) &= 0 \\ c_1 f_1(x) &= -\left(c_2 f_2(x) + \cdots + c_n f_n(x)\right) \\ f_1(x) &= -\frac{1}{c_1}\left(c_2 f_2(x) + \cdots + c_n f_n(x)\right) \end{aligned}$$

In other words, if the functions are linearly dependent then we can write at least one of them in terms of the other functions.

Okay, let's move on to the other topic of this section. There is an alternate method of computing the Wronskian. The following theorem gives this alternate method.

Abel's Theorem

If $y_1(t)$ and $y_2(t)$ are two solutions to

$$y'' + p(t)\,y' + q(t)\,y = 0$$

then the Wronskian of the two solutions is

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)\,\mathbf{e}^{-\int_{t_0}^{t} p(x)\,dx}$$

for some t_0.

Because we don't know the Wronskian and we don't know t_0 this won't do us a lot of good apparently. However, we can rewrite this as

$$W(y_1, y_2)(t) = c\,\mathbf{e}^{-\int p(t)\,dt} \qquad \textbf{(3)}$$

where the original Wronskian sitting in front of the exponential is absorbed into the c and the evaluation of the integral at t_0 will put a constant in the exponential that can also be brought out and absorbed into the constant c. If you don't recall how to do this go back and take a look at the linear, first order differential equation section as we did something similar there.

With this rewrite we can compute the Wronskian up to a multiplicative constant, which isn't too bad. Notice as well that we don't actually need the two solutions to do this. All we need is the coefficient of the first derivative from the differential equation (provided the coefficient of the second derivative is one of course…).

Let's take a look at a quick example of this.

Example 4 Without solving, determine the Wronskian of two solutions to the following differential equation.

$$t^4 y'' - 2t^3 y' - t^8 y = 0$$

Solution

The first thing that we need to do is divide the differential equation by the coefficient of the second derivative as that needs to be a one. This gives us

$$y'' - \frac{2}{t} y' - t^4 y = 0$$

Now, using (3) the Wronskian is

$$W = c\mathbf{e}^{-\int -\frac{2}{t}dt} = c\mathbf{e}^{2\ln t} = c\mathbf{e}^{\ln t^2} = ct^2$$

Nonhomogeneous Differential Equations

It's now time to start thinking about how to solve nonhomogeneous differential equations. A second order, linear nonhomogeneous differential equation is

$$y'' + p(t)\,y' + q(t)\,y = g(t) \qquad \textbf{(1)}$$

where $g(t)$ is a non-zero function. Note that we didn't go with constant coefficients here because everything that we're going to do in this section doesn't require it. Also, we're using a coefficient of 1 on the second derivative just to make some of the work a little easier to write down. It is not required to be a 1.

Before talking about how to solve one of these we need to get some basics out of the way, which is the point of this section.

First, we will call

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad \textbf{(2)}$$

the associated homogeneous differential equation to (1).

Now, let's take a look at the following theorem.

Theorem

Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to (1) and that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to the associated homogeneous differential equation (2) then,

$$Y_1(t) - Y_2(t)$$

is a solution to (2) and it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Note the notation used here. Capital letters referred to solutions to (1) while lower case letters referred to solutions to (2). This is a fairly common convention when dealing with nonhomogeneous differential equations.

This theorem is easy enough to prove so let's do that. To prove that $Y_1(t)$ - $Y_2(t)$ is a solution to (2) all we need to do is plug this into the differential equation and check it.

$$\begin{aligned}
(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) &= 0 \\
Y_1'' + p(t)Y_1' + q(t)Y_1 - \left(Y_2'' + p(t)Y_2' + q(t)Y_2\right) &= 0 \\
g(t) - g(t) &= 0 \\
0 &= 0
\end{aligned}$$

We used the fact that $Y_1(t)$ and $Y_2(t)$ are two solutions to (1) in the third step. Because they are solutions to (1) we know that

$$\begin{aligned}
Y_1'' + p(t)Y_1' + q(t)Y_1 &= g(t) \\
Y_2'' + p(t)Y_2' + q(t)Y_2 &= g(t)
\end{aligned}$$

So, we were able to prove that the difference of the two solutions is a solution to (2).

Proving that

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

is even easier. Since $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions to (2) we know that they form a general solution and so any solution to (2) can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Well, $Y_1(t) - Y_2(t)$ is a solution to (2), as we've shown above, therefore it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

So, what does this theorem do for us? We can use this theorem to write down the form of the general solution to (1). Let's suppose that $y(t)$ is the general solution to (1) and that $Y_P(t)$ is any solution to (1) that we can get our hands on. Then using the second part of our theorem we know that

$$y(t) - Y_P(t) = c_1 y_1(t) + c_2 y_2(t)$$

where $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for (2). Solving for $y(t)$ gives,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_P(t)$$

We will call

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

the complementary solution and $Y_P(t)$ a particular solution. The general solution to a differential equation can then be written as.

$$y(t) = y_c(t) + Y_P(t)$$

So, to solve a nonhomogeneous differential equation, we will need to solve the homogeneous differential equation, (2), which for constant coefficient differential equations is pretty easy to do, and we'll need a solution to (1).

This seems to be a circular argument. In order to write down a solution to (1) we need a solution. However, this isn't the problem that it seems to be. There are ways to find a solution to (1). They just won't, in general, be the general solution. In fact, the next two sections are devoted to exactly that, finding a particular solution to a nonhomogeneous differential equation.

There are two common methods for finding particular solutions : Undetermined Coefficients and Variation of Parameters. Both have their advantages and disadvantages as you will see in the next couple of sections.

Undetermined Coefficients

In this section we will take a look at the first method that can be used to find a particular solution to a nonhomogeneous differential equation.

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

One of the main advantages of this method is that it reduces the problem down to an algebra problem. The algebra can get messy on occasion, but for most of the problems it will not be terribly difficult. Another nice thing about this method is that the complementary solution will not be explicitly required, although as we will see knowledge of the complementary solution will be needed in some cases and so we'll generally find that as well.

There are two disadvantages to this method. First, it will only work for a fairly small class of *g(t)*'s. The class of *g(t)*'s for which the method works, does include some of the more common functions, however, there are many functions out there for which undetermined coefficients simply won't work. Second, it is generally only useful for constant coefficient differential equations.

The method is quite simple. All that we need to do is look at *g(t)* and make a guess as to the form of $Y_P(t)$ leaving the coefficient(s) undetermined (and hence the name of the method). Plug the guess into the differential equation and see if we can determine values of the coefficients. If we can determine values for the coefficients then we guessed correctly, if we can't find values for the coefficients then we guessed incorrectly.

It's usually easier to see this method in action rather than to try and describe it, so let's jump into some examples.

Example 1 Determine a particular solution to

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t}$$

Solution

The point here is to find a particular solution, however the first thing that we're going to do is find the complementary solution to this differential equation. Recall that the complementary solution comes from solving,

$$y'' - 4y' - 12y = 0$$

The characteristic equation for this differential equation and its roots are.

$$r^2 - 4r - 12 = (r-6)(r+2) = 0 \qquad \Rightarrow \qquad r_1 = -2, \quad r_2 = 6$$

The complementary solution is then,

$$y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t}$$

At this point the reason for doing this first will not be apparent, however we want you in the habit of finding it before we start the work to find a particular solution. Eventually, as we'll see, having the complementary solution in hand will be helpful and so it's best to be in the habit of finding it first prior to doing the work for undetermined coefficients.

Now, let's proceed with finding a particular solution. As mentioned prior to the start of this example we need to make a guess as to the form of a particular solution to this differential equation. Since *g(t)* is an exponential and we know that exponentials never just appear or disappear in the differentiation process it seems that a likely form of the particular solution would be

$$Y_P(t) = A\mathbf{e}^{5t}$$

Now, all that we need to do is do a couple of derivatives, plug this into the differential equation and see if we can determine what *A* needs to be.

Plugging into the differential equation gives

$$\begin{aligned} 25A\mathbf{e}^{5t} - 4\left(5A\mathbf{e}^{5t}\right) - 12\left(A\mathbf{e}^{5t}\right) &= 3\mathbf{e}^{5t} \\ -7A\mathbf{e}^{5t} &= 3\mathbf{e}^{5t} \end{aligned}$$

So, in order for our guess to be a solution we will need to choose *A* so that the coefficients of the exponentials on either side of the equal sign are the same. In other words we need to choose *A* so that,

$$-7A = 3 \qquad \Rightarrow \qquad A = -\frac{3}{7}$$

Okay, we found a value for the coefficient. This means that we guessed correctly. A particular solution to the differential equation is then,

$$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t}$$

Before proceeding any further let's again note that we started off the solution above by finding the complementary solution. This is not technically part the method of Undetermined Coefficients however, as we'll eventually see, having this in hand before we make our guess for the particular solution can save us a lot of work and/or headache. Finding the complementary solution first is simply a good habit to have so we'll try to get you in the habit over the course of the next few examples. At this point do not worry about why it is a good habit. We'll eventually see why it is a good habit.

Now, back to the work at hand. Notice in the last example that we kept saying "a" particular solution, not "the" particular solution. This is because there are other possibilities out there for the particular solution we've just managed to find one of them. Any of them will work when it comes to writing down the general solution to the differential equation.

Speaking of which… This section is devoted to finding particular solutions and most of the examples will be finding only the particular solution. However, we should do at least one full blown IVP to make sure that we can say that we've done one.

Example 2 Solve the following IVP

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t} \qquad y(0) = \frac{18}{7} \qquad y'(0) = -\frac{1}{7}$$

Solution

We know that the general solution will be of the form,

$$y(t) = y_c(t) + Y_P(t)$$

and we already have both the complementary and particular solution from the first example so we don't really need to do any extra work for this problem.

One of the more common mistakes in these problems is to find the complementary solution and then, because we're probably in the habit of doing it, apply the initial conditions to the complementary solution to find the constants. This however, is incorrect. The complementary solution is only the solution to the homogeneous differential equation and we are after a solution to the nonhomogeneous differential equation and the initial conditions must satisfy that solution instead of the complementary solution.

So, we need the general solution to the nonhomogeneous differential equation. Taking the complementary solution and the particular solution that we found in the previous example we get the following for a general solution and its derivative.

$$y(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t} - \frac{3}{7}\mathbf{e}^{5t}$$

$$y'(t) = -2c_1\mathbf{e}^{-2t} + 6c_2\mathbf{e}^{6t} - \frac{15}{7}\mathbf{e}^{5t}$$

Now, apply the initial conditions to these.

$$\frac{18}{7} = y(0) = c_1 + c_2 - \frac{3}{7}$$

$$-\frac{1}{7} = y'(0) = -2c_1 + 6c_2 - \frac{15}{7}$$

Solving this system gives $c_1 = 2$ and $c_2 = 1$. The actual solution is then.

$$y(t) = 2\mathbf{e}^{-2t} + \mathbf{e}^{6t} - \frac{3}{7}\mathbf{e}^{5t}$$

This will be the only IVP in this section so don't forget how these are done for nonhomogeneous differential equations!

Let's take a look at another example that will give the second type of *g(t)* for which undetermined coefficients will work.

Example 3 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = \sin(2t)$$

Solution

Again, let's note that we should probably find the complementary solution before we proceed onto the guess for a particular solution. However, because the homogeneous differential equation for this example is the same as that for the first example we won't bother with that here.

Now, let's take our experience from the first example and apply that here. The first example had an exponential function in the *g(t)* and our guess was an exponential. This differential equation has a sine so let's try the following guess for the particular solution.

$$Y_P(t) = A\sin(2t)$$

Differentiating and plugging into the differential equation gives,

$$-4A\sin\left(2t\right)-4\left(2A\cos\left(2t\right)\right)-12\left(A\sin\left(2t\right)\right)=\sin\left(2t\right)$$

Collecting like terms yields

$$-16A\sin\left(2t\right)-8A\cos\left(2t\right)=\sin\left(2t\right)$$

We need to pick A so that we get the same function on both sides of the equal sign. This means that the coefficients of the sines and cosines must be equal. Or,

$$\begin{aligned}\cos\left(2t\right): &\qquad -8A=0 &\qquad \Rightarrow &\qquad A=0\\ \sin\left(2t\right): &\qquad -16A=1 &\qquad \Rightarrow &\qquad A=-\frac{1}{16}\end{aligned}$$

Notice two things. First, since there is no cosine on the right hand side this means that the coefficient must be zero on that side. More importantly we have a serious problem here. In order for the cosine to drop out, as it must in order for the guess to satisfy the differential equation, we need to set $A = 0$, but if $A = 0$, the sine will also drop out and that can't happen. Likewise, choosing A to keep the sine around will also keep the cosine around.

What this means is that our initial guess was wrong. If we get multiple values of the same constant or are unable to find the value of a constant then we have guessed wrong.

One of the nicer aspects of this method is that when we guess wrong our work will often suggest a fix. In this case the problem was the cosine that cropped up. So, to counter this let's add a cosine to our guess. Our new guess is

$$Y_P\left(t\right)=A\cos\left(2t\right)+B\sin\left(2t\right)$$

Plugging this into the differential equation and collecting like terms gives,

$$\begin{aligned}-4A\cos\left(2t\right)-4B\sin\left(2t\right)-4\left(-2A\sin\left(2t\right)+2B\cos\left(2t\right)\right)-&\\ 12\left(A\cos\left(2t\right)+B\sin\left(2t\right)\right)&=\sin\left(2t\right)\\ \left(-4A-8B-12A\right)\cos\left(2t\right)+\left(-4B+8A-12B\right)\sin\left(2t\right)&=\sin\left(2t\right)\\ \left(-16A-8B\right)\cos\left(2t\right)+\left(8A-16B\right)\sin\left(2t\right)&=\sin\left(2t\right)\end{aligned}$$

Now, set the coefficients equal

$$\begin{aligned}\cos\left(2t\right): &\qquad -16A-8B=0\\ \sin\left(2t\right): &\qquad 8A-16B=1\end{aligned}$$

Solving this system gives us

$$A=\frac{1}{40}\qquad\qquad B=-\frac{1}{20}$$

We found constants and this time we guessed correctly. A particular solution to the differential equation is then,

$$Y_P\left(t\right)=\frac{1}{40}\cos\left(2t\right)-\frac{1}{20}\sin\left(2t\right)$$

Notice that if we had had a cosine instead of a sine in the last example then our guess would have been the same. In fact, if both a sine and a cosine had shown up we will see that the same guess will also work.

Let's take a look at the third and final type of basic *g(t)* that we can have. There are other types of *g(t)* that we can have, but as we will see they will all come back to two types that we've already done as well as the next one.

Example 4 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = 2t^3 - t + 3$$

Solution

Once, again we will generally want the complementary solution in hand first, but again we're working with the same homogeneous differential equation (you'll eventually see why we keep working with the same homogeneous problem) so we'll again just refer to the first example.

For this example *g(t)* is a cubic polynomial. For this we will need the following guess for the particular solution.

$$Y_P(t) = At^3 + Bt^2 + Ct + D$$

Notice that even though *g(t)* doesn't have a t^2 in it our guess will still need one! So, differentiate and plug into the differential equation.

$$6At + 2B - 4\left(3At^2 + 2Bt + C\right) - 12\left(At^3 + Bt^2 + Ct + D\right) = 2t^3 - t + 3$$

$$-12At^3 + \left(-12A - 12B\right)t^2 + \left(6A - 8B - 12C\right)t + 2B - 4C - 12D = 2t^3 - t + 3$$

Now, as we've done in the previous examples we will need the coefficients of the terms on both sides of the equal sign to be the same so set coefficients equal and solve.

$$\begin{array}{llcl} t^3: & -12A = 2 & \Rightarrow & A = -\frac{1}{6} \\ t^2: & -12A - 12B = 0 & \Rightarrow & B = \frac{1}{6} \\ t^1: & 6A - 8B - 12C = -1 & \Rightarrow & C = -\frac{1}{9} \\ t^0: & 2B - 4C - 12D = 3 & \Rightarrow & D = -\frac{5}{27} \end{array}$$

Notice that in this case it was very easy to solve for the constants. The first equation gave *A*. Then once we knew *A* the second equation gave *B*, *etc*. A particular solution for this differential equation is then

$$Y_P(t) = -\frac{1}{6}t^3 + \frac{1}{6}t^2 - \frac{1}{9}t - \frac{5}{27}$$

Now that we've gone over the three basic kinds of functions that we can use undetermined coefficients on let's summarize.

$g(t)$	$Y_P(t)$ guess
$a\mathbf{e}^{\beta t}$	$A\mathbf{e}^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
$a\cos(\beta t)+b\sin(\beta t)$	$A\cos(\beta t)+B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots A_1 t + A_0$

Notice that there are really only three kinds of functions given above. If you think about it the single cosine and single sine functions are really special cases of the case where both the sine and cosine are present. Also, we have not yet justified the guess for the case where both a sine and a cosine show up. We will justify this later.

We now need move on to some more complicated functions. The more complicated functions arise by taking products and sums of the basic kinds of functions. Let's first look at products.

Example 5 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = t\mathbf{e}^{4t}$$

Solution

You're probably getting tired of the opening comment, but again find the complementary solution first really a good idea but again we've already done the work in the first example so we won't do it again here. We promise that eventually you'll see why we keep using the same homogeneous problem and why we say it's a good idea to have the complementary solution in hand first. At this point all we're trying to do is reinforce the habit of finding the complementary solution first.

Okay, let's start off by writing down the guesses for the individual pieces of the function. The guess for the t would be

$$At + B$$

while the guess for the exponential would be

$$C\mathbf{e}^{4t}$$

Now, since we've got a product of two functions it seems like taking a product of the guesses for the individual pieces might work. Doing this would give

$$C\mathbf{e}^{4t}\left(At + B\right)$$

However, we will have problems with this. As we will see, when we plug our guess into the differential equation we will only get two equations out of this. The problem is that with this guess we've got three unknown constants. With only two equations we won't be able to solve for all the constants.

This is easy to fix however. Let's notice that we could do the following

$$C\mathbf{e}^{4t}\left(At + B\right) = \mathbf{e}^{4t}\left(ACt + BC\right)$$

If we multiply the C through, we can see that the guess can be written in such a way that there are really only two constants. So, we will use the following for our guess.

$$Y_P\left(t\right) = \mathbf{e}^{4t}\left(At + B\right)$$

Notice that this is nothing more than the guess for the t with an exponential tacked on for good measure.

Now that we've got our guess, let's differentiate, plug into the differential equation and collect like terms.

$$\begin{aligned} \mathbf{e}^{4t}\left(16At + 16B + 8A\right) - 4\left(\mathbf{e}^{4t}\left(4At + 4B + A\right)\right) - 12\left(\mathbf{e}^{4t}\left(At + B\right)\right) &= t\mathbf{e}^{4t} \\ \left(16A - 16A - 12A\right)t\mathbf{e}^{4t} + \left(16B + 8A - 16B - 4A - 12B\right)\mathbf{e}^{4t} &= t\mathbf{e}^{4t} \\ -12At\mathbf{e}^{4t} + \left(4A - 12B\right)\mathbf{e}^{4t} &= t\mathbf{e}^{4t} \end{aligned}$$

Note that when we're collecting like terms we want the coefficient of each term to have only constants in it. Following this rule we will get two terms when we collect like terms. Now, set coefficients equal.

$$\begin{aligned} t\mathbf{e}^{4t}: &\qquad -12A = 1 &\qquad \Rightarrow \qquad A = -\frac{1}{12} \\ \mathbf{e}^{4t}: &\qquad 4A - 12B = 0 &\qquad \Rightarrow \qquad B = -\frac{1}{36} \end{aligned}$$

A particular solution for this differential equation is then

$$Y_P\left(t\right) = \mathbf{e}^{4t}\left(-\frac{t}{12} - \frac{1}{36}\right) = -\frac{1}{36}\left(3t + 1\right)\mathbf{e}^{4t}$$

This last example illustrated the general rule that we will follow when products involve an exponential. When a product involves an exponential we will first strip out the exponential and write down the guess for the portion of the function without the exponential, then we will go back and tack on the exponential without any leading coefficient.

Let's take a look at some more products. In the interest of brevity we will just write down the guess for a particular solution and not go through all the details of finding the constants. Also, because we aren't going to give an actual differential equation we can't deal with finding the complementary solution first.

Example 6 Write down the form of the particular solution to

$$y'' + p\left(t\right)y' + q\left(t\right)y = g\left(t\right)$$

for the following *g(t)*'s.

(a) $g\left(t\right) = 16\mathbf{e}^{7t}\sin\left(10t\right)$ [Solution]

(b) $g\left(t\right) = \left(9t^2 - 103t\right)\cos t$ [Solution]

(c) $g\left(t\right) = -\mathbf{e}^{-2t}\left(3 - 5t\right)\cos\left(9t\right)$ [Solution]

Solution

(a) $g(t) = 16\mathbf{e}^{7t}\sin(10t)$

So, we have an exponential in the function. Remember the rule. We will ignore the exponential and write down a guess for 16 sin(10*t*) then put the exponential back in.

The guess for the sine is

$$A\cos(10t) + B\sin(10t)$$

Now, for the actual guess for the particular solution we'll take the above guess and tack an exponential onto it. This gives,

$$Y_P(t) = \mathbf{e}^{7t}\left(A\cos(10t) + B\sin(10t)\right)$$

One final note before we move onto the next part. The 16 in front of the function has absolutely no bearing on our guess. Any constants multiplying the whole function are ignored.

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(b) $g(t) = \left(9t^2 - 103t\right)\cos t$

We will start this one the same way that we initially started the previous example. The guess for the polynomial is

$$At^2 + Bt + C$$

and the guess for the cosine is

$$D\cos t + E\sin t$$

If we multiply the two guesses we get.

$$\left(At^2 + Bt + C\right)\left(D\cos t + E\sin t\right)$$

Let's simplify things up a little. First multiply the polynomial through as follows.

$$\left(At^2 + Bt + C\right)\left(D\cos t\right) + \left(At^2 + Bt + C\right)\left(E\sin t\right)$$

$$\left(ADt^2 + BDt + CD\right)\cos t + \left(AEt^2 + BEt + CE\right)\sin t$$

Notice that everywhere one of the unknown constants occurs it is in a product of unknown constants. This means that if we went through and used this as our guess the system of equations that we would need to solve for the unknown constants would have products of the unknowns in them. These types of systems are generally very difficult to solve.

So, to avoid this we will do the same thing that we did in the previous example. Everywhere we see a product of constants we will rename it and call it a single constant. The guess that we'll use for this function will be.

$$Y_P(t) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t$$

This is a general rule that we will use when faced with a product of a polynomial and a trig function. We write down the guess for the polynomial and then multiply that by a cosine. We then write down the guess for the polynomial again, using different coefficients, and multiply this by a sine.

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(c) $g(t) = -\mathbf{e}^{-2t}(3-5t)\cos(9t)$

This final part has all three parts to it. First we will ignore the exponential and write down a guess for.

$$-(3-5t)\cos(9t)$$

The minus sign can also be ignored. The guess for this is

$$(At+B)\cos(9t)+(Ct+D)\sin(9t)$$

Now, tack an exponential back on and we're done.

$$Y_P(t) = \mathbf{e}^{-2t}(At+B)\cos(9t)+\mathbf{e}^{-2t}(Ct+D)\sin(9t)$$

Notice that we put the exponential on both terms.

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There a couple of general rules that you need to remember for products.

1. If *g(t)* contains an exponential, ignore it and write down the guess for the remainder. Then tack the exponential back on without any leading coefficient.

2. For products of polynomials and trig functions you first write down the guess for just the polynomial and multiply that by the appropriate cosine. Then add on a new guess for the polynomial with different coefficients and multiply that by the appropriate sine.

If you can remember these two rules you can't go wrong with products. Writing down the guesses for products is usually not that difficult. The difficulty arises when you need to actually find the constants.

Now, let's take a look at sums of the basic components and/or products of the basic components. To do this we'll need the following fact.

Fact

If $Y_{P1}(t)$ is a particular solution for

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and if $Y_{P2}(t)$ is a particular solution for

$$y'' + p(t)y' + q(t)y = g_2(t)$$

then $Y_{P1}(t) + Y_{P2}(t)$ is a particular solution for

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$$

This fact can be used to both find particular solutions to differential equations that have sums in then and to write down guess for functions that have sums in them.

Example 7 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = 3\mathbf{e}^{5t} + \sin(2t) + t\mathbf{e}^{4t}$$

Solution

This example is the reason that we've been using the same homogeneous differential equation for all the previous examples. There is nothing to do with this problem. All that we need to do it go back to the appropriate examples above and get the particular solution from that example and add them all together.

Doing this gives

$$Y_P(t) = -\frac{3}{7}\mathbf{e}^{5t} + \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t) - \frac{1}{36}(3t+1)\mathbf{e}^{4t}$$

Let's take a look at a couple of other examples. As with the products we'll just get guesses here and not worry about actually finding the coefficients.

Example 8 Write down the form of the particular solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

for the following *g(t)*'s.

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$ [Solution]

(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$ [Solution]

(c) $g(t) = \mathbf{e}^{7t} + 6$ [Solution]

(d) $g(t) = 6t^2 - 7\sin(3t) + 9$ [Solution]

(e) $g(t) = 10\mathbf{e}^{t} - 5t\mathbf{e}^{-8t} + 2\mathbf{e}^{-8t}$ [Solution]

(f) $g(t) = t^2\cos t - 5t\sin t$ [Solution]

(g) $g(t) = 5\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\cos(6t) - \sin(6t)$ [Solution]

Solution

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

This first one we've actually already told you how to do. This is in the table of the basic functions. However we wanted to justify the guess that we put down there. Using the fact on sums of function we would be tempted to write down a guess for the cosine and a guess for the sine. This would give.

$$\underbrace{A\cos(6t) + B\sin(6t)}_{\text{guess for the cosine}} + \underbrace{C\cos(6t) + D\sin(6t)}_{\text{guess for the sine}}$$

So, we would get a cosine from each guess and a sine from each guess. The problem with this as a guess is that we are only going to get two equations to solve after plugging into the differential equation and yet we have 4 unknowns. We will never be able to solve for each of the constants.

To fix this notice that we can combine some terms as follows.

$$(A+C)\cos(6t) + (B+D)\sin(6t)$$

Upon doing this we can see that we've really got a single cosine with a coefficient and a single sine with a coefficient and so we may as well just use

$$Y_P(t) = A\cos(6t) + B\sin(6t)$$

The general rule of thumb for writing down guesses for functions that involve sums is to always combine like terms into single terms with single coefficients. This will greatly simplify the work required to find the coefficients.

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(b) $g(t) = -2\sin t + \sin(14t) - 5\cos(14t)$

For this one we will get two sets of sines and cosines. This will arise because we have two different arguments in them. We will get on set for the sine with just a *t* as its argument and we'll get another set for the sine and cosine with the 14*t* as their arguments.

The guess for this function is

$$Y_P(t) = A\cos t + B\sin t + C\cos(14t) + D\sin(14t)$$

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(c) $g(t) = \mathbf{e}^{7t} + 6$

The main point of this problem is dealing with the constant. But that isn't too bad. We just wanted to make sure that an example of that is somewhere in the notes. If you recall that a constant is nothing more than a zeroth degree polynomial the guess becomes clear.

The guess for this function is

$$Y_p(t) = A\mathbf{e}^{7t} + B$$

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(d) $g(t) = 6t^2 - 7\sin(3t) + 9$

This one can be a little tricky if you aren't paying attention. Let's first rewrite the function

$$g(t) = 6t^2 - 7\sin(3t) + 9 \qquad \text{as}$$

$$g(t) = 6t^2 + 9 - 7\sin(3t)$$

All we did was move the 9. However upon doing that we see that the function is really a sum of a quadratic polynomial and a sine. The guess for this is then

$$Y_P(t) = At^2 + Bt + C + D\cos(3t) + E\sin(3t)$$

If we don't do this and treat the function as the sum of three terms we would get

$$At^2 + Bt + C + D\cos(3t) + E\sin(3t) + G$$

and as with the first part in this example we would end up with two terms that are essentially the same (the *C* and the *G*) and so would need to be combined. An added step that isn't really necessary if we first rewrite the function.

Look for problems where rearranging the function can simplify the initial guess.

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(e) $g(t) = 10\mathbf{e}^{t} - 5t\mathbf{e}^{-8t} + 2\mathbf{e}^{-8t}$

So, this look like we've got a sum of three terms here. Let's write down a guess for that.

$$A\mathbf{e}^{t} + (Bt + C)\mathbf{e}^{-8t} + D\mathbf{e}^{-8t}$$

Notice however that if we were to multiply the exponential in the second term through we would end up with two terms that are essentially the same and would need to be combined. This is a case where the guess for one term is completely contained in the guess for a different term. When this happens we just drop the guess that's already included in the other term.

So, the guess here is actually.

$$Y_P(t) = A\mathbf{e}^{t} + (Bt + C)\mathbf{e}^{-8t}$$

Notice that this arose because we had two terms in our *g(t)* whose only difference was the polynomial that sat in front of them. When this happens we look at the term that contains the largest degree polynomial, write down the guess for that and don't bother writing down the guess for the other term as that guess will be completely contained in the first guess.

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(f) $g(t) = t^2 \cos t - 5t \sin t$

In this case we've got two terms whose guess without the polynomials in front of them would be the same. Therefore, we will take the one with the largest degree polynomial in front of it and write down the guess for that one and ignore the other term. So, the guess for the function is

$$Y_P(t) = \left(At^2 + Bt + C\right)\cos t + \left(Dt^2 + Et + F\right)\sin t$$

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(g) $g(t) = 5\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\cos(6t) - \sin(6t)$

This last part is designed to make sure you understand the general rule that we used in the last two parts. This time there really are three terms and we will need a guess for each term. The guess here is

$$Y_P(t) = A\mathbf{e}^{-3t} + \mathbf{e}^{-3t}\left(B\cos(6t) + C\sin(6t)\right) + D\cos(6t) + E\sin(6t)$$

We can only combine guesses if they are identical up to the constant. So we can't combine the first exponential with the second because the second is really multiplied by a cosine and a sine and so the two exponentials are in fact different functions. Likewise, the last sine and cosine can't be combined with those in the middle term because the sine and cosine in the middle term are in fact multiplied by an exponential and so are different.

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So, when dealing with sums of functions make sure that you look for identical guesses that may or may not be contained in other guesses and combine them. This will simplify your work later on.

We have one last topic in this section that needs to be dealt with. In the first few examples we were constantly harping on the usefulness of having the complementary solution in hand before making the guess for a particular solution. We never gave any reason for this other that "trust us". It is now time to see why having the complementary solution in hand first is useful. This is best shown with an example so let's jump into one.

Example 9 Find a particular solution for the following differential equation.

$$y'' - 4y' - 12y = \mathbf{e}^{6t}$$

Solution

This problem seems almost too simple to be given this late in the section. This is especially true given the ease of finding a particular solution for $g(t)$'s that are just exponential functions. Also, because the point of this example is to illustrate why it is generally a good idea to have the complementary solution in hand first we'll let's go ahead and recall the complementary solution first. Here it is,

$$y_c(t) = c_1\mathbf{e}^{-2t} + c_2\mathbf{e}^{6t}$$

Now, without worrying about the complementary solution for a couple more seconds let's go ahead and get to work on the particular solution. There is not much to the guess here. From our previous work we know that the guess for the particular solution should be,

$$Y_P(t) = A\mathbf{e}^{6t}$$

Plugging this into the differential equation gives,

$$\begin{aligned} 36A\mathbf{e}^{6t} - 24A\mathbf{e}^{6t} - 12A\mathbf{e}^{6t} &= \mathbf{e}^{6t} \\ 0 &= \mathbf{e}^{6t} \end{aligned}$$

Hmmmm…. Something seems wrong here. Clearly an exponential can't be zero. So, what went wrong? We finally need the complementary solution. Notice that the second term in the complementary solution (listed above) is exactly our guess for the form of the particular solution and now recall that both portions of the complementary solution are solutions to the homogeneous differential equation,

$$y'' - 4y' - 12y = 0$$

In other words, we had better have gotten zero by plugging our guess into the differential equation, it is a solution to the homogeneous differential equation!

So, how do we fix this? The way that we fix this is to add a *t* to our guess as follows.

$$Y_P(t) = At\mathbf{e}^{6t}$$

Plugging this into our differential equation gives,

$$\begin{aligned} \left(12A\mathbf{e}^{6t} + 36At\mathbf{e}^{6t}\right) - 4\left(A\mathbf{e}^{6t} + 6At\mathbf{e}^{6t}\right) - 12At\mathbf{e}^{6t} &= \mathbf{e}^{6t} \\ \left(36A - 24A - 12A\right)t\mathbf{e}^{6t} + \left(12A - 4A\right)\mathbf{e}^{6t} &= \mathbf{e}^{6t} \\ 8A\mathbf{e}^{6t} &= \mathbf{e}^{6t} \end{aligned}$$

Now, we can set coefficients equal.

$$8A = 1 \qquad \Rightarrow \qquad A = \frac{1}{8}$$

So, the particular solution in this case is,

$$Y_P(t) = \frac{t}{8}\mathbf{e}^{6t}$$

So, what did we learn from this last example. While technically we don't need the complementary solution to do undetermined coefficients, you can go through a lot of work only to figure out at the end that you needed to add in a t to the guess because it appeared in the complementary solution. This work is avoidable if we first find the complementary solution and comparing our guess to the complementary solution and seeing if any portion of your guess shows up in the complementary solution.

If a portion of your guess does show up in the complementary solution then we'll need to modify that portion of the guess by adding in a t to the portion of the guess that is causing the problems. We do need to be a little careful and make sure that we add the t in the correct place however. The following set of examples will show you how to do this.

Example 10 Write down the guess for the particular solution to the given differential equation. Do not find the coefficients.

(a) $y'' + 3y' - 28y = 7t + \mathbf{e}^{-7t} - 1$ [Solution]

(b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$ [Solution]

(c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$ [Solution]

(d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$ [Solution]

(e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + \left(t^2 + 5\right)\mathbf{e}^{-4t}$ [Solution]

Solution

In these solutions we'll leave the details of checking the complementary solution to you.

(a) $y'' + 3y' - 28y = 7t + \mathbf{e}^{-7t} - 1$

The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{4t} + c_2\mathbf{e}^{-7t}$$

Remembering to put the "-1" with the $7t$ gives a first guess for the particular solution.

$$Y_P(t) = At + B + C\mathbf{e}^{-7t}$$

Notice that the last term in the guess is the last term in the complementary solution. The first two terms however aren't a problem and don't appear in the complementary solution. Therefore, we will only add a t onto the last term.

The correct guess for the form of the particular solution is.

$$Y_P(t) = At + B + Ct\mathbf{e}^{-7t}$$

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(b) $y'' - 100y = 9t^2\mathbf{e}^{10t} + \cos t - t\sin t$

The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{10t} + c_2\mathbf{e}^{-10t}$$

A first guess for the particular solution is

$$Y_P(t) = \left(At^2 + Bt + C\right)\mathbf{e}^{10t} + \left(Et + F\right)\cos t + \left(Gt + H\right)\sin t$$

Notice that if we multiplied the exponential term through the parenthesis that we would end up getting part of the complementary solution showing up. Since the problem part arises from the first term the *whole* first term will get multiplied by *t*. The second and third terms are okay as they are.

The correct guess for the form of the particular solution in this case is.

$$Y_P(t) = t\left(At^2 + Bt + C\right)\mathbf{e}^{10t} + \left(Et + F\right)\cos t + \left(Gt + H\right)\sin t$$

So, in general, if you were to multiply out a guess and if any term in the result shows up in the complementary solution, then the whole term will get a *t* not just the problem portion of the term.

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(c) $4y'' + y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

The complementary solution is

$$y_c(t) = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)$$

A first guess for the particular solution is

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + \left(Ct + D\right)\cos\left(\frac{t}{2}\right) + \left(Et + F\right)\sin\left(\frac{t}{2}\right)$$

In this case both the second and third terms contain portions of the complementary solution. The first term doesn't however, since upon multiplying out, both the sine and the cosine would have an exponential with them and that isn't part of the complementary solution. We only need to worry about terms showing up in the complementary solution if the only difference between the complementary solution term and the particular guess term is the constant in front of them.

So, in this case the second and third terms will get a *t* while the first won't

The correct guess for the form of the particular solution is.

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + t\left(Ct + D\right)\cos\left(\frac{t}{2}\right) + t\left(Et + F\right)\sin\left(\frac{t}{2}\right)$$

[Return to Problems]

(d) $4y'' + 16y' + 17y = \mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$

To get this problem we changed the differential equation from the last example and left the *g(t)* alone. The complementary solution this time is

$$y_c(t) = c_1\mathbf{e}^{-2t}\cos\left(\frac{t}{2}\right) + c_2\mathbf{e}^{-2t}\sin\left(\frac{t}{2}\right)$$

As with the last part, a first guess for the particular solution is

$$Y_P(t) = \mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct + D)\cos\left(\frac{t}{2}\right) + (Et + F)\sin\left(\frac{t}{2}\right)$$

This time however it is the first term that causes problems and not the second or third. In fact, the first term is exactly the complementary solution and so it will need a t. Recall that we will only have a problem with a term in our guess if it only differs from the complementary solution by a constant. The second and third terms in our guess don't have the exponential in them and so they don't differ from the complementary solution by only a constant.

The correct guess for the form of the particular solution is.

$$Y_P(t) = t\mathbf{e}^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct + D)\cos\left(\frac{t}{2}\right) + (Et + F)\sin\left(\frac{t}{2}\right)$$

[Return to Problems]

(e) $y'' + 8y' + 16y = \mathbf{e}^{-4t} + (t^2 + 5)\mathbf{e}^{-4t}$

The complementary solution is

$$y_c(t) = c_1\mathbf{e}^{-4t} + c_2 t\mathbf{e}^{-4t}$$

The two terms in *g(t)* are identical with the exception of a polynomial in front of them. So this means that we only need to look at the term with the highest degree polynomial in front of it. A first guess for the particular solution is

$$Y_P(t) = (At^2 + Bt + C)\mathbf{e}^{-4t}$$

Notice that if we multiplied the exponential term through the parenthesis the last two terms would be the complementary solution. Therefore, we will need to multiply this whole thing by a t.

The next guess for the particular solution is then.

$$Y_P(t) = t(At^2 + Bt + C)\mathbf{e}^{-4t}$$

This still causes problems however. If we multiplied the t and the exponential through, the last term will still be in the complementary solution. In this case, unlike the previous ones, a t wasn't sufficient to fix the problem. So, we will add in another t to our guess.

The correct guess for the form of the particular solution is.

$$Y_P(t) = t^2\left(At^2 + Bt + C\right)\mathbf{e}^{-4t}$$

Upon multiplying this out none of the terms are in the complementary solution and so it will be okay.

[Return to Problems]

As this last set of examples has shown, we really should have the complementary solution in hand before even writing down the first guess for the particular solution. By doing this we can compare our guess to the complementary solution and if any of the terms from your particular solution show up we will know that we'll have problems. Once the problem is identified we can add a t to the problem term(s) and compare our new guess to the complementary solution. If there are no problems we can proceed with the problem, if there are problems add in another t and compare again.

Can you see a general rule as to when a t will be needed and when a t^2 will be needed for second order differential equations?

Variation of Parameters

In the last section we looked at the method of undetermined coefficients for finding a particular solution to

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = g(t) \tag{1}$$

and we saw that while it reduced things down to just an algebra problem, the algebra could become quite messy. On top of that undetermined coefficients will only work for a fairly small class of functions.

The method of Variation of Parameters is a much more general method that can be used in many more cases. However, there are two disadvantages to the method. First, the complementary solution is absolutely required to do the problem. This is in contrast to the method of undetermined coefficients where it was advisable to have the complementary solution on hand, but was not required. Second, as we will see, in order to complete the method we will be doing a couple of integrals and there is no guarantee that we will be able to do the integrals. So, while it will always be possible to write down a formula to get the particular solution, we may not be able to actually find it if the integrals are too difficult or if we are unable to find the complementary solution.

We're going to derive the formula for variation of parameters. We'll start off by acknowledging that the complementary solution to (1) is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Remember as well that this is the general solution to the homogeneous differential equation.

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = 0 \tag{2}$$

Also recall that in order to write down the complementary solution we know that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions.

What we're going to do is see if we can find a pair of functions, $u_1(t)$ and $u_2(t)$ so that

$$Y_P(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

will be a solution to (1). We have two unknowns here and so we'll need two equations eventually. One equation is easy. Our proposed solution must satisfy the differential equation, so we'll get the first equation by plugging our proposed solution into (1). The second equation can come from a variety of places. We are going to get our second equation simply by making an assumption that will make our work easier. We'll say more about this shortly.

So, let's start. If we're going to plug our proposed solution into the differential equation we're going to need some derivatives so let's get those. The first derivative is

$$Y_P'(t) = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

Here's the assumption. Simply to make the first derivative easier to deal with we are going to assume that whatever $u_1(t)$ and $u_2(t)$ are they will satisfy the following.

$$u_1' y_1 + u_2' y_2 = 0 \tag{3}$$

Now, there is no reason ahead of time to believe that this can be done. However, we will see that this will work out. We simply make this assumption on the hope that it won't cause problems down the road and to make the first derivative easier so don't get excited about it.

With this assumption the first derivative becomes.

$$Y_P'(t) = u_1 y_1' + u_2 y_2'$$

The second derivative is then,

$$Y_P''(t) = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Plug the solution and its derivatives into (1).

$$p(t)\left(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''\right) + q(t)\left(u_1 y_1' + u_2 y_2'\right) + r(t)\left(u_1 y_1 + u_2 y_2\right) = g(t)$$

Rearranging a little gives the following.

$$\begin{aligned} p(t)\left(u_1' y_1' + u_2' y_2'\right) + u_1(t)\big(p(t)\, y_1'' + q(t)\, y_1' + r(t)\, y_1\big) + \\ u_2(t)\big(p(t)\, y_2'' + q(t)\, y_2' + r(t)\, y_2\big) = g(t) \end{aligned}$$

Now, both $y_1(t)$ and $y_2(t)$ are solutions to (2) and so the second and third terms are zero. Acknowledging this and rearranging a little gives us,

$$p(t)\left(u_1' y_1' + u_2' y_2'\right) + u_1(t)(0) + u_2(t)(0) = g(t)$$

$$u_1' y_1' + u_2' y_2' = \frac{g(t)}{p(t)} \tag{4}$$

We've almost got the two equations that we need. Before proceeding we're going to go back and make a further assumption. The last equation, (4), is actually the one that we want, however, in order to make things simpler for us we are going to assume that the function $p(t) = 1$.

In other words, we are going to go back and start working with the differential equation,

$$y'' + q(t)\, y' + r(t)\, y = g(t)$$

If the coefficient of the second derivative isn't one divide it out so that it becomes a one. The formula that we're going to be getting will assume this! Upon doing this the two equations that we want so solve for the unknown functions are

$$u_1' y_1 + u_2' y_2 = 0 \tag{5}$$

$$u_1' y_1' + u_2' y_2' = g(t) \tag{6}$$

Note that in this system we know the two solutions and so the only two unknowns here are u_1' and u_2'. Solving this system is actually quite simple. First, solve (5) for u_1' and plug this into (6) and do some simplification.

$$u_1' = -\frac{u_2' y_2}{y_1} \tag{7}$$

$$\left(-\frac{u_2' y_2}{y_1}\right) y_1' + u_2' y_2' = g\left(t\right)$$

$$u_2'\left(y_2' - \frac{y_2 y_1'}{y_1}\right) = g\left(t\right)$$

$$u_2'\left(\frac{y_1 y_2' - y_2 y_1'}{y_1}\right) = g\left(t\right)$$

$$u_2' = \frac{y_1 g\left(t\right)}{y_1 y_2' - y_2 y_1'} \qquad \textbf{(8)}$$

So, we now have an expression for u_2'. Plugging this into (7) will give us an expression for u_1'.

$$u_1' = -\frac{y_2 g\left(t\right)}{y_1 y_2' - y_2 y_1'} \qquad \textbf{(9)}$$

Next, let's notice that

$$W\left(y_1, y_2\right) = y_1 y_2' - y_2 y_1' \neq 0$$

Recall that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions and so we know that the Wronskian won't be zero!

Finally, all that we need to do is integrate (8) and (9) in order to determine what $u_1(t)$ and $u_2(t)$ are. Doing this gives,

$$u_1\left(t\right) = -\int \frac{y_2 g\left(t\right)}{W\left(y_1, y_2\right)}\, dt \qquad\qquad u_2\left(t\right) = \int \frac{y_1 g\left(t\right)}{W\left(y_1, y_2\right)}\, dt$$

So, provided we can do these integrals, a particular solution to the differential equation is

$$\begin{aligned} Y_P\left(t\right) &= y_1 u_1 + y_2 u_2 \\ &= -y_1 \int \frac{y_2 g\left(t\right)}{W\left(y_1, y_2\right)}\, dt + y_2 \int \frac{y_1 g\left(t\right)}{W\left(y_1, y_2\right)}\, dt \end{aligned}$$

So, let's summarize up what we've determined here.

Variation of Parameters

Consider the differential equation,

$$y'' + q\left(t\right) y' + r\left(t\right) y = g\left(t\right)$$

Assume that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for

$$y'' + q\left(t\right) y' + r\left(t\right) y = 0$$

Then a particular solution to the nonhomogeneous differential equation is,

$$Y_P\left(t\right) = -y_1 \int \frac{y_2 g\left(t\right)}{W\left(y_1, y_2\right)}\, dt + y_2 \int \frac{y_1 g\left(t\right)}{W\left(y_1, y_2\right)}\, dt$$

Depending on the person and the problem, some will find the formula easier to memorize and use, while others will find the process used to get the formula easier. The examples in this section will be done using the formula.

Before proceeding with a couple of examples let's first address the issues involving the constants of integration that will arise out of the integrals. Putting in the constants of integration will give the following.

$$\begin{aligned} Y_P(t) &= -y_1\left(\int \frac{y_2 g(t)}{W(y_1,y_2)}\,dt + c\right) + y_2\left(\int \frac{y_1 g(t)}{W(y_1,y_2)}\,dt + k\right) \\ &= -y_1\int \frac{y_2 g(t)}{W(y_1,y_2)}\,dt + y_2\int \frac{y_1 g(t)}{W(y_1,y_2)}\,dt + \left(-c y_1 + k y_2\right) \end{aligned}$$

The final quantity in the parenthesis is nothing more than the complementary solution with $c_1 = -c$ and $c_2 = k$ and we know that if we plug this into the differential equation it will simplify out to zero since it is the solution to the homogeneous differential equation. In other words, these terms add nothing to the particular solution and so we will go ahead and assume that $c = 0$ and $k = 0$ in all the examples.

One final note before we proceed with examples. Do not worry about which of your two solutions in the complementary solution is $y_1(t)$ and which one is $y_2(t)$. It doesn't matter. You will get the same answer no matter which one you choose to be $y_1(t)$ and which one you choose to be $y_2(t)$.

Let's work a couple of examples now.

Example 1 Find a general solution to the following differential equation.

$$2y'' + 18y = 6\tan(3t)$$

Solution

First, since the formula for variation of parameters requires a coefficient of a one in front of the second derivative let's take care of that before we forget. The differential equation that we'll actually be solving is

$$y'' + 9y = 3\tan(3t)$$

We'll leave it to you to verify that the complementary solution for this differential equation is

$$y_c(t) = c_1\cos(3t) + c_2\sin(3t)$$

So, we have

$$y_1(t) = \cos(3t) \qquad\qquad y_2(t) = \sin(3t)$$

The Wronskian of these two functions is

$$W = \begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix} = 3\cos^2(3t) + 3\sin^2(3t) = 3$$

The particular solution is then,

$$\begin{aligned}
Y_P(t) &= -\cos(3t)\int\frac{3\sin(3t)\tan(3t)}{3}\,dt + \sin(3t)\int\frac{3\cos(3t)\tan(3t)}{3}\,dt \\
&= -\cos(3t)\int\frac{\sin^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int\sin(3t)\,dt \\
&= -\cos(3t)\int\frac{1-\cos^2(3t)}{\cos(3t)}\,dt + \sin(3t)\int\sin(3t)\,dt \\
&= -\cos(3t)\int\sec(3t)-\cos(3t)\,dt + \sin(3t)\int\sin(3t)\,dt \\
&= -\frac{\cos(3t)}{3}\left(\ln\left|\sec(3t)+\tan(3t)\right| - \sin(3t)\right) + \frac{\sin(3t)}{3}\left(-\cos(3t)\right) \\
&= -\frac{\cos(3t)}{3}\ln\left|\sec(3t)+\tan(3t)\right|
\end{aligned}$$

The general solution is,

$$y(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{\cos(3t)}{3}\ln\left|\sec(3t)+\tan(3t)\right|$$

Example 2 Find a general solution to the following differential equation.

$$y'' - 2y' + y = \frac{\mathbf{e}^t}{t^2+1}$$

Solution

We first need the complementary solution for this differential equation. We'll leave it to you to verify that the complementary solution is,

$$y_c(t) = c_1\mathbf{e}^t + c_2 t\mathbf{e}^t$$

So, we have

$$y_1(t) = \mathbf{e}^t \qquad\qquad y_2(t) = t\mathbf{e}^t$$

The Wronskian of these two functions is

$$W = \begin{vmatrix} \mathbf{e}^t & t\mathbf{e}^t \\ \mathbf{e}^t & \mathbf{e}^t + t\mathbf{e}^t \end{vmatrix} = \mathbf{e}^t\left(\mathbf{e}^t + t\mathbf{e}^t\right) - \mathbf{e}^t\left(t\mathbf{e}^t\right) = \mathbf{e}^{2t}$$

The particular solution is then,

$$\begin{aligned}
Y_P(t) &= -\mathbf{e}^t\int\frac{t\mathbf{e}^t\,\mathbf{e}^t}{\mathbf{e}^{2t}\left(t^2+1\right)}\,dt + t\mathbf{e}^t\int\frac{\mathbf{e}^t\,\mathbf{e}^t}{\mathbf{e}^{2t}\left(t^2+1\right)}\,dt \\
&= -\mathbf{e}^t\int\frac{t}{t^2+1}\,dt + t\mathbf{e}^t\int\frac{1}{t^2+1}\,dt \\
&= -\frac{1}{2}\mathbf{e}^t\ln\left(1+t^2\right) + t\mathbf{e}^t\tan^{-1}(t)
\end{aligned}$$

The general solution is,

$$y(t) = c_1\mathbf{e}^t + c_2 t\mathbf{e}^t - \frac{1}{2}\mathbf{e}^t\ln\left(1+t^2\right) + t\mathbf{e}^t\tan^{-1}(t)$$

This method can also be used on non-constant coefficient differential equations, provided we know a fundamental set of solutions for the associated homogeneous differential equation.

Example 3 Find the general solution to

$$ty'' - (t+1)y' + y = t^2$$

given that

$$y_1(t) = \mathbf{e}^t \qquad\qquad y_2(t) = t+1$$

form a fundamental set of solutions for the homogeneous differential equation.

Solution

As with the first example, we first need to divide out by a *t*.

$$y'' - \left(1 + \frac{1}{t}\right)y' + \frac{1}{t}y = t$$

The Wronskian for the fundamental set of solutions is

$$W = \begin{vmatrix} \mathbf{e}^t & t+1 \\ \mathbf{e}^t & 1 \end{vmatrix} = \mathbf{e}^t - \mathbf{e}^t(t+1) = -t\mathbf{e}^t$$

The particular solution is.

$$\begin{aligned} Y_P(t) &= -\mathbf{e}^t \int \frac{(t+1)t}{-t\mathbf{e}^t}\,dt + (t+1)\int \frac{\mathbf{e}^t(t)}{-t\mathbf{e}^t}\,dt \\ &= \mathbf{e}^t \int (t+1)\mathbf{e}^{-t}\,dt - (t+1)\int dt \\ &= \mathbf{e}^t\left(-\mathbf{e}^{-t}(t+2)\right) - (t+1)t \\ &= -t^2 - 2t - 2 \end{aligned}$$

The general solution for this differential equation is.

$$y(t) = c_1\mathbf{e}^t + c_2(t+1) - t^2 - 2t - 2$$

We need to address one more topic about the solution to the previous example. The solution can be simplified down somewhat if we do the following.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^t + c_2(t+1) - t^2 - 2t - 2 \\ &= c_1\mathbf{e}^t + c_2(t+1) - t^2 - 2(t+1) \\ &= c_1\mathbf{e}^t + (c_2 - 2)(t+1) - t^2 \end{aligned}$$

Now, since c_2 is an unknown constant subtracting 2 from it won't change that fact. So we can just write the $c_2 - 2$ as c_2 and be done with it. Here is a simplified version of the solution for this example.

$$y(t) = c_1\mathbf{e}^t + c_2(t+1) - t^2$$

This isn't always possible to do, but when it is you can simplify future work.

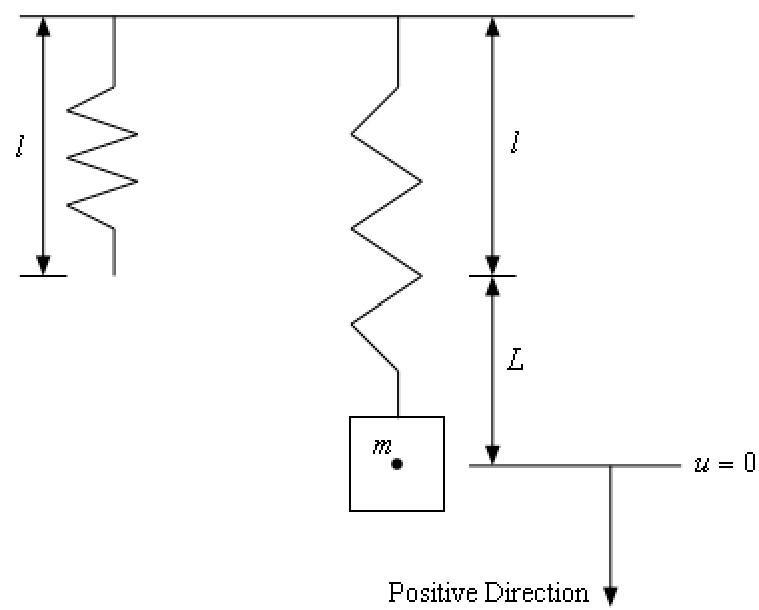
Mechanical Vibrations

It's now time to take a look at an application of second order differential equations. We're going to take a look at mechanical vibrations. In particular we are going to look at a mass that is hanging from a spring.

Vibrations can occur in pretty much all branches of engineering and so what we're going to be doing here can be easily adapted to other situations, usually with just a change in notation.

Let's get the situation setup. We are going to start with a spring of length l, called the natural length, and we're going to hook an object with mass m up to it. When the object is attached to the spring the spring will stretch a length of L. We will call the equilibrium position the position of the center of gravity for the object as it hangs on the spring with no movement.

Below is sketch of the spring with and without the object attached to it.



As denoted in the sketch we are going to assume that all forces, velocities, and displacements in the downward direction will be positive. All forces, velocities, and displacements in the upward direction will be negative.

Also, as shown in the sketch above, we will measure all displacement of the mass from its equilibrium position. Therefore, the $u = 0$ position will correspond to the center of gravity for the mass as it hangs on the spring and is at rest (*i.e.* no movement).

Now, we need to develop a differential equation that will give the displacement of the object at any time t. First, recall Newton's Second Law of Motion.

$$ma = F$$

In this case we will use the second derivative of the displacement, u, for the acceleration and so Newton's Second Law becomes,

$$mu'' = F\left(t, u, u'\right)$$

We now need to determine all the forces that will act upon the object. There are four forces that we will assume act upon the object. Two that will always act on the object and two that may or may not act upon the object.

Here is a list of the forces that will act upon the object.

1. **Gravity, F_g**
   The force due to gravity will always act upon the object of course. This force is
   $$F_g = mg$$

2. **Spring, F_s**
   We are going to assume that Hooke's Law will govern the force that the spring exerts on the object. This force will always be present as well and is
   $$F_s = -k\left(L+u\right)$$

   Hooke's Law tells us that the force exerted by a spring will be the spring constant, $k > 0$, times the displacement of the spring from its natural length. For our set up the displacement from the spring's natural length is $L + u$ and the minus sign is in there to make sure that the force always has the correct direction.

   Let's make sure that this force does what we expect it to. If the object is at rest in its equilibrium position the displacement is L and the force is simply $F_s = -kL$ which will act in the upward position as it should since the spring has been stretched from its natural length.

   If the spring has been stretched further down from the equilibrium position then $L+u$ will be positive and F_s will be negative acting to pull the object back up as it should be.

   Next, if the object has been moved up past its equilibrium point, but not yet to its natural length then u will be negative, but still less than L and so $L + u$ will be positive and once again F_s will be negative acting to pull the object up.

   Finally, if the object has been moved upwards so that the spring is now compressed, then u will be negative and greater than L. Therefore, $L + u$ will be negative and now F_s will be positive acting to push the object down.

   So, it looks like this force will act as we expect that it should.

3. **Damping, F_d**
   The next force that we need to consider is damping. This force may or may not be present for any given problem.

   Dampers work to counteract any movement. There are several ways to define a damping force. The one that we'll use is the following.
   $$F_d = -\gamma\, u'$$
   where, $\gamma > 0$ is the damping coefficient. Let's think for a minute about how this force will act. If the object is moving downward, then the velocity (u') will be positive and so F_d will be negative and acting to pull the object back up. Likewise, if the object is

moving upward, the velocity (u') will be negative and so F_d will be positive and acting to push the object back down.

In other words, the damping force as we've defined it will always act to counter the current motion of the object and so will act to damp out any motion in the object.

4. **External Forces, *F(t)***
This is the catch all force. If there are any other forces that we decide we want to act on our object we lump them in here and call it good. We typically call $F(t)$ the forcing function.

Putting all of these together gives us the following for Newton's Second Law.

$$mu'' = mg - k\left(L+u\right) - \gamma u' + F\left(t\right)$$

Or, upon rewriting, we get,

$$mu'' + \gamma u' + ku = mg - kL + F\left(t\right)$$

Now, when the object is at rest in its equilibrium position there are exactly two forces acting on the object, the force due to gravity and the force due to the spring. Also, since the object is at rest (*i.e.* not moving) these two forces must be canceling each other out. This means that we must have,

$$mg = kL \tag{1}$$

Using this in Newton's Second Law gives us the final version of the differential equation that we'll work with.

$$mu'' + \gamma u' + ku = F\left(t\right) \tag{2}$$

Along with this differential equation we will have the following initial conditions.

$$\begin{aligned} u\left(0\right) &= u_0 && \text{Initial displacement from the equilibrium position.} \\ u'\left(0\right) &= u'_0 && \text{Initial velocity.} \end{aligned} \tag{3}$$

Note that we'll also be using (1) to determine the spring constant, *k*.

Okay. Let's start looking at some specific cases.

Free, Undamped Vibrations
This is the simplest case that we can consider. Free or unforced vibrations means that $F(t) = 0$ and undamped vibrations means that $\gamma = 0$. In this case the differential equation becomes,

$$mu'' + ku = 0$$

This is easy enough to solve in general. The characteristic equation has the roots,

$$r = \pm\, i\sqrt{\frac{k}{m}}$$

This is usually reduced to,

$$r = \pm \omega_0 \, i$$

where,

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and ω_0 is called the natural frequency. Recall as well that $m > 0$ and $k > 0$ and so we can guarantee that this quantity will be complex. The solution in this case is then

$$u(t) = c_1 \cos(\omega_0 \, t) + c_2 \sin(\omega_0 \, t) \tag{4}$$

We can write (4) in the following form,

$$u(t) = R\cos(\omega_0 t - \delta) \tag{5}$$

where R is the amplitude of the displacement and δ is the phase shift or phase angle of the displacement.

When the displacement is in the form of (5) it is usually easier to work with. However, it's easier to find the constants in (4) from the initial conditions than it is to find the amplitude and phase shift in (5) from the initial conditions. So, in order to get the equation into the form in (5) we will first put the equation in the form in (4), find the constants, c_1 and c_2 and then convert this into the form in (5).

So, assuming that we have c_1 and c_2 how do we determine R and δ? Let's start with (5) and use a trig identity to write it as

$$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t) \tag{6}$$

Now, R and δ are constants and so if we compare (6) to (4) we can see that

$$c_1 = R\cos\delta \qquad\qquad c_2 = R\sin\delta$$

We can find R in the following way.

$$c_1^2 + c_2^2 = R^2\cos^2\delta + R^2\sin^2\delta = R^2$$

Taking the square root of both sides and assuming that R is positive will give

$$R = \sqrt{c_1^2 + c_2^2} \tag{7}$$

Finding δ is just as easy. We'll start with

$$\frac{c_2}{c_1} = \frac{R\sin\delta}{R\cos\delta} = \tan\delta$$

Taking the inverse tangent of both sides gives,

$$\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right) \tag{8}$$

Before we work any examples let's talk a little bit about units of mass and the British vs. metric system differences.

Recall that the weight of the object is given by

$$W = mg$$

where m is the mass of the object and g is the gravitational acceleration. For the examples in this problem we'll be using the following values for g.

$$\text{British} : g = 32 \;\; ft/s^2$$

$$\text{Metric} : g = 9.8 \; m/s^2$$

This is not the standard 32.2 ft/s^2 or 9.81 m/s^2, but using these will make some of the numbers come out a little nicer.

In the metric system the mass of objects is given in kilograms (kg) and there is nothing for us to do. However, in the British system we tend to be given the weight of an object in pounds (yes, pounds are the units of weight not mass…) and so we'll need to compute the mass for these problems.

At this point we should probably work an example of all this to see how this stuff works.

Example 1 A 16 lb object stretches a spring $\frac{8}{9}$ ft by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 inches upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement at any time t, $u(t)$.

Solution

We first need to set up the IVP for the problem. This requires us to get our hands on m and k.

This is the British system so we'll need to compute the mass.

$$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$$

Now, let's get k. We can use the fact that $mg = kL$ to find k. Don't forget that we'll need all of our length units the same. We'll use feet for the unit of measurement for this problem.

$$k = \frac{mg}{L} = \frac{16}{8/9} = 18$$

We can now set up the IVP.

$$\frac{1}{2}u'' + 18u = 0 \qquad\qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

For the initial conditions recall that upward displacement/motion is negative while downward displacement/motion is positive. Also, since we decided to do everything in feet we had to convert the initial displacement to feet.

Now, to solve this we can either go through the characteristic equation or we can just jump straight to the formula that we derived above. We'll do it that way. First, we need the natural frequency,

$$\omega_0 = \sqrt{\frac{18}{\frac{1}{2}}} = \sqrt{36} = 6$$

The general solution, along with its derivative, is then,

$$u(t) = c_1 \cos(6t) + c_2 \sin(6t)$$
$$u'(t) = -6c_1 \sin(6t) + 6c_2 \cos(6t)$$

Applying the initial conditions gives

$$-\frac{1}{2} = u(0) = c_1 \qquad\qquad c_1 = -\frac{1}{2}$$
$$1 = u'(0) = 6c_2 \cos(6t) \qquad c_2 = \frac{1}{6}$$

The displacement at any time *t* is then

$$u(t) = -\frac{1}{2}\cos(6t) + \frac{1}{6}\sin(6t)$$

Now, let's convert this to a single cosine. First let's get the amplitude, *R*.

$$R = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{6}\right)^2} = \frac{\sqrt{10}}{6} = 0.52705$$

You can use either the exact value here or a decimal approximation. Often the decimal approximation will be easier.

Now let's get the phase shift.

$$\delta = \tan^{-1}\left(\frac{\frac{1}{6}}{-\frac{1}{2}}\right) = -0.32175$$

We need to be careful with this part. The phase angle found above is in Quadrant IV, but there is also an angle in Quadrant II that would work as well. We get this second angle by adding π onto the first angle. So, we actually have two angles. They are

$$\delta_1 = -0.32175$$
$$\delta_2 = \delta_1 + \pi = 2.81984$$

We need to decide which of these phase shifts is correct, because only one will be correct. To do this recall that

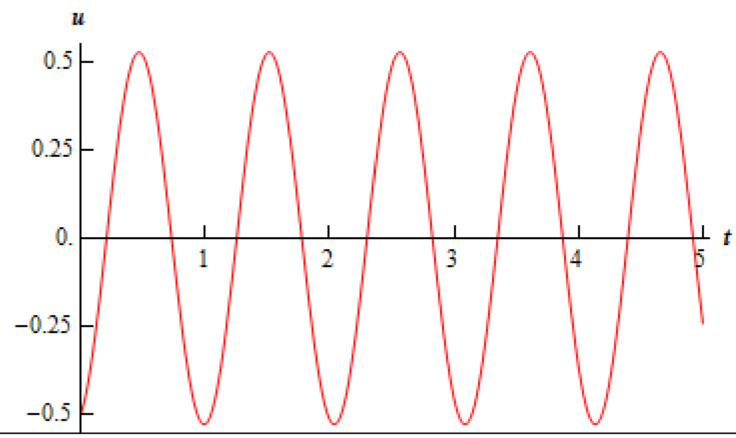
$$c_1 = R\cos\delta$$
$$c_2 = R\sin\delta$$

Now, since we are assuming that *R* is positive this means that the sign of $\cos\delta$ will be the same as the sign of c_1 and the sign of $\sin\delta$ will be the same as the sign of c_2. So, for this particular case we must have $\cos\delta < 0$ and $\sin\delta > 0$. This means that the phase shift must be in Quadrant II and so the second angle is the one that we need.

So, after all of this the displacement at any time t is.

$$u(t) = 0.52705\cos(6t - 2.81984)$$

Here is a sketch of the displacement for the first 5 seconds.



Now, let's take a look at a slightly more realistic situation. No vibration will go on forever. So let's add in a damper and see what happens now.

Free, Damped Vibrations

We are still going to assume that there will be no external forces acting on the system, with the exception of damping of course. In this case the differential equation will be.

$$mu'' + \gamma u' + ku = 0$$

where m, δ, and k are all positive constants. Upon solving for the roots of the characteristic equation we get the following.

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

We will have three cases here.

1. $\gamma^2 - 4mk = 0$

   In this case we will get a double root out of the characteristic equation and the displacement at any time t will be.

   $$u(t) = c_1\mathbf{e}^{-\frac{\gamma t}{2m}} + c_2 t\mathbf{e}^{-\frac{\gamma t}{2m}}$$

   Notice that as $t \to \infty$ the displacement will approach zero and so the damping in this case will do what it's supposed to do.

   This case is called **critical damping** and will happen when the damping coefficient is,

$$\begin{aligned}\gamma^2 - 4mk &= 0 \\ \gamma^2 &= 4mk \\ \gamma &= 2\sqrt{mk} = \gamma_{CR}\end{aligned}$$

The value of the damping coefficient that gives critical damping is called the critical damping coefficient and denoted by γ_{CR}.

2. $\gamma^2 - 4mk > 0$

In this case let's rewrite the roots a little.

$$\begin{aligned}r_{1,2} &= \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \\ &= \frac{-\gamma \pm \gamma\sqrt{1 - \frac{4mk}{\gamma^2}}}{2m} \\ &= -\frac{\gamma}{2m}\left(1 \pm \sqrt{1 - \frac{4mk}{\gamma^2}}\right)\end{aligned}$$

Also notice that from our initial assumption that we have,

$$\begin{aligned}\gamma^2 &> 4mk \\ 1 &> \frac{4mk}{\gamma^2}\end{aligned}$$

Using this we can see that the fraction under the square root above is less than one. Then if the quantity under the square root is less than one, this means that the square root of this quantity is also going to be less than one. In other words,

$$\sqrt{1 - \frac{4mk}{\gamma^2}} < 1$$

Why is this important? Well, the quantity in the parenthesis is now one plus/minus a number that is less than one. This means that the quantity in the parenthesis is guaranteed to be positive and so the two roots in this case are guaranteed to be negative. Therefore the displacement at any time t is,

$$u(t) = c_1 \mathbf{e}^{r_1 t} + c_2 \mathbf{e}^{r_2 t}$$

and will approach zero as $t \to \infty$. So, once again the damper does what it is supposed to do.

This case will occur when

$$\begin{aligned}\gamma^2 &> 4mk \\ \gamma &> 2\sqrt{mk} \\ \gamma &> \gamma_{CR}\end{aligned}$$

and is called **over damping.**

3. $\gamma^2 - 4mk < 0$

In this case we will get complex roots out of the characteristic equation.

$$r_{1,2} = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m} = \lambda \pm \mu\, i$$

where the real part is guaranteed to be negative and so the displacement is

$$\begin{aligned} u(t) &= c_1 \mathbf{e}^{\lambda t} \cos(\mu t) + c_2 \mathbf{e}^{\lambda t} \sin(\mu t) \\ &= \mathbf{e}^{\lambda t} \left(c_1 \cos(\mu t) + c_2 \sin(\mu t) \right) \\ &= R \mathbf{e}^{\lambda t} \cos(\mu t - \delta) \end{aligned}$$

Notice that we reduced the sine and cosine down to a single cosine in this case as we did in the undamped case. Also, since $\lambda < 0$ the displacement will approach zero as $t \to \infty$ and the damper will also work as it's supposed to in this case.

We will get this case will occur when

$$\begin{aligned} \gamma^2 &< 4mk \\ \gamma &< 2\sqrt{mk} \\ \gamma &< \gamma_{CR} \end{aligned}$$

and is called **under damping.**

Let's take a look at a couple of examples here with damping.

Example 2 Take the spring and mass system from the first example and attach a damper to it that will exert a force of 12 lbs when the velocity is 2 ft/s. Find the displacement at any time *t*, *u(t)*.

Solution

The mass and spring constant were already found in the first example so we won't do the work here. We do need to find the damping coefficient however. To do this we will use the formula for the damping force given above with one modification. The original damping force formula is,

$$F_d = -\gamma u'$$

However, remember that the force and the velocity are always acting in opposite directions. So, if the velocity is upward (*i.e.* negative) the force will be downward (*i.e.* positive) and so the minus in the formula will cancel against the minus in the velocity. Likewise, if the velocity is downward (*i.e.* positive) the force will be upwards (*i.e.* negative) and in this case the minus sign in the formula will cancel against the minus in the force. In other words, we can drop the minus sign in the formula and use

$$F_d = \gamma u'$$

and then just ignore any signs for the force and velocity.

Doing this gives us the following for the damping coefficient

$$12 = \gamma(2) \qquad \Rightarrow \qquad \gamma = 6$$

The IVP for this example is then,

$$\frac{1}{2}u'' + 6u' + 18u = 0 \qquad\qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

Before solving let's check to see what kind of damping we've got. To do this all we need is the critical damping coefficient.

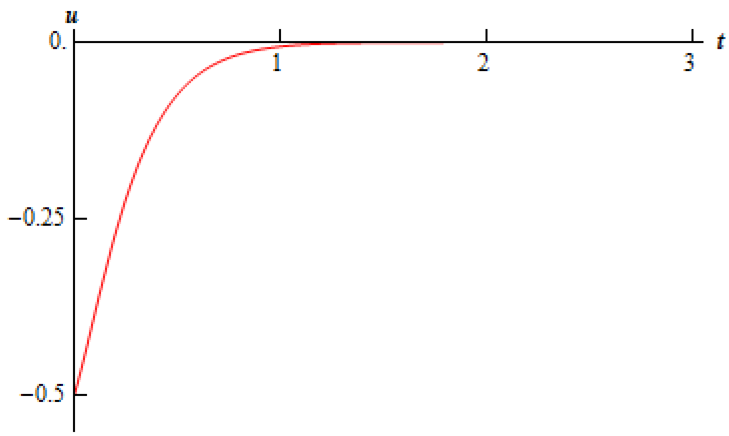
$$\gamma_{CR} = 2\sqrt{km} = 2\sqrt{(18)\left(\frac{1}{2}\right)} = 2\sqrt{9} = 6$$

So, it looks like we've got critical damping. Note that this means that when we go to solve the differential equation we should get a double root.

Speaking of solving, let's do that. I'll leave the details to you to check that the displacement at any time *t* is.

$$u(t) = -\frac{1}{2}\mathbf{e}^{-6t} - 2t\mathbf{e}^{-6t}$$

Here is a sketch of the displacement during the first 3 seconds.



Notice that the "vibration" in the system is not really a true vibration as we tend to think of them. In the critical damping case there isn't going to be a real oscillation about the equilibrium point that we tend to associate with vibrations. The damping in this system is strong enough to force the "vibration" to die out before it ever really gets a chance to do much in the way of oscillation.

Example 3 Take the spring and mass system from the first example and this time let's attach a damper to it that will exert a force of 17 lbs when the velocity is 2 ft/s. Find the displacement at any time *t*, *u(t)*.

Solution
So, the only difference between this example and the previous example is damping force. So let's find the damping coefficient

$$17 = \gamma(2) \qquad \Rightarrow \qquad \gamma = \frac{17}{2} = 8.5 > \gamma_{CR}$$

So it looks like we've got over damping this time around so we should expect to get two real distinct roots from the characteristic equation and they should both be negative. The IVP for this example is,

$$\frac{1}{2}u'' + \frac{17}{2}u' + 18u = 0 \qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

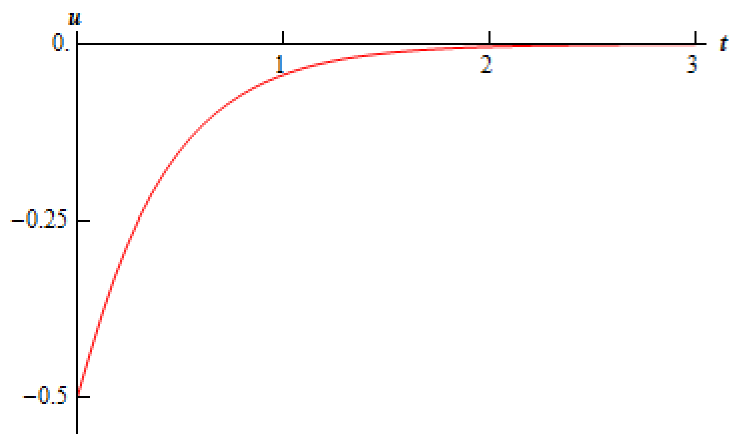
This one's a little messier than the previous example so we'll do a couple of the steps, leaving it to you to fill in the blanks. The roots of the characteristic equation are

$$r_{1,2} = \frac{-17 \pm \sqrt{145}}{2} = -2.4792,\ -14.5208$$

In this case it will be easier to just convert to decimals and go that route. Note that, as predicted we got two real, distinct and negative roots. The general and actual solution for this example are then,

$$u(t) = c_1\mathbf{e}^{-2.4792\,t} + c_2\mathbf{e}^{-14.5208\,t}$$
$$u(t) = -0.5198\mathbf{e}^{-2.4792\,t} + 0.0199\mathbf{e}^{-14.5208\,t}$$

Here's a sketch of the displacement for this example.



Notice an interesting thing here about the displacement here. Even though we are "over" damped in this case, it actually takes longer for the vibration to die out than in the critical damping case. Sometimes this happens, although it will not always be the case that over damping will allow the vibration to continue longer than the critical damping case.

Also notice that, as with the critical damping case, we don't get a vibration in the sense that we usually think of them. Again, the damping is strong enough to force the vibration do die out quick enough so that we don't see much, if any, of the oscillation that we typically associate with vibrations.

Let's take a look at one more example before moving on the next type of vibrations.

Example 4 Take the spring and mass system from the first example and for this example let's attach a damper to it that will exert a force of 5 lbs when the velocity is 2 ft/s. Find the displacement at any time *t*, *u(t)*.

Solution
So, let's get the damping coefficient.

$$5 = \gamma(2) \qquad \Rightarrow \qquad \gamma = \frac{5}{2} = 2.5 < \gamma_{CR}$$

So it's under damping this time. That shouldn't be too surprising given the first two examples. The IVP for this example is,

$$\frac{1}{2}u'' + \frac{5}{2}u' + 18u = 0 \qquad\qquad u(0) = -\frac{1}{2} \qquad u'(0) = 1$$

In this case the roots of the characteristic equation are

$$r_{1,2} = \frac{-5 \pm \sqrt{119}\, i}{2}$$

They are complex as we expected to get since we are in the under damped case. The general solution and actual solution are

$$u(t) = \mathbf{e}^{-\frac{5t}{2}}\left(c_1 \cos\left(\frac{\sqrt{119}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{119}}{2}t\right)\right)$$

$$u(t) = \mathbf{e}^{-\frac{5t}{2}}\left(-0.5\cos\left(\frac{\sqrt{119}}{2}t\right) - 0.04583\sin\left(\frac{\sqrt{119}}{2}t\right)\right)$$

Let's convert this to a single cosine as we did in the undamped case.

$$R = \sqrt{(-0.5)^2 + (-0.04583)^2} = 0.502096$$

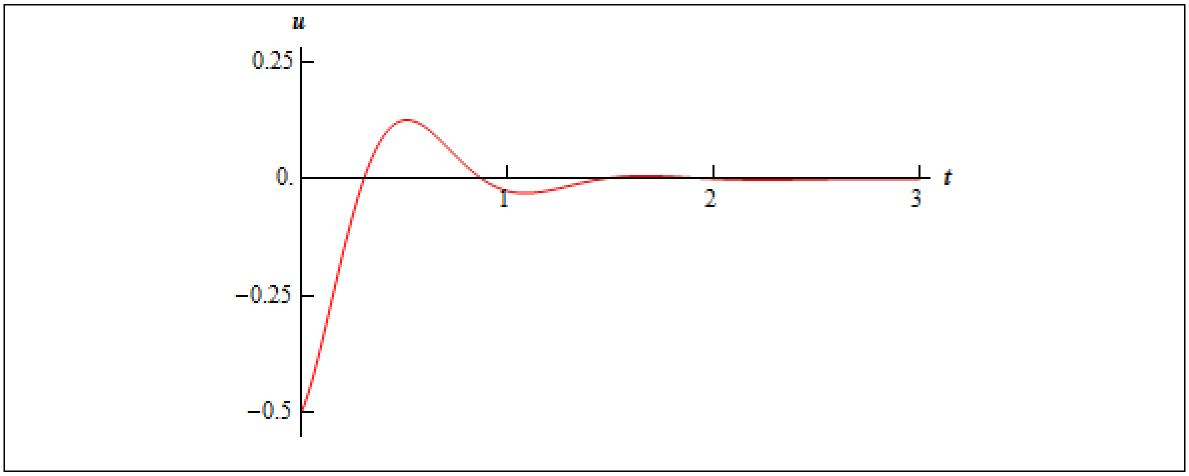
$$\delta_1 = \tan^{-1}\left(\frac{-0.04583}{-0.5}\right) = 0.09051 \qquad \text{OR} \qquad \delta_2 = \delta_1 + \pi = 3.2321$$

As with the undamped case we can use the coefficients of the cosine and the sine to determine which phase shift that we should use. The coefficient of the cosine (c_1) is negative and so $\cos\delta$ must also be negative. Likewise, the coefficient of the sine (c_2) is also negative and so $\sin\delta$ must also be negative. This means that δ must be in the Quadrant III and so the second angle is the one that we want.

The displacement is then

$$u(t) = 0.502096\mathbf{e}^{-\frac{5t}{2}}\cos\left(\frac{\sqrt{119}}{2}t - 3.2321\right)$$

Here is a sketch of this displacement.



In this case we finally got what we usually consider to be a true vibration. In fact that is the point of critical damping. As we increase the damping coefficient, the critical damping coefficient will be the first one in which a true oscillation in the displacement will not occur. For all values of the damping coefficient larger than this (*i.e.* over damping) we will also not see a true oscillation in the displacement.

From a physical standpoint critical (and over) damping is usually preferred to under damping. Think of the shock absorbers in your car. When you hit a bump you don't want to spend the next few minutes bouncing up and down while the vibration set up by the bump die out. You would like there to be as little movement as possible. In other words, you will want to set up the shock absorbers in your car so get at the least critical damping so that you can avoid the oscillations that will arise from an under damped case.

It's now time to look at systems in which we allow other external forces to act on the object in the system.

Undamped, Forced Vibrations

We will first take a look at the undamped case. The differential equation in this case is

$$mu'' + ku = F(t)$$

This is just a nonhomogeneous differential equation and we know how to solve these. The general solution will be

$$u(t) = u_c(t) + U_P(t)$$

where the complementary solution is the solution to the free, undamped vibration case. To get the particular solution we can use either undetermined coefficients or variation of parameters depending on which we find easier for a given forcing function.

There is a particular type of forcing function that we should take a look at since it leads to some interesting results. Let's suppose that the forcing function is a simple periodic function of the form

$$F(t) = F_0 \cos(\omega t) \qquad \text{OR} \qquad F(t) = F_0 \sin(\omega t)$$

For the purposes of this discussion we'll use the first one. Using this, the IVP becomes,

$$mu'' + ku = F_0 \cos(\omega t)$$

The complementary solution, as pointed out above, is just

$$u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where ω_0 is the natural frequency.

We will need to be careful in finding a particular solution. The reason for this will be clear if we use undetermined coefficients. With undetermined coefficients our guess for the form of the particular solution would be,

$$U_P(t) = A\cos(\omega t) + B\sin(\omega t)$$

Now, this guess will be problems if $\omega_0 = \omega$. If this were to happen the guess for the particular solution is exactly the complementary solution and so we'd need to add in a *t*. Of course if we don't have $\omega_0 = \omega$ then there will be nothing wrong with the guess.

So, we will need to look at this in two cases.

1. $\omega_0 \neq \omega$

   In this case our initial guess is okay since it won't be the complementary solution. Upon differentiating the guess and plugging it into the differential equation and simplifying we get,

   $$\left(-m\omega^2 A + kA\right)\cos(wt) + \left(-m\omega^2 B + kB\right)\sin(wt) = F_0 \cos(wt)$$

   Setting coefficients equal gives us,

   $$\begin{aligned} \cos(\omega t): &\qquad \left(-m\omega^2 + k\right)A = F_0 &\qquad \Rightarrow &\qquad A = \frac{F_0}{k - m\omega^2} \\ \sin(\omega t): &\qquad \left(-m\omega^2 + k\right)B = 0 &\qquad \Rightarrow &\qquad B = 0 \end{aligned}$$

   The particular solution is then

   $$\begin{aligned} U_P(t) &= \frac{F_0}{k - m\omega^2}\cos(\omega t) \\ &= \frac{F_0}{m\left(\frac{k}{m} - \omega^2\right)}\cos(\omega t) \\ &= \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos(\omega t) \end{aligned}$$

   Note that we rearranged things a little. Depending on the form that you'd like the displacement to be in we can have either of the following.

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$
$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

If we used the sine form of the forcing function we could get a similar formula.

2. $\omega_0 = \omega$
In this case we will need to add in a t to the guess for the particular solution.
$$U_P(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$$

Note that we went ahead and acknowledge that $\omega_0 = \omega$ in our guess. Acknowledging this will help with some simplification that we'll need to do later on. Differentiating our guess, plugging it into the differential equation and simplifying gives us the following.
$$\left(-m\omega_0^2 + k\right) At\cos(wt) + \left(-m\omega_0^2 + k\right) Bt\sin(wt) +$$
$$2m\omega_0 B\cos(\omega t) - 2m\omega_0 A\sin(wt) = F_0 \cos(wt)$$

Before setting coefficients equal, let's remember the definition of the natural frequency and note that
$$-m\omega_0^2 + k = -m\left(\sqrt{\frac{k}{m}}\right)^2 + k = -m\left(\frac{k}{m}\right) + k = 0$$

So, the first two terms actually drop out (which is a very good thing…) and this gives us,
$$2m\omega_0 B\cos(\omega t) - 2m\omega_0 A\sin(wt) = F_0 \cos(wt)$$

Now let's set coefficient equal.
$$\cos(\omega t): \qquad 2m\omega_0 B = F_0 \qquad \Rightarrow \qquad B = \frac{F_0}{2m\omega_0}$$
$$\sin(\omega t): \qquad 2m\omega_0 A = 0 \qquad \Rightarrow \qquad A = 0$$

In this case the particular will be,
$$U_P(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$

The displacement for this case is then
$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$
$$u(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$
depending on the form that you prefer for the displacement.

So, what was the point of the two cases here? Well in the first case, $\omega_0 \neq \omega$ our displacement function consists of two cosines and is nice and well behaved for all time.

In contrast, the second case, $\omega_0 = \omega$ will have some serious issues at *t* increases. The addition of the *t* in the particular solution will mean that we are going to see an oscillation that grows in amplitude as *t* increases. This case is called **resonance** and we would generally like to avoid this at all costs.

In this case resonance arose by assuming that the forcing function was,

$$F(t) = F_0 \cos(\omega_0 t)$$

We would also have the possibility of resonance if we assumed a forcing function of the form.

$$F(t) = F_0 \sin(\omega_0 t)$$

We should also take care to not assume that a forcing function will be in one of these two forms. Forcing functions can come in a wide variety of forms. If we do run into a forcing function different from the one that used here you will have to go through undetermined coefficients or variation of parameters to determine the particular solution.

Example 5 A 3 kg object is attached to spring and will stretch the spring 392 mm by itself. There is no damping in the system and a forcing function of the form

$$F(t) = 10\cos(\omega t)$$

is attached to the object and the system will experience resonance. If the object is initially displaced 20 cm downward from its equilibrium position and given a velocity of 10 cm/sec upward find the displacement at any time *t*.

Solution

Since we are in the metric system we won't need to find mass as it's been given to us. Also, for all calculations we'll be converting all lengths over to meters.

The first thing we need to do is find *k*.

$$k = \frac{mg}{L} = \frac{(3)(9.8)}{0.392} = 75$$

Now, we are told that the system experiences resonance so let's go ahead and get the natural frequency so we can completely set up the IVP.

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5$$

The IVP for this is then

$$3u'' + 75u = 10\cos(5t) \qquad u(0) = 0.2 \qquad u'(0) = -0.1$$

Solution wise there isn't a whole lot to do here. The complementary solution is the free undamped solution which is easy to get and for the particular solution we can just use the formula that we derived above.

The general solution is then,

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{10}{2(3)(5)} t \sin(5t)$$

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t) + \frac{1}{3} t \sin(5t)$$

Applying the initial conditions gives the displacement at any time *t*. We'll leave the details to you to check.

$$u(t) = \frac{1}{5}\cos(5t) - \frac{1}{50}\sin(5t) + \frac{1}{3} t \sin(5t)$$

The last thing that we'll do is combine the first two terms into a single cosine.

$$R = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{50}\right)^2} = 0.200998$$

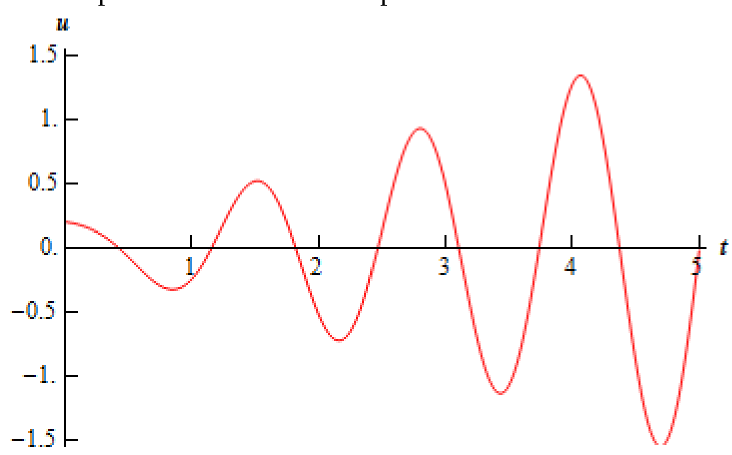
$$\delta_1 = \tan^{-1}\left(\frac{-\,{}^{1}\!/_{50}}{{}^{1}\!/_{5}}\right) = -0.099669 \qquad\qquad \delta_2 = \delta_1 + \pi = 3.041924$$

In this case the coefficient of the cosine is positive and the coefficient of the sine is negative. This forces $\cos\delta$ to be positive and $\sin\delta$ to be negative. This means that the phase shift needs to be in Quadrant IV and so the first one is the correct phase shift this time.

The displacement then becomes,

$$u(t) = 0.200998\cos(5t + 0.099669) + \frac{1}{3} t \sin(5t)$$

Here is a sketch of the displacement for this example.



It's now time to look at the final vibration case.

Forced, Damped Vibrations

This is the full blown case where we consider every last possible force that can act upon the system. The differential equation for this case is,

$$mu'' + \gamma u' + ku = F(t)$$

The displacement function this time will be,

$$u(t) = u_c(t) + U_P(t)$$

where the complementary solution will be the solution to the free, damped case and the particular solution will be found using undetermined coefficients or variation of parameter, whichever is most convenient to use.

There are a couple of things to note here about this case. First, from our work back in the free, damped case we know that the complementary solution will approach zero as *t* increases. Because of this the complementary solution is often called the **transient solution** in this case.

Also, because of this behavior the displacement will start to look more and more like the particular solution as *t* increases and so the particular solution is often called the **steady state solution** or **forced response**.

Let's work one final example before leaving this section. As with the previous examples, we're going to leave most of the details out for you to check.

Example 6 Take the system from the last example and add in a damper that will exert a force of 45 Newtons when then velocity is 50 cm/sec.

Solution

So, all we need to do is compute the damping coefficient for this problem then pull everything else down from the previous problem. The damping coefficient is

$$\begin{aligned} F_d &= \gamma u' \\ 45 &= \gamma(0.5) \\ \gamma &= 90 \end{aligned}$$

The IVP for this problem is.

$$3u'' + 90u' + 75u = 10\cos(5t) \qquad u(0) = 0.2 \qquad u'(0) = -0.1$$

The complementary solution for this example is

$$\begin{aligned} u_c(t) &= c_1 \mathbf{e}^{\left(-15+10\sqrt{2}\right)t} + c_2 \mathbf{e}^{\left(-15-10\sqrt{2}\right)t} \\ u_c(t) &= c_1 \mathbf{e}^{-0.8579t} + c_2 \mathbf{e}^{-29.1421t} \end{aligned}$$

For the particular solution we the form will be,

$$U_P(t) = A\cos(5t) + B\sin(5t)$$

Plugging this into the differential equation and simplifying gives us,

$$450B\cos(5t) - 450A\sin(5t) = 10\cos(5t)$$

Setting coefficient equal gives,

$$U_P(t) = \frac{1}{45}\sin(5t)$$

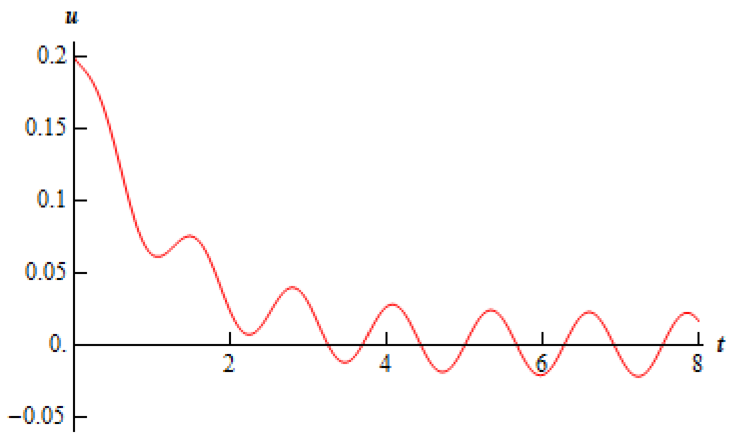
The general solution is then

$$u(t) = c_1\mathbf{e}^{-0.8579t} + c_2\mathbf{e}^{-29.1421t} + \frac{1}{45}\sin(5t)$$

Applying the initial condition gives

$$u(t) = 0.1986\mathbf{e}^{-0.8579t} + 0.001398\mathbf{e}^{-29.1421t} + \frac{1}{45}\sin(5t)$$

Here is a sketch of the displacement for this example.



Laplace Transforms

Introduction

In this chapter we will be looking at how to use Laplace transforms to solve differential equations. There are many kinds of transforms out there in the world. Laplace transforms and Fourier transforms are probably the main two kinds of transforms that are used. As we will see in later sections we can use Laplace transforms to reduce a differential equation to an algebra problem. The algebra can be messy on occasion, but it will be simpler than actually solving the differential equation directly in many cases. Laplace transforms can also be used to solve IVP's that we can't use any previous method on.

For "simple" differential equations such as those in the first few sections of the last chapter Laplace transforms will be more complicated than we need. In fact, for most homogeneous differential equations such as those in the last chapter Laplace transforms is significantly longer and not so useful. Also, many of the "simple" nonhomogeneous differential equations that we saw in the Undetermined Coefficients and Variation of Parameters are still simpler (or at the least no more difficult than Laplace transforms) to do as we did them there. However, at this point, the amount of work required for Laplace transforms is starting to equal the amount of work we did in those sections.

Laplace transforms comes into its own when the forcing function in the differential equation starts getting more complicated. In the previous chapter we looked only at nonhomogeneous differential equations in which $g(t)$ was a fairly simple continuous function. In this chapter we will start looking at $g(t)$'s that are not continuous. It is these problems where the reasons for using Laplace transforms start to become clear.

We will also see that, for some of the more complicated nonhomogeneous differential equations from the last chapter, Laplace transforms are actually easier on those problems as well.

Here is a brief rundown of the sections in this chapter.

Solving IVP's with Laplace Transforms – Here's how we used Laplace transforms to solve IVP's.

Nonconstant Coefficient IVP's – We will see how Laplace transforms can be used to solve some nonconstant coefficient IVP's

IVP's with Step Functions – Solving IVP's that contain step functions. This is the section where the reason for using Laplace transforms really becomes apparent.

Dirac Delta Function – One last function that often shows up in Laplace transform problems.

Convolution Integral – A brief introduction to the convolution integral and an application for Laplace transforms.

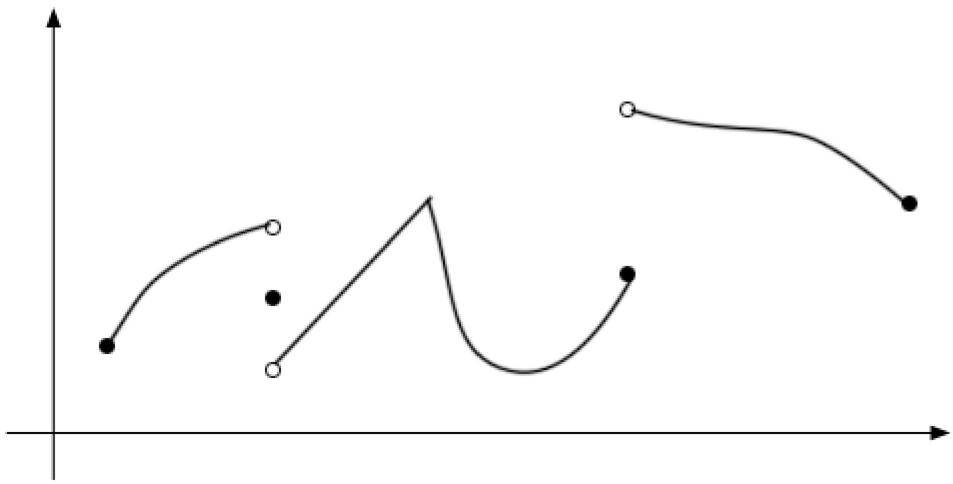
Table of Laplace Transforms – This is a small table of Laplace Transforms that we'll be using here.

The Definition

You know, it's always a little scary when we devote a whole section just to the definition of something. Laplace transforms (or just transforms) can seem scary when we first start looking at them. However, as we will see, they aren't as bad as they may appear at first.

Before we start with the definition of the Laplace transform we need to get another definition out of the way.

A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.* the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval. Below is a sketch of a piecewise continuous function.



In other words, a piecewise continuous function is a function that has a finite number of breaks in it and doesn't blow up to infinity anywhere.

Now, let's take a look at the definition of the Laplace transform.

Definition

Suppose that *f(t)* is a piecewise continuous function. The Laplace transform of *f(t)* is denoted $\mathfrak{L}\left\{ f\left(t\right)\right\}$ and defined as

$$\mathfrak{L}\left\{ f\left(t\right)\right\} = \int_0^\infty \mathbf{e}^{-st} f\left(t\right) dt \qquad \textbf{(1)}$$

There is an alternate notation for Laplace transforms. For the sake of convenience we will often denote Laplace transforms as,

$$\mathfrak{L}\left\{ f\left(t\right)\right\} = F\left(s\right)$$

With this alternate notation, note that the transform is really a function of a new variable, s, and that all the t's will drop out in the integration process.

Now, the integral in the definition of the transform is called an improper integral and it would probably be best to recall how these kinds of integrals work before we actually jump into computing some transforms.

Example 1 If $c \neq 0$, evaluate the following integral.

$$\int_0^{\infty} \mathbf{e}^{ct}\, dt$$

Solution

Remember that you need to convert improper integrals to limits as follows,

$$\int_0^{\infty} \mathbf{e}^{ct}\, dt = \lim_{n\to\infty} \int_0^{n} \mathbf{e}^{ct}\, dt$$

Now, do the integral, then evaluate the limit.

$$\begin{aligned}\int_0^{\infty} \mathbf{e}^{ct}\, dt &= \lim_{n\to\infty} \int_0^{n} \mathbf{e}^{ct}\, dt \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathbf{e}^{ct} \right)\Bigg|_0^n \\ &= \lim_{n\to\infty} \left(\frac{1}{c}\mathbf{e}^{cn} - \frac{1}{c} \right)\end{aligned}$$

Now, at this point, we've got to be careful. The value of *c* will affect our answer. We've already assumed that *c* was non-zero, now we need to worry about the sign of *c*. If *c* is positive the exponential will go to infinity. On the other hand, if *c* is negative the exponential will go to zero.

So, the integral is only convergent (*i.e.* the limit exists and is finite) provided *c<0*. In this case we get,

$$\int_0^{\infty} \mathbf{e}^{ct}\, dt = -\frac{1}{c} \qquad \text{provided } c < 0 \qquad \textbf{(2)}$$

Now that we remember how to do these, let's compute some Laplace transforms. We'll start off with probably the simplest Laplace transform to compute.

Example 2 Compute $\mathfrak{L}\{1\}$.

Solution

There's not really a whole lot do here other than plug the function *f(t)* = 1 into (1)

$$\mathfrak{L}\{1\} = \int_0^{\infty} \mathbf{e}^{-st}\, dt$$

Now, at this point notice that this is nothing more than the integral in the previous example with $c = -s$. Therefore, all we need to do is reuse (2) with the appropriate substitution. Doing this gives,

$$\mathfrak{L}\{1\} = \int_0^{\infty} \mathbf{e}^{-st}\, dt = -\frac{1}{-s} \qquad \text{provided } -s < 0$$

Or, with some simplification we have,

$$\mathfrak{L}\{1\} = \frac{1}{s} \qquad \text{provided } s > 0$$

Notice that we had to put a restriction on *s* in order to actually compute the transform. All Laplace transforms will have restrictions on *s*. At this stage of the game, this restriction is something that we tend to ignore, but we really shouldn't ever forget that it's there.

Let's do another example.

Example 3 Compute $\mathfrak{L}\left\{\mathbf{e}^{at}\right\}$

Solution
Plug the function into the definition of the transform and do a little simplification.

$$\mathfrak{L}\left\{\mathbf{e}^{at}\right\}=\int_0^\infty \mathbf{e}^{-st}\mathbf{e}^{at}\,dt=\int_0^\infty \mathbf{e}^{(a-s)t}\,dt$$

Once again, notice that we can use (2) provided $c=a-s$. So let's do this.

$$\begin{aligned}\mathfrak{L}\left\{\mathbf{e}^{at}\right\}&=\int_0^\infty \mathbf{e}^{(a-s)t}\,dt\\&=-\frac{1}{a-s} &&\text{provided } a-s<0\\&=\frac{1}{s-a} &&\text{provided } s>a\end{aligned}$$

Let's do one more example that doesn't come down to an application of (2).

Example 4 Compute $\mathfrak{L}\{\sin(at)\}$.

Solution
Note that we're going to leave it to you to check most of the integration here. Plug the function into the definition. This time let's also use the alternate notation.

$$\begin{aligned}\mathfrak{L}\left\{\sin(at)\right\}&=F(s)\\&=\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\\&=\lim_{n\to\infty}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\end{aligned}$$

Now, if we integrate by parts we will arrive at,

$$F(s)=\lim_{n\to\infty}\left(-\left(\frac{1}{a}\mathbf{e}^{-st}\cos(at)\right)\Bigg|_0^n-\frac{s}{a}\int_0^n \mathbf{e}^{-st}\cos(at)\,dt\right)$$

Now, evaluate the first term to simplify it a little and integrate by parts again on the integral. Doing this arrives at,

$$F(s)=\lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right)-\frac{s}{a}\left(\left(\frac{1}{a}\mathbf{e}^{-st}\sin(at)\right)\Bigg|_0^n+\frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right)$$

Now, evaluate the second term, take the limit and simplify.

$$\begin{aligned} F(s) &= \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right)-\frac{s}{a}\left(\frac{1}{a}\mathbf{e}^{-sn}\sin(an)+\frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right)\\ &= \frac{1}{a}-\frac{s}{a}\left(\frac{s}{a}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\right)\\ &= \frac{1}{a}-\frac{s^2}{a^2}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt \end{aligned}$$

Now, notice that in the limits we had to assume that $s>0$ in order to do the following two limits.

$$\lim_{n\to\infty}\mathbf{e}^{-sn}\cos(an)=0$$
$$\lim_{n\to\infty}\mathbf{e}^{-sn}\sin(an)=0$$

Without this assumption, we get a divergent integral again. Also, note that when we got back to the integral we just converted the upper limit back to infinity. The reason for this is that, if you think about it, this integral is nothing more than the integral that we started with. Therefore, we now get,

$$F(s)=\frac{1}{a}-\frac{s^2}{a^2}F(s)$$

Now, simply solve for $F(s)$ to get,

$$\mathfrak{L}\left\{\sin(at)\right\}=F(s)=\frac{a}{s^2+a^2} \qquad \text{provided } s>0$$

As this example shows, computing Laplace transforms is often messy.

Before moving on to the next section, we need to do a little side note. On occasion you will see the following as the definition of the Laplace transform.

$$\mathfrak{L}\left\{f(t)\right\}=\int_{-\infty}^{\infty}\mathbf{e}^{-st}f(t)\,dt$$

Note the change in the lower limit from zero to negative infinity. In these cases there is almost always the assumption that the function $f(t)$ is in fact defined as follows,

$$f(t)=\begin{cases}0 & \text{if } t<0\\ f(t) & \text{if } t\geq 0\end{cases}$$

In other words, it is assumed that the function is zero if $t<0$. In this case the first half of the integral will drop out since the function is zero and we will get back to the definition given in (1). A Heaviside function is usually used to make the function zero for $t<0$. We will be looking at these in a later section.

Laplace Transforms

As we saw in the last section computing Laplace transforms directly can be fairly complicated. Usually we just use a table of transforms when actually computing Laplace transforms. The table that is provided here is not an inclusive table, but does include most of the commonly used Laplace transforms and most of the commonly needed formulas pertaining to Laplace transforms.

Before doing a couple of examples to illustrate the use of the table let's get a quick fact out of the way.

Fact

Given *f(t)* and *g(t)* then,

$$\mathfrak{L}\left\{af\left(t\right)+bg\left(t\right)\right\}=a\,F\left(s\right)+b\,G\left(s\right)$$

for any constants *a* and *b*.

In other words, we don't worry about constants and we don't worry about sums or differences of functions in taking Laplace transforms. All that we need to do is take the transform of the individual functions, then put any constants back in and add or subtract the results back up.

So, let's do a couple of quick examples.

Example 1 Find the Laplace transforms of the given functions.

(a) $f\left(t\right)=6\mathbf{e}^{-5t}+\mathbf{e}^{3t}+5t^{3}-9$ [Solution]

(b) $g\left(t\right)=4\cos\left(4t\right)-9\sin\left(4t\right)+2\cos\left(10t\right)$ [Solution]

(c) $h\left(t\right)=3\sinh\left(2t\right)+3\sin\left(2t\right)$ [Solution]

(d) $g\left(t\right)=\mathbf{e}^{3t}+\cos\left(6t\right)-\mathbf{e}^{3t}\cos\left(6t\right)$ [Solution]

Solution

Okay, there's not really a whole lot to do here other than go to the table, transform the individual functions up, put any constants back in and then add or subtract the results.

We'll do these examples in a little more detail than is typically used since this is the first time we're using the tables.

(a) $f\left(t\right)=6\mathbf{e}^{-5t}+\mathbf{e}^{3t}+5t^{3}-9$

$$\begin{aligned}F\left(s\right)&=6\frac{1}{s-\left(-5\right)}+\frac{1}{s-3}+5\frac{3!}{s^{3+1}}-9\frac{1}{s}\\&=\frac{6}{s+5}+\frac{1}{s-3}+\frac{30}{s^{4}}-\frac{9}{s}\end{aligned}$$

[Return to Problems]

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$\begin{aligned} G(s) &= 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2} \\ &= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100} \end{aligned}$$

[Return to Problems]

(c) $h(t) = 3\sinh(2t) + 3\sin(2t)$

$$\begin{aligned} H(s) &= 3\frac{2}{s^2-(2)^2} + 3\frac{2}{s^2+(2)^2} \\ &= \frac{6}{s^2-4} + \frac{6}{s^2+4} \end{aligned}$$

[Return to Problems]

(d) $g(t) = \mathbf{e}^{3t} + \cos(6t) - \mathbf{e}^{3t}\cos(6t)$

$$\begin{aligned} G(s) &= \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2} \\ &= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36} \end{aligned}$$

[Return to Problems]

Make sure that you pay attention to the difference between a "normal" trig function and hyperbolic functions. The only difference between them is the "+ a^2" for the "normal" trig functions becomes a "- a^2" in the hyperbolic function! It's very easy to get in a hurry and not pay attention and grab the wrong formula. If you don't recall the definition of the hyperbolic functions see the notes for the table.

Let's do one final set of examples.

Example 2 Find the transform of each of the following functions.

(a) $f(t) = t\cosh(3t)$ [Solution]

(b) $h(t) = t^2\sin(2t)$ [Solution]

(c) $g(t) = t^{\frac{3}{2}}$ [Solution]

(d) $f(t) = (10t)^{\frac{3}{2}}$ [Solution]

(e) $f(t) = tg'(t)$ [Solution]

Solution

(a) $f(t) = t\cosh(3t)$

This function is not in the table of Laplace transforms. However we can use #30 in the table to compute its transform. This will correspond to #30 if we take $n=1$.

$$F(s) = \mathfrak{L}\{tg(t)\} = -G'(s), \quad \text{where } g(t) = \cosh(3t)$$

So, we then have,

$$G(s) = \frac{s}{s^2-9} \qquad\qquad G'(s) = -\frac{s^2+9}{(s^2-9)^2}$$

Using #30 we then have,

$$F(s) = \frac{s^2+9}{(s^2-9)^2}$$

[Return to Problems]

(b) $h(t) = t^2 \sin(2t)$

This part will also use #30 in the table. In fact we could use #30 in one of two ways. We could use it with $n=1$.

$$H(s) = \mathfrak{L}\{tf(t)\} = -F'(s), \qquad \text{where } f(t) = t\sin(2t)$$

Or we could use it with $n=2$.

$$H(s) = \mathfrak{L}\{t^2 f(t)\} = F''(s), \qquad \text{where } f(t) = \sin(2t)$$

Since it's less work to do one derivative, let's do it the first way. So using #9 we have,

$$F(s) = \frac{4s}{(s^2+4)^2} \qquad\qquad F'(s) = -\frac{12s^2-16}{(s^2+4)^3}$$

The transform is then,

$$H(s) = \frac{12s^2-16}{(s^2+4)^3}$$

[Return to Problems]

(c) $g(t) = t^{\frac{3}{2}}$

This part can be done using either #6 (with $n=2$) or #32 (along with #5). We will use #32 so we can see an example of this. In order to use #32 we'll need to notice that

$$\int_0^t \sqrt{v}\, dv = \frac{2}{3}t^{\frac{3}{2}} \qquad \Rightarrow \qquad t^{\frac{3}{2}} = \frac{3}{2}\int_0^t \sqrt{v}\, dv$$

Now, using #5,

$$f(t) = \sqrt{t} \qquad\qquad F(s) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

we get the following.

$$G(s) = \frac{3}{2}\left(\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\right)\left(\frac{1}{s}\right) = \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}$$

This is what we would have gotten had we used #6.

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(d) $f(t) = (10t)^{\frac{3}{2}}$

For this part we will use #24 along with the answer from the previous part. To see this note that if

$$g(t) = t^{\frac{3}{2}}$$

then

$$f(t) = g(10t)$$

Therefore, the transform is.

$$\begin{aligned}
F(s) &= \frac{1}{10} G\left(\frac{s}{10}\right) \\
&= \frac{1}{10}\left(\frac{3\sqrt{\pi}}{4\left(\frac{s}{10}\right)^{\frac{5}{2}}}\right) \\
&= 10^{\frac{3}{2}} \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}}
\end{aligned}$$

[Return to Problems]

(e) $f(t) = tg'(t)$

This final part will again use #30 from the table as well as #35.

$$\begin{aligned}
\mathfrak{L}\{tg'(t)\} &= -\frac{d}{ds}\mathfrak{L}\{g'\} \\
&= -\frac{d}{ds}\{sG(s) - g(0)\} \\
&= -\left(G(s) + sG'(s) - 0\right) \\
&= -G(s) - sG'(s)
\end{aligned}$$

Remember that *g(0)* is just a constant so when we differentiate it we will get zero!

[Return to Problems]

As this set of examples has shown us we can't forget to use some of the general formulas in the table to derive new Laplace transforms for functions that aren't explicitly listed in the table!

Inverse Laplace Transforms

Finding the Laplace transform of a function is not terribly difficult if we've got a table of transforms in front of us to use as we saw in the last section. What we would like to do now is go the other way.

We are going to be given a transform, *F(s)*, and ask what function (or functions) did we have originally. As you will see this can be a more complicated and lengthy process than taking transforms. In these cases we say that we are finding the **Inverse Laplace Transform** of *F(s)* and use the following notation.

$$f(t) = \mathfrak{L}^{-1}\left\{F(s)\right\}$$

As with Laplace transforms, we've got the following fact to help us take the inverse transform.

Fact

Given the two Laplace transforms *F(s)* and *G(s)* then

$$\mathfrak{L}^{-1}\left\{aF(s) + bG(s)\right\} = a\mathfrak{L}^{-1}\left\{F(s)\right\} + b\mathfrak{L}^{-1}\left\{G(s)\right\}$$

for any constants *a* and *b*.

So, we take the inverse transform of the individual transforms, put any constants back in and then add or subtract the results back up.

Let's take a look at a couple of fairly simple inverse transforms.

Example 1 Find the inverse transform of each of the following.

(a) $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$ [Solution]

(b) $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$ [Solution]

(c) $F(s) = \frac{6s}{s^2+25} + \frac{3}{s^2+25}$ [Solution]

(d) $G(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49}$ [Solution]

Solution

I've always felt that the key to doing inverse transforms is to look at the denominator and try to identify what you've got based on that. If there is only one entry in the table that has that particular denominator, the next step is to make sure the numerator is correctly set up for the inverse transform process. If it isn't, correct it (this is always easy to do) and then take the inverse transform.

If there is more than one entry in the table has a particular denominator, then the numerators of each will be different, so go up to the numerator and see which one you've got. If you need to correct the numerator to get it into the correct form and then take the inverse transform.

So, with this advice in mind let's see if we can take some inverse transforms.

(a) $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$

From the denominator of the first term it looks like the first term is just a constant. The correct numerator for this term is a "1" so we'll just factor the 6 out before taking the inverse transform. The second term appears to be an exponential with $a = 8$ and the numerator is exactly what it needs to be. The third term also appears to be an exponential, only this time $a = 3$ and we'll need to factor the 4 out before taking the inverse transforms.

So, with a little more detail than we'll usually put into these,

$$\begin{aligned} F(s) &= 6\,\frac{1}{s} - \frac{1}{s-8} + 4\frac{1}{s-3} \\ f(t) &= 6(1) - \mathbf{e}^{8t} + 4\left(\mathbf{e}^{3t}\right) \\ &= 6 - \mathbf{e}^{8t} + 4\mathbf{e}^{3t} \end{aligned}$$

[Return to Problems]

(b) $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$

The first term in this case looks like an exponential with $a = -2$ and we'll need to factor out the 19. Be careful with negative signs in these problems, it's very easy to lose track of them.

The second term almost looks like an exponential, except that it's got a *3s* instead of just an *s* in the denominator. It is an exponential, but in this case we'll need to factor a 3 out of the denominator before taking the inverse transform.

The denominator of the third term appears to be #3 in the table with $n = 4$. The numerator however, is not correct for this. There is currently a 7 in the numerator and we need a 4! = 24 in the numerator. This is very easy to fix. Whenever a numerator is off by a multiplicative constant, as in this case, all we need to do is put the constant that we need in the numerator. We will just need to remember to take it back out by dividing by the same constant.

So, let's first rewrite the transform.

$$\begin{aligned} H(s) &= \frac{19}{s-(-2)} - \frac{1}{3\left(s-\frac{5}{3}\right)} + \frac{7\,\frac{4!}{4!}}{s^{4+1}} \\ &= 19\frac{1}{s-(-2)} - \frac{1}{3}\,\frac{1}{s-\frac{5}{3}} + \frac{7}{4!}\,\frac{4!}{s^{4+1}} \end{aligned}$$

So, what did we do here? We factored the 19 out of the first term. We factored the 3 out of the denominator of the second term since it can't be there for the inverse transform and in the third term we factored everything out of the numerator except the 4! since that is the portion that we need in the numerator for the inverse transform process.

Let's now take the inverse transform.

$$h(t) = 19\mathbf{e}^{-2t} - \frac{1}{3}\mathbf{e}^{\frac{5t}{3}} + \frac{7}{24}t^4$$

[Return to Problems]

(c) $F(s) = \frac{6s}{s^2+25} + \frac{3}{s^2+25}$

In this part we've got the same denominator in both terms and our table tells us that we've either got #7 or #8. The numerators will tell us which we've actually got. The first one has an *s* in the numerator and so this means that the first term must be #8 and we'll need to factor the 6 out of the numerator in this case. The second term has only a constant in the numerator and so this term must be #7, however, in order for this to be exactly #7 we'll need to multiply/divide a 5 in the numerator to get it correct for the table.

The transform becomes,

$$\begin{aligned} F(s) &= 6\frac{s}{s^2+(5)^2} + \frac{3\frac{5}{5}}{s^2+(5)^2} \\ &= 6\frac{s}{s^2+(5)^2} + \frac{3}{5}\frac{5}{s^2+(5)^2} \end{aligned}$$

Taking the inverse transform gives,

$$f(t) = 6\cos(5t) + \frac{3}{5}\sin(5t)$$

[Return to Problems]

(d) $G(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49}$

In this case the first term will be a sine once we factor a 3 out of the denominator, while the second term appears to be a hyperbolic sine (#17). Again, be careful with the difference between these two. Both of the terms will also need to have their numerators fixed up. Here is the transform once we're done rewriting it.

$$\begin{aligned} G(s) &= \frac{1}{3}\frac{8}{s^2+4} + \frac{3}{s^2-49} \\ &= \frac{1}{3}\frac{(4)(2)}{s^2+(2)^2} + \frac{3\frac{7}{7}}{s^2-(7)^2} \end{aligned}$$

Notice that in the first term we took advantage of the fact that we could get the 2 in the numerator that we needed by factoring the 8. The inverse transform is then,

$$g(t) = \frac{4}{3}\sin(2t) + \frac{3}{7}\sinh(7t)$$

[Return to Problems]

So, probably the best way to identify the transform is by looking at the denominator. If there is more than one possibility use the numerator to identify the correct one. Fix up the numerator if needed to get it into the form needed for the inverse transform process. Finally, take the inverse transform.

Let's do some slightly harder problems. These are a little more involved than the first set.

Example 2 Find the inverse transform of each of the following.

(a) $F(s) = \dfrac{6s-5}{s^2+7}$ [Solution]

(b) $F(s) = \dfrac{1-3s}{s^2+8s+21}$ [Solution]

(c) $G(s) = \dfrac{3s-2}{2s^2-6s-2}$ [Solution]

(d) $H(s) = \dfrac{s+7}{s^2-3s-10}$ [Solution]

Solution

(a) $F(s) = \dfrac{6s-5}{s^2+7}$

From the denominator of this one it appears that it is either a sine or a cosine. However, the numerator doesn't match up to either of these in the table. A cosine wants just an *s* in the numerator with at most a multiplicative constant, while a sine wants only a constant and no *s* in the numerator.

We've got both in the numerator. This is easy to fix however. We will just split up the transform into two terms and then do inverse transforms.

$$F(s) = \frac{6s}{s^2+7} - \frac{5\frac{\sqrt{7}}{\sqrt{7}}}{s^2+7}$$

$$f(t) = 6\cos\left(\sqrt{7}t\right) - \frac{5}{\sqrt{7}}\sin\left(\sqrt{7}t\right)$$

Do not get too used to always getting the perfect squares in sines and cosines that we saw in the first set of examples. More often than not (at least in my class) they won't be perfect squares!

[Return to Problems]

(b) $F(s) = \dfrac{1-3s}{s^2+8s+21}$

In this case there are no denominators in our table that look like this. We can however make the denominator look like one of the denominators in the table by completing the square on the denominator. So, let's do that first.

$$\begin{aligned} s^2+8s+21 &= s^2+8s+16-16+21 \\ &= s^2+8s+16+5 \\ &= (s+4)^2+5 \end{aligned}$$

Recall that in completing the square you take half the coefficient of the *s*, square this, and then add and subtract the result to the polynomial. After doing this the first three terms should factor as a perfect square.

So, the transform can be written as the following.

$$F(s) = \frac{1-3s}{(s+4)^2+5}$$

Okay, with this rewrite it looks like we've got [#19](#) and/or [#20](#)'s from our table of transforms. However, note that in order for it to be a #19 we want just a constant in the numerator and in order to be a #20 we need an *s – a* in the numerator. We've got neither of these so we'll have to correct the numerator to get it into proper form.

In correcting the numerator always get the *s – a* first. This is the important part. We will also need to be careful of the 3 that sits in front of the *s*. One way to take care of this is to break the term into two pieces, factor the 3 out of the second and then fix up the numerator of this term. This will work, however it will put three terms into our answer and there are really only two terms.

So, we will leave the transform as a single term and correct it as follows,

$$\begin{aligned} F(s) &= \frac{1-3(s+4-4)}{(s+4)^2+5} \\ &= \frac{1-3(s+4)+12}{(s+4)^2+5} \\ &= \frac{-3(s+4)+13}{(s+4)^2+5} \end{aligned}$$

We needed an $s + 4$ in the numerator, so we put that in. We just needed to make sure and take the 4 back out by subtracting it back out. Also, because of the 3 multiplying the *s* we needed to do all this inside a set of parenthesis. Then we partially multiplied the 3 through the second term and combined the constants. With the transform in this form, we can break it up into two transforms each of which are in the tables and so we can do inverse transforms on them,

$$\begin{aligned} F(s) &= -3\frac{s+4}{(s+4)^2+5} + \frac{13\frac{\sqrt{5}}{\sqrt{5}}}{(s+4)^2+5} \\ f(t) &= -3\mathbf{e}^{-4t}\cos\left(\sqrt{5}t\right) + \frac{13}{\sqrt{5}}\mathbf{e}^{-4t}\sin\left(\sqrt{5}t\right) \end{aligned}$$

[Return to Problems]

(c) $G(s) = \dfrac{3s-2}{2s^2-6s-2}$

This one is similar to the last one. We just need to be careful with the completing the square however. The first thing that we should do is factor a 2 out of the denominator, then complete the square. Remember that when completing the square a coefficient of 1 on the s^2 term is needed! So, here's the work for this transform.

$$G(s)=\frac{3s-2}{2\left(s^2-3s-1\right)}$$
$$=\frac{1}{2}\frac{3s-2}{s^2-3s+\frac{9}{4}-\frac{9}{4}-1}$$
$$=\frac{1}{2}\frac{3s-2}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$

So, it looks like we've got #21 and #22 with a corrected numerator. Here's the work for that and the inverse transform.

$$G(s)=\frac{1}{2}\frac{3\left(s-\frac{3}{2}+\frac{3}{2}\right)-2}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$
$$=\frac{1}{2}\frac{3\left(s-\frac{3}{2}\right)+\frac{5}{2}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$
$$=\frac{1}{2}\left(\frac{3\left(s-\frac{3}{2}\right)}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}+\frac{\frac{5}{2}\frac{\sqrt{13}}{\sqrt{13}}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}\right)$$
$$g(t)=\frac{1}{2}\left(3\mathbf{e}^{\frac{3t}{2}}\cosh\left(\frac{\sqrt{13}}{2}t\right)+\frac{5}{\sqrt{13}}\mathbf{e}^{\frac{3t}{2}}\sinh\left(\frac{\sqrt{13}}{2}t\right)\right)$$

In correcting the numerator of the second term, notice that I only put in the square root since we already had the "over 2" part of the fraction that we needed in the numerator.

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(d) $H(s)=\dfrac{s+7}{s^2-3s-10}$

This one appears to be similar to the previous two, but it actually isn't. The denominators in the previous two couldn't be easily factored. In this case the denominator does factor and so we need to deal with it differently. Here is the transform with the factored denominator.

$$H(s)=\frac{s+7}{(s+2)(s-5)}$$

The denominator of this transform seems to suggest that we've got a couple of exponentials, however in order to be exponentials there can only be a single term in the denominator and no s's in the numerator.

To fix this we will need to do partial fractions on this transform. In this case the partial fraction decomposition will be

$$H(s)=\frac{A}{s+2}+\frac{B}{s-5}$$

Don't remember how to do partial fractions? In this example we'll show you one way of getting the values of the constants and after this example we'll review how to get the correct form of the partial fraction decomposition.

Okay, so let's get the constants. There is a method for finding the constants that will always work, however it can lead to more work than is sometimes required. Eventually, we will need that method, however in this case there is an easier way to find the constants.

Regardless of the method used, the first step is to actually add the two terms back up. This gives the following.

$$\frac{s+7}{(s+2)(s-5)} = \frac{A(s-5)+B(s+2)}{(s+2)(s-5)}$$

Now, this needs to be true for any *s* that we should choose to put in. So, since the denominators are the same we just need to get the numerators equal. Therefore, set the numerators equal.

$$s+7 = A(s-5)+B(s+2)$$

Again, this must be true for ANY value of *s* that we want to put in. So, let's take advantage of that. If it must be true for any value of *s* then it must be true for $s=-2$, to pick a value at random. In this case we get,

$$5 = A(-7)+B(0) \qquad \Rightarrow A = -\frac{5}{7}$$

We found *A* by appropriately picking *s*. We can *B* in the same way if we chose $s=5$.

$$12 = A(0)+B(7) \qquad \Rightarrow \qquad B = \frac{12}{7}$$

This will not always work, but when it does it will usually simplify the work considerably.

So, with these constants the transform becomes,

$$H(s) = \frac{-\frac{5}{7}}{s+2} + \frac{\frac{12}{7}}{s-5}$$

We can now easily do the inverse transform to get,

$$h(t) = -\frac{5}{7}\mathbf{e}^{-2t} + \frac{12}{7}\mathbf{e}^{5t}$$

[Return to Problems]

The last part of this example needed partial fractions to get the inverse transform. When we finally get back to differential equations and we start using Laplace transforms to solve them, you will quickly come to understand that partial fractions are a fact of life in these problems. Almost every problem will require partial fractions to one degree or another.

Note that we could have done the last part of this example as we had done the previous two parts. If we had we would have gotten hyperbolic functions. However, recalling the definition of the hyperbolic functions we could have written the result in the form we got from the way we worked our problem. However, most students have a better feel for exponentials than they do for hyperbolic functions and so it's usually best to just use partial fractions and get the answer in terms of exponentials. It may be a little more work, but it will give a nicer (and easier to work with) form of the answer.

Be warned that in my class I've got a rule that if the denominator can be factored with integer coefficients then it must be.

So, let's remind you how to get the correct partial fraction decomposition. The first step is to factor the denominator as much as possible. Then for each term in the denominator we will use the following table to get a term or terms for our partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Notice that the first and third cases are really special cases of the second and fourth cases respectively.

So, let's do a couple more examples to remind you how to do partial fractions.

Example 3 Find the inverse transform of each of the following.

(a) $G(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$ [Solution]

(b) $F(s)=\dfrac{2-5s}{(s-6)(s^2+11)}$ [Solution]

(c) $G(s)=\dfrac{25}{s^3(s^2+4s+5)}$ [Solution]

Solution

(a) $G(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

Here's the partial fraction decomposition for this part.

$$G(s)=\frac{A}{s+3}+\frac{B}{s-4}+\frac{C}{5s-1}$$

Now, this time we won't go into quite the detail as we did in the last example. We are after the numerator of the partial fraction decomposition and this is usually easy enough to do in our heads. Therefore, we will go straight to setting numerators equal.

$$86s-78=A(s-4)(5s-1)+B(s+3)(5s-1)+C(s+3)(s-4)$$

As with the last example, we can easily get the constants by correctly picking values of *s*.

$$\begin{array}{llll} s=-3 & -336=A(-7)(-16) & \Rightarrow & A=-3 \\ s=\frac{1}{5} & -\frac{304}{5}=C\left(\frac{16}{5}\right)\left(-\frac{19}{5}\right) & \Rightarrow & C=5 \\ s=4 & 266=B(7)(19) & \Rightarrow & B=2 \end{array}$$

So, the partial fraction decomposition for this transform is,

$$G(s)=-\frac{3}{s+3}+\frac{2}{s-4}+\frac{5}{5s-1}$$

Now, in order to actually take the inverse transform we will need to factor a 5 out of the denominator of the last term. The corrected transform as well as its inverse transform is.

$$G(s)=-\frac{3}{s+3}+\frac{2}{s-4}+\frac{1}{s-\frac{1}{5}}$$

$$g(t)=-3\mathbf{e}^{-3t}+2\mathbf{e}^{4t}+\mathbf{e}^{\frac{t}{5}}$$

[Return to Problems]

(b) $F(s)=\dfrac{2-5s}{(s-6)(s^2+11)}$

So, for the first time we've got a quadratic in the denominator. Here's the decomposition for this part.

$$F(s)=\frac{A}{s-6}+\frac{Bs+C}{s^2+11}$$

Setting numerators equal gives,

$$2-5s=A(s^2+11)+(Bs+C)(s-6)$$

Okay, in this case we could use $s=6$ to quickly find *A*, but that's all it would give. In this case we will need to go the "long" way around to getting the constants. Note that this way will always work, but is sometimes more work than is required.

The "long" way is to completely multiply out the right side and collect like terms.

$$\begin{aligned} 2-5s&=A(s^2+11)+(Bs+C)(s-6) \\ &=As^2+11A+Bs^2-6B+Cs-6C \\ &=(A+B)s^2+(-6B+C)s+11A-6C \end{aligned}$$

In order for these two to be equal the coefficients of the s^2, *s* and the constants must all be equal.

So, setting coefficients equal gives the following system of equations that can be solved.

$$\left.\begin{array}{l} s^2: \quad A+B=0 \\ s^1: -6B+C=-5 \\ s^0: 11A-6C=2 \end{array}\right\} \quad \Rightarrow \quad A=-\frac{28}{47}, \quad B=\frac{28}{47}, \quad C=-\frac{67}{47}$$

Notice that I used s^0 to denote the constants. This is habit on my part and isn't really required, it's just what I'm used to doing. Also, the coefficients are fairly messy fractions in this case. Get used to that. They will often be like this when we get back into solving differential equations.

There is a way to make our life a little easier as well with this. Since all of the fractions have a denominator of 47 we'll factor that out as we plug them back into the decomposition. This will make dealing with them much easier. The partial fraction decomposition is then,

$$\begin{aligned} F(s) &= \frac{1}{47}\left(-\frac{28}{s-6}+\frac{28s-67}{s^2+11}\right) \\ &= \frac{1}{47}\left(-\frac{28}{s-6}+\frac{28s}{s^2+11}-\frac{67\frac{\sqrt{11}}{\sqrt{11}}}{s^2+11}\right) \end{aligned}$$

The inverse transform is then.

$$f(t)=\frac{1}{47}\left(-28\mathbf{e}^{6t}+28\cos\left(\sqrt{11}t\right)-\frac{67}{\sqrt{11}}\sin\left(\sqrt{11}t\right)\right)$$

[Return to Problems]

(c) $G(s)=\dfrac{25}{s^3\left(s^2+4s+5\right)}$

With this last part do not get excited about the s^3. We can think of this term as

$$s^3=(s-0)^3$$

and it becomes a linear term to a power. So, the partial fraction decomposition is

$$G(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+4s+5}$$

Setting numerators equal and multiplying out gives.

$$\begin{aligned} 25 &= As^2\left(s^2+4s+5\right)+Bs\left(s^2+4s+5\right)+C\left(s^2+4s+5\right)+(Ds+E)s^3 \\ &= (A+D)s^4+(4A+B+E)s^3+(5A+4B+C)s^2+(5B+4C)s+5C \end{aligned}$$

Setting coefficients equal gives the following system.

$$\left.\begin{array}{rr} s^4: & A+D=0 \\ s^3: & 4A+B+E=0 \\ s^2: & 5A+4B+C=0 \\ s^1: & 5B+4C=0 \\ s^0: & 5C=25 \end{array}\right\} \quad \Rightarrow \quad A=\frac{11}{5}, B=-4, C=5, D=-\frac{11}{5}, E=-\frac{24}{5}$$

This system looks messy, but it's easier to solve than it might look. First we get *C* for free from the last equation. We can then use the fourth equation to find *B*. The third equation will then give *A*, *etc*.

When plugging into the decomposition we'll get everything with a denominator of 5, then factor that out as we did in the previous part in order to make things easier to deal with.

$$G(s)=\frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25}{s^3}-\frac{11s+24}{s^2+4s+5}\right)$$

Note that we also factored a minus sign out of the last two terms. To complete this part we'll need to complete the square on the later term and fix up a couple of numerators. Here's that work.

$$\begin{aligned} G(s) &= \frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25}{s^3}-\frac{11s+24}{s^2+4s+5}\right) \\ &= \frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25}{s^3}-\frac{11(s+2-2)+24}{(s+2)^2+1}\right) \\ &= \frac{1}{5}\left(\frac{11}{s}-\frac{20}{s^2}+\frac{25\frac{2!}{2!}}{s^3}-\frac{11(s+2)}{(s+2)^2+1}-\frac{2}{(s+2)^2+1}\right) \end{aligned}$$

The inverse transform is then.

$$g(t)=\frac{1}{5}\left(11-20t+\frac{25}{2}t^2-11\mathbf{e}^{-2t}\cos(t)-2\mathbf{e}^{-2t}\sin(t)\right)$$

[Return to Problems]

So, one final time. Partial fractions are a fact of life when using Laplace transforms to solve differential equations. Make sure that you can deal with them.

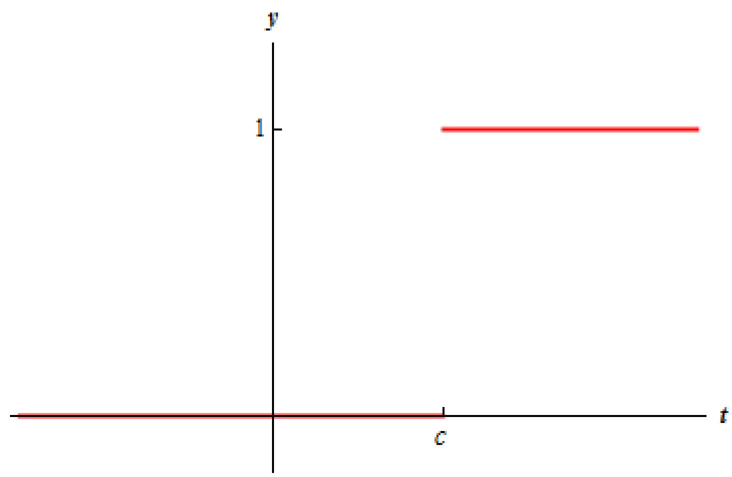
Step Functions

Before proceeding into solving differential equations we should take a look at one more function. Without Laplace transforms it would be much more difficult to solve differential equations that involve this function in *g(t)*.

The function is the Heaviside function and is defined as,

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

Here is a graph of the Heaviside function.



Heaviside functions are often called step functions. Here is some alternate notation for Heaviside functions.

$$u_c(t) = u(t-c) = H(t-c)$$

We can think of the Heaviside function as a switch that is off until $t = c$ at which point it turns on and takes a value of 1. So what if we want a switch that will turn on and takes some other value, say 4, or -7?

Heaviside functions can only take values of 0 or 1, but we can use them to get other kinds of switches. For instance $4u_c(t)$ is a switch that is off until $t = c$ and then turns on and takes a value of 4. Likewise, $-7u_c(t)$ will be a switch that will take a value of -7 when it turns on.

Now, suppose that we want a switch that is on (with a value of 1) and then turns off at $t = c$. We can use Heaviside functions to represent this as well. The following function will exhibit this kind of behavior.

$$1 - u_c(t) = \begin{cases} 1 - 0 = 1 & \text{if } t < c \\ 1 - 1 = 0 & \text{if } t \geq c \end{cases}$$

Prior to $t = c$ the Heaviside is off and so has a value of zero. The function as whole then for $t < c$ has a value of 1. When we hit $t = c$ the Heaviside function will turn on and the function will now take a value of 0.

We can also modify this so that it has values other than 1 when it is on. For instance,

$$3 - 3u_c(t)$$

will be a switch that has a value of 3 until it turns off at $t = c$.

We can also use Heaviside functions to represent much more complicated switches.

Example 1 Write the following function (or switch) in terms of Heaviside functions.

$$f(t) = \begin{cases} -4 & \text{if } t < 6 \\ 25 & \text{if } 6 \le t < 8 \\ 16 & \text{if } 8 \le t < 30 \\ 10 & \text{if } t \ge 30 \end{cases}$$

Solution

There are three sudden shifts in this function and so (hopefully) it's clear that we're going to need three Heaviside functions here, one for each shift in the function. Here's the function in terms of Heaviside functions.

$$f(t) = -4 + 29u_6(t) - 9u_8(t) - 6u_{30}(t)$$

It's fairly easy to verify this.

In the first interval, $t < 6$ all three Heaviside functions are off and the function has the value

$$f(t) = -4$$

Notice that when we **know** that Heaviside functions are on or off we tend to not write them at all as we did in this case.

In the next interval, $6 \le t < 8$ the first Heaviside function is now on while the remaining two are still off. So, in this case the function has the value.

$$f(t) = -4 + 29 = 25$$

In the third interval, $8 \le t < 30$ the first two Heaviside functions are one while the last remains off. Here the function has the value.

$$f(t) = -4 + 29 - 9 = 16$$

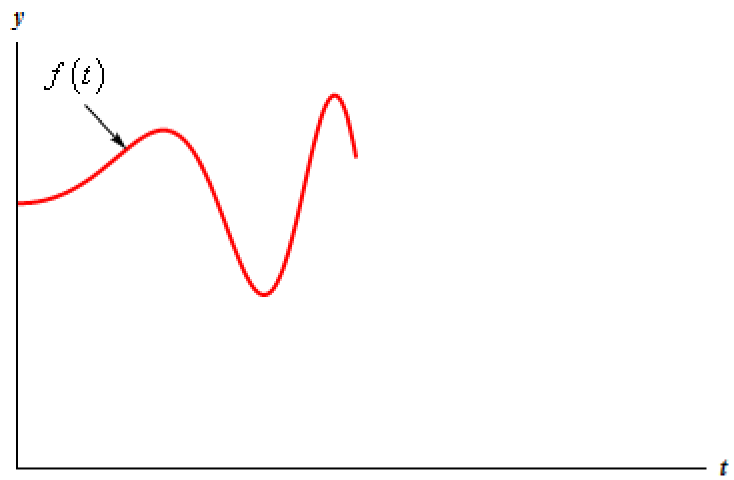
In the last interval, $t \ge 30$ all three Heaviside function are one and the function has the value.

$$f(t) = -4 + 29 - 9 - 6 = 10$$

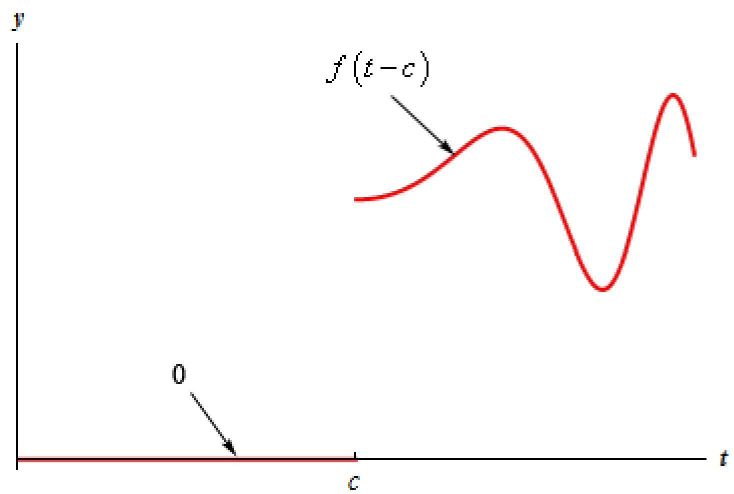
So, the function has the correct value in all the intervals.

All of this is fine, but if we continue the idea of using Heaviside function to represent switches, we really need to acknowledge that most switches will not turn on and take constant values. Most switches will turn on and vary continually with the value of *t*.

So, let's consider the following function.



We would like a switch that is off until $t = c$ and then turns on and takes the values above. By this we mean that when $t = c$ we want the switch to turn on and take the value of *f(0)* and when $t = c + 4$ we want the switch to turn on and take the value of *f(4), etc.* In other words, we want the switch to look like the following,



Notice that in order to take the values that we want the switch to take it needs to turn on and take the values of $f(t-c)$! We can use Heaviside functions to help us represent this switch as well. Using Heaviside functions this switch can be written as

$$g(t) = u_c(t)\, f(t-c) \tag{1}$$

Okay, we've talked a lot about Heaviside functions to this point, but we haven't even touched on Laplace transforms yet. So, let's start thinking about that. Let's determine the Laplace transform of (1). This is actually easy enough to derive so let's do that. Plugging (1) into the definition of the Laplace transform gives,

$$\begin{aligned} \mathfrak{L}\{u_c(t)\, f(t-c)\} &= \int_0^{\infty} \mathbf{e}^{-st} u_c(t)\, f(t-c)\, dt \\ &= \int_c^{\infty} \mathbf{e}^{-st} f(t-c)\, dt \end{aligned}$$

Notice that we took advantage of the fact that the Heaviside function will be zero if $t < c$ and 1 otherwise. This means that we can drop the Heaviside function and start the integral at c instead of 0. Now use the substitution $u = t - c$ and the integral becomes,

$$\begin{aligned}\mathfrak{L}\{u_c(t)\, f(t-c)\} &= \int_0^\infty \mathbf{e}^{-s(u+c)} f(u)\,du \\ &= \int_0^\infty \mathbf{e}^{-su}\mathbf{e}^{-cs} f(u)\,du\end{aligned}$$

The second exponential has no u's in it and so it can be factored out of the integral. Note as well that in the substitution process the lower limit of integration went back to 0.

$$\mathfrak{L}\{u_c(t)\, f(t-c)\} = \mathbf{e}^{-cs}\int_0^\infty \mathbf{e}^{-su} f(u)\,du$$

Now, the integral left is nothing more than the integral that we would need to compute if we were going to find the Laplace transform of *f(t)*. Therefore, we get the following formula

$$\mathfrak{L}\{u_c(t)\, f(t-c)\} = \mathbf{e}^{-cs}F(s) \qquad \textbf{(2)}$$

In order to use (2) the function *f(t)* must be shifted by *c*, the same value that is used in the Heaviside function. Also note that we only take the transform of *f(t)* and not *f(t-c)*! We can also turn this around to get a useful formula for inverse Laplace transforms.

$$\mathfrak{L}^{-1}\{\mathbf{e}^{-cs}F(s)\} = u_c(t)\, f(t-c) \qquad \textbf{(3)}$$

We can use (2) to get the Laplace transform of a Heaviside function by itself. To do this we will consider the function in (2) to be *f(t) = 1*. Doing this gives us

$$\mathfrak{L}\{u_c(t)\} = \mathfrak{L}\{u_c(t)\bullet 1\} = \mathbf{e}^{-cs}\mathfrak{L}\{1\} = \frac{1}{s}\mathbf{e}^{-cs} = \frac{\mathbf{e}^{-cs}}{s}$$

Putting all of this together leads to the following two formulas.

$$\mathfrak{L}\{u_c(t)\} = \frac{\mathbf{e}^{-cs}}{s} \qquad\qquad \mathfrak{L}^{-1}\left\{\frac{\mathbf{e}^{-cs}}{s}\right\} = u_c(t) \qquad \textbf{(4)}$$

Let's do some examples.

Example 2 Find the Laplace transform of each of the following.

(a) $g(t) = 10u_{12}(t) + 2(t-6)^3 u_6(t) - (7 - \mathbf{e}^{12-3t})u_4(t)$ [Solution]

(b) $f(t) = -t^2 u_3(t) + \cos(t) u_5(t)$ [Solution]

(c) $h(t) = \begin{cases} t^4 & \text{if } t < 5 \\ t^4 + 3\sin\left(\dfrac{t}{10} - \dfrac{1}{2}\right) & \text{if } t \ge 5 \end{cases}$ [Solution]

(d) $f(t) = \begin{cases} t & \text{if } t < 6 \\ -8 + (t-6)^2 & \text{if } t \ge 6 \end{cases}$ [Solution]

Solution

In all of these problems remember that the function **MUST** be in the form

$$u_c(t)\, f(t-c)$$

before we start taking transforms. If it isn't in that form we will have to put it into that form!

(a) $g(t) = 10u_{12}(t) + 2(t-6)^3 u_6(t) - \left(7 - \mathbf{e}^{12-3t}\right)u_4(t)$

So there are three terms in this function. The first is simply a Heaviside function and so we can use (4) on this term. The second and third terms however have functions with them and we need to identify the functions that are shifted for each of these. In the second term it appears that we are using the following function,

$$f(t) = 2t^3 \qquad \Rightarrow \qquad f(t-6) = 2(t-6)^3$$

and this has been shifted by the correct amount.

The third term uses the following function,

$$f(t) = 7 - \mathbf{e}^{-3t} \qquad \Rightarrow \qquad f(t-4) = 7 - \mathbf{e}^{-3(t-4)} = 7 - \mathbf{e}^{12-3t}$$

which has also been shifted by the correct amount.

With these functions identified we can now take the transform of the function.

$$\begin{aligned} G(s) &= \frac{10\mathbf{e}^{-12s}}{s} + \mathbf{e}^{-6s}\frac{2(3!)}{s^{3+1}} - \left(\frac{7}{s} - \frac{1}{s+3}\right)\mathbf{e}^{-4s} \\ &= \frac{10\mathbf{e}^{-12s}}{s} + \frac{12\mathbf{e}^{-6s}}{s^{3+1}} - \left(\frac{7}{s} - \frac{1}{s+3}\right)\mathbf{e}^{-4s} \end{aligned}$$

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(b) $f(t) = -t^2 u_3(t) + \cos(t)\, u_5(t)$

This part is going to cause some problems. There are two terms and neither has been shifted by the proper amount. The first term needs to be shifted by 3 and the second needs to be shifted by 5. So, since they haven't been shifted, we will need to force the issue. We will need to add in the shifts, and then take them back out of course. Here they are.

$$f(t) = -(t-3+3)^2 u_3(t) + \cos(t-5+5)\, u_5(t)$$

Now we still have some potential problems here. The first function is still not really shifted correctly, so we'll need to use

$$(a+b)^2 = a^2 + 2ab + b^2$$

to get this shifted correctly.

The second term can be dealt with in one of two ways. The first would be to use the formula

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

to break it up into cosines and sines with arguments of t-5 which will be shifted as we expect. There is an easier way to do this one however. From our table of Laplace transforms we have #16 and using that we can see that if

$$g(t) = \cos(t+5) \qquad \Rightarrow \qquad g(t-5) = \cos(t-5+5)$$

This will make our life a little easier so we'll do it this way.

Now, breaking up the first term and leaving the second term alone gives us,

$$f(t) = -\left((t-3)^2 + 6(t-3) + 9\right)u_3(t) + \cos(t-5+5)u_5(t)$$

Okay, so it looks like the two functions that have been shifted here are

$$g(t) = t^2 + 6t + 9$$
$$g(t) = \cos(t+5)$$

Taking the transform then gives,

$$F(s) = -\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)\mathbf{e}^{-3s} + \left(\frac{s\cos(5) - \sin(5)}{s^2+1}\right)\mathbf{e}^{-5s}$$

It's messy, especially the second term, but there it is. Also, do not get excited about the $\cos(5)$ and $\sin(5)$. They are just numbers.

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(c) $h(t) = \begin{cases} t^4 & \text{if } t < 5 \\ t^4 + 3\sin\left(\frac{t}{10} - \frac{1}{2}\right) & \text{if } t \geq 5 \end{cases}$

This one isn't as bad as it might look on the surface. The first thing that we need to do is write it in terms of Heaviside functions.

$$\begin{aligned} h(t) &= t^4 + 3u_5(t)\sin\left(\frac{t}{10} - \frac{1}{2}\right) \\ &= t^4 + 3u_5(t)\sin\left(\frac{1}{10}(t-5)\right) \end{aligned}$$

Since the t^4 is in both terms there isn't anything to do when we add in the Heaviside function. The only thing that gets added in is the sine term. Notice as well that the sine has been shifted by the proper amount.

All we need to do now is to take the transform.

$$\begin{aligned} H(s) &= \frac{4!}{s^5} + \frac{3\left(\frac{1}{10}\right)\mathbf{e}^{-5s}}{s^2 + \left(\frac{1}{10}\right)^2} \\ &= \frac{24}{s^5} + \frac{\frac{3}{10}\mathbf{e}^{-5s}}{s^2 + \frac{1}{100}} \end{aligned}$$

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(d) $f(t) = \begin{cases} t & \text{if } t < 6 \\ -8 + (t-6)^2 & \text{if } t \geq 6 \end{cases}$

Again, the first thing that we need to do is write the function in terms of Heaviside functions.

$$f(t) = t + \left(-8 - t + (t-6)^2\right) u_6(t)$$

We had to add in a "-8" in the second term since that appears in the second part and we also had to subtract a t in the second term since the t in the first portion is no longer there. This subtraction of the t adds a problem because the second function is no longer correctly shifted. This is easier to fix than the previous example however.

Here is the corrected function.

$$\begin{aligned} f(t) &= t + \left(-8 - (t-6+6) + (t-6)^2\right) u_6(t) \\ &= t + \left(-8 - (t-6) - 6 + (t-6)^2\right) u_6(t) \\ &= t + \left(-14 - (t-6) + (t-6)^2\right) u_6(t) \end{aligned}$$

So, in the second term it looks like we are shifting

$$g(t) = t^2 - t - 14$$

The transform is then,

$$F(s) = \frac{1}{s^2} + \left(\frac{2}{s^3} - \frac{1}{s^2} - \frac{14}{s}\right) \mathbf{e}^{-6s}$$

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Without the Heaviside function taking Laplace transforms is not a terribly difficult process provided we have our trusty table of transforms. However, with the advent of Heaviside functions, taking transforms can become a fairly messy process on occasion.

So, let's do some inverse Laplace transforms to see how they are done.

Example 3 Find the inverse Laplace transform of each of the following.

(a) $H(s) = \dfrac{s\mathbf{e}^{-4s}}{(3s+2)(s-2)}$ [Solution]

(b) $G(s) = \dfrac{5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}}{(s+2)(s^2+9)}$ [Solution]

(c) $F(s) = \dfrac{4s + \mathbf{e}^{-s}}{(s-1)(s+2)}$ [Solution]

(d) $G(s) = \dfrac{3s + 8\mathbf{e}^{-20s} - 2s\mathbf{e}^{-3s} + 6\mathbf{e}^{-7s}}{s^2(s+3)}$ [Solution]

Solution
All of these will use (3) somewhere in the process. Notice that in order to use this formula the exponential doesn't really enter into the mix until the very end. The vast majority of the process is finding the inverse transform of the stuff without the exponential.

In these problems we are not going to go into detail on many of the inverse transforms. If you need a refresher on some of the basics of inverse transforms go back and take a look at the previous section.

(a) $H(s) = \dfrac{s\mathbf{e}^{-4s}}{(3s+2)(s-2)}$

In light of the comments above let's first rewrite the transform in the following way.

$$H(s) = \mathbf{e}^{-4s}\frac{s}{(3s+2)(s-2)} = \mathbf{e}^{-4s}F(s)$$

Now, this problem really comes down to needing *f(t)*. So, let's do that. We'll need to partial fraction *F(s)* up. Here's the partial fraction decomposition.

$$F(s) = \frac{A}{3s+2} + \frac{B}{s-2}$$

Setting numerators equal gives,

$$s = A(s-2) + B(3s+2)$$

We'll find the constants here by selecting values of *s*. Doing this gives,

$$s = 2 \qquad 2 = 8B \qquad \Rightarrow \qquad B = \frac{1}{4}$$

$$s = -\frac{2}{3} \qquad -\frac{2}{3} = -\frac{8}{3}A \qquad \Rightarrow \qquad A = \frac{1}{4}$$

So, the partial fraction decomposition becomes,

$$F(s) = \frac{\frac{1}{4}}{3\left(s+\frac{2}{3}\right)} + \frac{\frac{1}{4}}{s-2}$$

Notice that we factored a 3 out of the denominator in order to actually do the inverse transform. The inverse transform of this is then,

$$f(t) = \frac{1}{12}\mathbf{e}^{-\frac{2t}{3}} + \frac{1}{4}\mathbf{e}^{2t}$$

Now, let's go back and do the actual problem. The original transform was,

$$H(s) = \mathbf{e}^{-4s}F(s)$$

Note that we didn't bother to plug in *F(s)*. There really isn't a reason to plug it back in. Let's just use (3) to write down the inverse transform in terms of symbols. The inverse transform is,

$$h(t) = u_4(t)\, f(t-4)$$

where, *f(t)* is,

$$f(t) = \frac{1}{12}\mathbf{e}^{-\frac{2t}{3}} + \frac{1}{4}\mathbf{e}^{2t}$$

This is all the farther that we'll go with the answer. There really isn't any reason to plug in *f(t)* at this point. It would make the function longer and definitely messier. We will give almost all of our answers to these types of inverse transforms in this form.

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(b) $G(s) = \dfrac{5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}}{(s+2)(s^2+9)}$

This problem is not as difficult as it might at first appear to be. Because there are two exponentials we will need to deal with them separately eventually. Now, this might lead us to conclude that the best way to deal with this function is to split it up as follows,

$$G(s) = \mathbf{e}^{-6s}\frac{5}{(s+2)(s^2+9)} - \mathbf{e}^{-11s}\frac{3}{(s+2)(s^2+9)}$$

Notice that we factored out the exponential, as we did in the last example, since we would need to do that eventually anyway. This is where a fairly common complication arises. Many people will call the first function *F(s)* and the second function *H(s)* and the partial fraction both of them.

However, if instead of just factoring out the exponential we would also factor out the coefficient we would get,

$$G(s) = 5\mathbf{e}^{-6s}\frac{1}{(s+2)(s^2+9)} - 3\mathbf{e}^{-11s}\frac{1}{(s+2)(s^2+9)}$$

Upon doing this we can see that the two functions are in fact the same function. The only difference is the constant that was in the numerator. So, the way that we'll do these problems is to first notice that both of the exponentials have only constants as coefficients. Instead of breaking things up then, we will simply factor out the whole numerator and get,

$$G(s) = \left(5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}\right)\frac{1}{(s+2)(s^2+9)} = \left(5\mathbf{e}^{-6s} - 3\mathbf{e}^{-11s}\right)F(s)$$

and now we will just partial fraction *F(s)*.

Here is the partial fraction decomposition.

$$F(s) = \frac{A}{s+2} + \frac{Bs+C}{s^2+9}$$

Setting numerators equal and combining gives us,

$$\begin{aligned} 1 &= A\left(s^2+9\right) + (s+2)(Bs+C) \\ &= (A+B)s^2 + (2B+C)s + 9A + 2C \end{aligned}$$

Setting coefficient equal and solving gives,

$$\left.\begin{aligned} s^2: &\quad A+B=0 \\ s^1: &\quad 2B+C=0 \\ s^0: &\ 9A+2C=1 \end{aligned}\right\} \qquad \Rightarrow \qquad A=\frac{1}{13},\quad B=-\frac{1}{13},\quad C=\frac{2}{13}$$

Substituting back into the transform gives and fixing up the numerators as needed gives,

$$\begin{aligned} F(s) &= \frac{1}{13}\left(\frac{1}{s+2}+\frac{-s+2}{s^2+9}\right) \\ &= \frac{1}{13}\left(\frac{1}{s+2}-\frac{s}{s^2+9}+\frac{2\frac{3}{3}}{s^2+9}\right) \end{aligned}$$

As we did in the previous section we factored out the common denominator to make our work a little simpler. Taking the inverse transform then gives,

$$f(t)=\frac{1}{13}\left(\mathbf{e}^{-2t}-\cos(3t)+\frac{2}{3}\sin(3t)\right)$$

At this point we can go back and start thinking about the original problem.

$$\begin{aligned} G(s) &= \left(5\mathbf{e}^{-6s}-3\mathbf{e}^{-11s}\right)F(s) \\ &= 5\mathbf{e}^{-6s}F(s)-3\mathbf{e}^{-11s}F(s) \end{aligned}$$

We'll also need to distribute the *F(s)* through as well in order to get the correct inverse transform. Recall that in order to use (3) to take the inverse transform you must have a single exponential times a single transform. This means that we must multiply the *F(s)* through the parenthesis. We can now take the inverse transform,

$$g(t)=5u_6(t)\,f(t-6)-3u_{11}(t)\,f(t-11)$$

where,

$$f(t)=\frac{1}{13}\left(\mathbf{e}^{-2t}-\cos(3t)+\frac{2}{3}\sin(3t)\right)$$

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(c) $F(s)=\dfrac{4s+\mathbf{e}^{-s}}{(s-1)(s+2)}$

In this case, unlike the previous part, we will need to break up the transform since one term has a constant in it and the other has an *s*. Note as well that we don't consider the exponential in this, only its coefficient. Breaking up the transform gives,

$$F(s)=\frac{4s}{(s-1)(s+2)}+\mathbf{e}^{-s}\frac{1}{(s-1)(s+2)}=G(s)+\mathbf{e}^{-s}H(s)$$

We will need to partial fraction both of these terms up. We'll start with *G(s)*.

$$G(s)=\frac{A}{s-1}+\frac{B}{s+2}$$

Setting numerators equal gives,

$$4s = A(s+2) + B(s-1)$$

Now, pick values of *s* to find the constants.

$$s = -2 \qquad -8 = -3B \qquad \Rightarrow \qquad B = \frac{8}{3}$$

$$s = 1 \qquad 4 = 3A \qquad \Rightarrow \qquad A = \frac{4}{3}$$

So *G(s)* and its inverse transform is,

$$G(s) = \frac{\frac{4}{3}}{s-1} + \frac{\frac{8}{3}}{s+2}$$

$$g(t) = \frac{4}{3}\mathbf{e}^{t} + \frac{8}{3}\mathbf{e}^{-2t}$$

Now, repeat the process for *H(s)*.

$$H(s) = \frac{A}{s-1} + \frac{B}{s+2}$$

Setting numerators equal gives,

$$1 = A(s+2) + B(s-1)$$

Now, pick values of *s* to find the constants.

$$s = -2 \qquad 1 = -3B \qquad \Rightarrow \qquad B = -\frac{1}{3}$$

$$s = 1 \qquad 1 = 3A \qquad \Rightarrow \qquad A = \frac{1}{3}$$

So *H(s)* and its inverse transform is,

$$H(s) = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2}$$

$$h(t) = \frac{1}{3}\mathbf{e}^{t} - \frac{1}{3}\mathbf{e}^{-2t}$$

Putting all of this together gives the following,

$$F(s) = G(s) + \mathbf{e}^{-s}H(s)$$

$$f(t) = g(t) + u_1(t)h(t-1)$$

where,

$$g(t) = \frac{4}{3}\mathbf{e}^{t} + \frac{8}{3}\mathbf{e}^{-2t} \qquad \text{and} \qquad h(t) = \frac{1}{3}\mathbf{e}^{t} - \frac{1}{3}\mathbf{e}^{-2t}$$

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(d) $G(s) = \dfrac{3s + 8\mathbf{e}^{-20s} - 2s\mathbf{e}^{-3s} + 6\mathbf{e}^{-7s}}{s^2(s+3)}$

This one looks messier than it actually is. Let's first rearrange the numerator a little.

$$G(s) = \frac{s\left(3 - 2\mathbf{e}^{-3s}\right) + \left(8\mathbf{e}^{-20s} + 6\mathbf{e}^{-7s}\right)}{s^2(s+3)}$$

In this form it looks like we can break this up into two pieces that will require partial fractions. When we break these up we should always try and break things up into as few pieces as possible for the partial fractioning. Doing this can save you a great deal of unnecessary work. Breaking up the transform as suggested above gives,

$$\begin{aligned} G(s) &= \left(3 - 2\mathbf{e}^{-3s}\right)\frac{1}{s(s+3)} + \left(8\mathbf{e}^{-20s} + 6\mathbf{e}^{-7s}\right)\frac{1}{s^2(s+3)} \\ &= \left(3 - 2\mathbf{e}^{-3s}\right)F(s) + \left(8\mathbf{e}^{-20s} + 6\mathbf{e}^{-7s}\right)H(s) \end{aligned}$$

Note that we canceled an *s* in *F(s)*. You should always simplify as much a possible before doing the partial fractions.

Let's partial fraction up *F(s)* first.

$$F(s) = \frac{A}{s} + \frac{B}{s+3}$$

Setting numerators equal gives,

$$1 = A(s+3) + Bs$$

Now, pick values of *s* to find the constants.

$$\begin{array}{llll} s = -3 & 1 = -3B & \Rightarrow & B = -\dfrac{1}{3} \\ s = 0 & 1 = 3A & \Rightarrow & A = \dfrac{1}{3} \end{array}$$

So *F(s)* and its inverse transform is,

$$\begin{aligned} F(s) &= \frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}}{s+3} \\ f(t) &= \frac{1}{3} - \frac{1}{3}\mathbf{e}^{-3t} \end{aligned}$$

Now partial fraction *H(s)*.

$$H(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Setting numerators equal gives,

$$1 = As(s+3) + B(s+3) + Cs^2$$

Pick values of s to find the constants.

$$s=-3 \qquad 1=9C \qquad \Rightarrow \qquad C=\frac{1}{9}$$

$$s=0 \qquad 1=3B \qquad \Rightarrow \qquad B=\frac{1}{3}$$

$$s=1 \qquad 1=4A+4B+C=4A+\frac{13}{9} \qquad \Rightarrow \qquad A=-\frac{1}{9}$$

So *H(s)* and its inverse transform is,

$$H(s)=-\frac{\frac{1}{9}}{s}+\frac{\frac{1}{3}}{s^2}+\frac{\frac{1}{9}}{s+3}$$

$$h(t)=-\frac{1}{9}+\frac{1}{3}t+\frac{1}{9}\mathbf{e}^{-3t}$$

Now, let's go back to the original problem, remembering to multiply the transform through the parenthesis.

$$G(s)=3F(s)-2\mathbf{e}^{-3s}F(s)+8\mathbf{e}^{-20s}H(s)+6\mathbf{e}^{-7s}H(s)$$

Taking the inverse transform gives,

$$g(t)=3f(t)-2u_3(t)f(t-3)+8u_{20}(t)h(t-20)+6u_7(t)h(t-7)$$

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So, as this example has shown, these can be a somewhat messy. However, the mess is really only that of notation and amount of work. The actual partial fraction work was identical to the previous sections work. The main difference in this section is we had to do more of it. As far as the inverse transform process goes. Again, the vast majority of that was identical to the previous section as well.

So, don't let the apparent messiness of these problems get you to decide that you can't do them. Generally they aren't as bad as they seem initially.

Solving IVP's with Laplace Transforms

It's now time to get back to differential equations. We've spent the last three sections learning how to take Laplace transforms and how to take inverse Laplace transforms. These are going to be invaluable skills for the next couple of sections so don't forget what we learned there.

Before proceeding into differential equations we will need one more formula. We will need to know how to take the Laplace transform of a derivative. First recall that $f^{(n)}$ denotes the n^{th} derivative of the function f. We now have the following fact.

Fact

Suppose that $f, f', f'',\ldots f^{(n-1)}$ are all continuous functions and $f^{(n)}$ is a piecewise continuous function. Then,

$$\mathfrak{L}\left\{f^{(n)}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

Since we are going to be dealing with second order differential equations it will be convenient to have the Laplace transform of the first two derivatives.

$$\mathfrak{L}\{y'\} = sY(s) - y(0)$$
$$\mathfrak{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0)$$

Notice that the two function evaluations that appear in these formulas, $y(0)$ and $y'(0)$, are often what we've been using for initial condition in out IVP's. So, this means that if we are to use these formulas to solve an IVP we will need initial conditions at $t = 0$.

While Laplace transforms are particularly useful for nonhomogeneous differential equations which have Heaviside functions in the forcing function we'll start off with a couple of fairly simple problems to illustrate how the process works.

Example 1 Solve the following IVP.

$$y'' - 10y' + 9y = 5t, \qquad y(0) = -1 \quad y'(0) = 2$$

Solution

The first step in using Laplace transforms to solve an IVP is to take the transform of every term in the differential equation.

$$\mathfrak{L}\{y''\} - 10\mathfrak{L}\{y'\} + 9\mathfrak{L}\{y\} = \mathfrak{L}\{5t\}$$

Using the appropriate formulas from our table of Laplace transforms gives us the following.

$$s^2 Y(s) - sy(0) - y'(0) - 10\left(sY(s) - y(0)\right) + 9Y(s) = \frac{5}{s^2}$$

Plug in the initial conditions and collect all the terms that have a $Y(s)$ in them.

$$\left(s^2 - 10s + 9\right)Y(s) + s - 12 = \frac{5}{s^2}$$

Solve for $Y(s)$.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

At this point it's convenient to recall just what we're trying to do. We are trying to find the solution, *y(t)*, to an IVP. What we've managed to find at this point is not the solution, but its Laplace transform. So, in order to find the solution all that we need to do is to take the inverse transform.

Before doing that let's notice that in its present form we will have to do partial fractions twice. However, if we combine the two terms up we will only be doing partial fractions once. Not only that, but the denominator for the combined term will be identical to the denominator of the first term. This means that we are going to partial fraction up a term with that denominator no matter what so we might as well make the numerator slightly messier and then just partial fraction once.

This is one of those things where we are apparently making the problem messier, but in the process we are going to save ourselves a fair amount of work!

Combining the two terms gives,

$$Y(s) = \frac{5+12s^2-s^3}{s^2(s-9)(s-1)}$$

The partial fraction decomposition for this transform is,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerators equal gives,

$$5+12s^2-s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Picking appropriate values of s and solving for the constants gives,

$$\begin{array}{llll} s=0 & 5 = 9B & \Rightarrow & B = \frac{5}{9} \\ s=1 & 16 = -8D & \Rightarrow & D = -2 \\ s=9 & 248 = 648C & \Rightarrow & C = \frac{31}{81} \\ s=2 & 45 = -14A + \frac{4345}{81} & \Rightarrow & A = \frac{50}{81} \end{array}$$

Plugging in the constants gives,

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Finally taking the inverse transform gives us the solution to the IVP.

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}\mathbf{e}^{9t} - 2\mathbf{e}^{t}$$

That was a fair amount of work for a problem that probably could have been solved much quicker using the techniques from the previous chapter. The point of this problem however, was to show how we would use Laplace transforms to solve an IVP.

There are a couple of things to note here about using Laplace transforms to solve an IVP. First, using Laplace transforms reduces a differential equation down to an algebra problem. In the case of the last example the algebra was probably more complicated than the straight forward approach from the last chapter. However, in later problems this will be reversed. The algebra, while still very messy, will often be easier than a straight forward approach.

Second, unlike the approach in the last chapter, we did not need to first find a general solution, differentiate this, plug in the initial conditions and then solve for the constants to get the solution. With Laplace transforms, the initial conditions are applied during the first step and at the end we get the actual solution instead of a general solution.

In many of the later problems Laplace transforms will make the problems significantly easier to work than if we had done the straight forward approach of the last chapter. Also, as we will see, there are some differential equations that simply can't be done using the techniques from the last chapter and so, in those cases, Laplace transforms will be our only solution.

Let's take a look at another fairly simple problem.

Example 2 Solve the following IVP.

$$2y'' + 3y' - 2y = t\mathbf{e}^{-2t}, \qquad y(0) = 0 \quad y'(0) = -2$$

Solution

As with the first example, let's first take the Laplace transform of all the terms in the differential equation. We'll the plug in the initial conditions to get,

$$2\left(s^2Y(s) - sy(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) - 2Y(s) = \frac{1}{(s+2)^2}$$

$$\left(2s^2 + 3s - 2\right)Y(s) + 4 = \frac{1}{(s+2)^2}$$

Now solve for *Y(s)*.

$$Y(s) = \frac{1}{(2s-1)(s+2)^3} - \frac{4}{(2s-1)(s+2)}$$

Now, as we did in the last example we'll go ahead and combine the two terms together as we will have to partial fraction up the first denominator anyway, so we may as well make the numerator a little more complex and just do a single partial fraction. This will give,

$$\begin{aligned} Y(s) &= \frac{1 - 4(s+2)^2}{(2s-1)(s+2)^3} \\ &= \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3} \end{aligned}$$

The partial fraction decomposition is then,

$$Y(s) = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

Setting numerator equal gives,

$$\begin{aligned} -4s^2 - 16s - 15 &= A(s+2)^3 + B(2s-1)(s+2)^2 + C(2s-1)(s+2) + D(2s-1) \\ &= (A+2B)s^3 + (6A+7B+2C)s^2 + (12A+4B+3C+2D)s \\ &\qquad + 8A - 4B - 2C - D \end{aligned}$$

In this case it's probably easier to just set coefficients equal and solve the resulting system of equation rather than pick values of *s*. So, here is the system and its solution.

$$\left.\begin{aligned} s^3 &: & A+2B &= 0 \\ s^2 &: & 6A+7B+2C &= -4 \\ s^1 &: & 12A+4B+3C+2D &= -16 \\ s^0 &: & 8A-4B-2C-D &= -15 \end{aligned}\right\} \qquad \Rightarrow \qquad \begin{aligned} A &= -\frac{192}{125} & B &= \frac{96}{125} \\ C &= -\frac{2}{25} & D &= -\frac{1}{5} \end{aligned}$$

We will get a common denominator of 125 on all these coefficients and factor that out when we go to plug them back into the transform. Doing this gives,

$$Y(s) = \frac{1}{125}\left(\frac{-192}{2\left(s-\frac{1}{2}\right)} + \frac{96}{s+2} - \frac{10}{(s+2)^2} - \frac{25\frac{2!}{2!}}{(s+2)^3}\right)$$

Notice that we also had to factor a 2 out of the denominator of the first term and fix up the numerator of the last term in order to get them to match up to the correct entries in our table of transforms.

Taking the inverse transform then gives,

$$y(t) = \frac{1}{125}\left(-96\mathbf{e}^{\frac{t}{2}} + 96\mathbf{e}^{-2t} - 10t\mathbf{e}^{-2t} - \frac{25}{2}t^2\mathbf{e}^{-2t}\right)$$

Example 3 Solve the following IVP.

$$y'' - 6y' + 15y = 2\sin(3t), \qquad y(0) = -1 \quad y'(0) = -4$$

Solution

Take the Laplace transform of everything and plug in the initial conditions.

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) - 6\left(sY(s) - y(0)\right) + 15Y(s) &= 2\frac{3}{s^2+9} \\ \left(s^2 - 6s + 15\right)Y(s) + s - 2 &= \frac{6}{s^2+9} \end{aligned}$$

Now solve for *Y(s)* and combine into a single term as we did in the previous two examples.

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{\left(s^2+9\right)\left(s^2-6s+15\right)}$$

Now, do the partial fractions on this. First let's get the partial fraction decomposition.

$$Y(s) = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

Now, setting numerators equal gives,

$$\begin{aligned} -s^3+2s^2-9s+24 &= (As+B)(s^2-6s+15)+(Cs+D)(s^2+9) \\ &= (A+C)s^3+(-6A+B+D)s^2+(15A-6B+9C)s+15B+9D \end{aligned}$$

Setting coefficients equal and solving for the constants gives,

$$\left.\begin{array}{ll} s^3: & A+C=-1 \\ s^2: & -6A+B+D=2 \\ s^1: & 15A-6B+9C=-9 \\ s^0: & 15B+9D=24 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{ll} A=\frac{1}{10} & B=\frac{1}{10} \\ C=-\frac{11}{10} & D=\frac{5}{2} \end{array}$$

Now, plug these into the decomposition, complete the square on the denominator of the second term and then fix up the numerators for the inverse transform process.

$$\begin{aligned} Y(s) &= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right) \\ &= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s-3+3)+25}{(s-3)^2+6}\right) \\ &= \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{1\frac{3}{3}}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right) \end{aligned}$$

Finally, take the inverse transform.

$$y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11\mathbf{e}^{3t}\cos\left(\sqrt{6}t\right) - \frac{8}{\sqrt{6}}\mathbf{e}^{3t}\sin\left(\sqrt{6}t\right)\right)$$

To this point we've only looked at IVP's in which the initial values were at $t = 0$. This is because we need the initial values to be at this point in order to take the Laplace transform of the derivatives. The problem with all of this is that there are IVP's out there in the world that have initial values at places other than $t = 0$. Laplace transforms would not be as useful as it is if we couldn't use it on these types of IVP's. So, we need to take a look at an example in which the initial conditions are not at $t = 0$ in order to see how to handle these kinds of problems.

Example 4 Solve the following IVP.

$$y'' + 4y' = \cos(t-3) + 4t, \qquad y(3)=0 \quad y'(3)=7$$

Solution

The first thing that we will need to do here is to take care of the fact that initial conditions are not at $t = 0$. The only way that we can take the Laplace transform of the derivatives is to have the initial conditions at $t = 0$.

This means that we will need to formulate the IVP in such a way that the initial conditions are at t

= 0. This is actually fairly simple to do, however we will need to do a change of variable to make it work. We are going to define

$$\eta = t - 3 \qquad \Rightarrow \qquad t = \eta + 3$$

Let's start with the original differential equation.

$$y''(t) + 4y'(t) = \cos(t-3) + 4t$$

Notice that we put in the *(t)* part on the derivatives to make sure that we get things correct here. We will next substitute in for *t*.

$$y''(\eta+3) + 4y'(\eta+3) = \cos(\eta) + 4(\eta+3)$$

Now, to simplify life a little let's define,

$$u(\eta) = y(\eta+3)$$

Then, by the chain rule, we get the following for the first derivative.

$$u'(\eta) = \frac{du}{d\eta} = \frac{dy}{dt}\frac{dt}{d\eta} = y'(\eta+3)$$

By a similar argument we get the following for the second derivative.

$$u''(\eta) = y''(\eta+3)$$

The initial conditions for $u(\eta)$ are,

$$u(0) = y(0+3) = y(3) = 0$$
$$u'(0) = y'(0+3) = y'(3) = 7$$

The IVP under these new variables is then,

$$u'' + 4u' = \cos(\eta) + 4\eta + 12, \qquad u(0) = 0 \quad u'(0) = 7$$

This is an IVP that we can use Laplace transforms on provided we replace all the *t*'s in our table with η's. So, taking the Laplace transform of this new differential equation and plugging in the new initial conditions gives,

$$s^2U(s) - su(0) - u'(0) + 4\left(sU(s) - u(0)\right) = \frac{s}{s^2+1} + \frac{4}{s^2} + \frac{12}{s}$$
$$\left(s^2+4s\right)U(s) - 7 = \frac{s}{s^2+1} + \frac{4+12s}{s^2}$$

Solving for *U(s)* gives,

$$\left(s^2+4s\right)U(s) = \frac{s}{s^2+1} + \frac{4+12s+7s^2}{s^2}$$
$$U(s) = \frac{1}{(s+4)\left(s^2+1\right)} + \frac{4+12s+7s^2}{s^3(s+4)}$$

Note that unlike the previous examples we did not completely combine all the terms this time. In all the previous examples we did this because the denominator of one of the terms was the common denominator for all the terms. Therefore, upon combining, all we did was make the numerator a little messier, and reduced the number of partial fractions required down from two to one. Note that all the terms in this transform that had only powers of s in the denominator were combined for exactly this reason.

In this transform however, if we combined both of the remaining terms into a single term we would be left with a fairly involved partial fraction problem. Therefore, in this case, it would probably be easier to just do partial fractions twice. We've done several partial fractions problems in this section and many partial fraction problems in the previous couple of sections so we're going to leave the details of the partial fractioning to you to check. Partial fractioning each of the terms in our transform gives us the following.

$$\frac{1}{(s+4)(s^2+1)} = \frac{\frac{1}{17}}{s+4} + \frac{1}{17}\left(\frac{-s+4}{s^2+1}\right)$$

$$\frac{4+12s+7s^2}{s^3(s+4)} = \frac{1}{s^3} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{17}{16}}{s+4}$$

Plugging these into our transform and combining like terms gives us

$$\begin{aligned} U(s) &= \frac{1}{s^3} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{273}{272}}{s+4} + \frac{1}{17}\left(\frac{-s+4}{s^2+1}\right) \\ &= \frac{1\frac{2!}{2!}}{s^3} + \frac{\frac{11}{4}}{s^2} + \frac{\frac{17}{16}}{s} - \frac{\frac{273}{272}}{s+4} + \frac{1}{17}\left(\frac{-s}{s^2+1} + \frac{4}{s^2+1}\right) \end{aligned}$$

Now, taking the inverse transform will give the solution to our new IVP. Don't forget to use η's instead of t's!

$$u(\eta) = \frac{1}{2}\eta^2 + \frac{11}{4}\eta + \frac{17}{16} - \frac{273}{272}\mathbf{e}^{-4\eta} + \frac{1}{17}\left(4\sin(\eta) - \cos(\eta)\right)$$

This is not the solution that we are after of course. We are after *y(t)*. However, we can get this by noticing that

$$y(t) = y(\eta+3) = u(\eta) = u(t-3)$$

So the solution to the original IVP is,

$$\begin{aligned} y(t) &= \frac{1}{2}(t-3)^2 + \frac{11}{4}(t-3) + \frac{17}{16} - \frac{273}{272}\mathbf{e}^{-4(t-3)} + \frac{1}{17}\left(4\sin(t-3) - \cos(t-3)\right) \\ y(t) &= \frac{1}{2}t^2 - \frac{1}{4}t - \frac{43}{16} - \frac{273}{272}\mathbf{e}^{-4(t-3)} + \frac{1}{17}\left(4\sin(t-3) - \cos(t-3)\right) \end{aligned}$$

So, we can now do IVP's that don't have initial conditions that are at $t = 0$. We also saw in the last example that it isn't always the best to combine all the terms into a single partial fraction problem as we have been doing prior to this example.

The examples worked in this section would have been just as easy, if not easier, if we had used techniques from the previous chapter. They were worked here using Laplace transforms to illustrate the technique and method.

Nonconstant Coefficient IVP's

In this section we are going to see how Laplace transforms can be used to solve some differential equations that do not have constant coefficients. This is not always an easy thing to do. However, there are some simple cases that can be done.

To do this we will need a quick fact.

Fact

If *f(t)* is a piecewise continuous function on $\left[0,\infty\right)$ of exponential order then,

$$\lim_{s\to\infty} F\left(s\right) = 0 \tag{1}$$

A function *f(t)* is said to be of exponential order α if there exists positive constants *T* and *M* such that

$$\left| f\left(t\right)\right| \le M\mathbf{e}^{\alpha t} \qquad\qquad \text{for all } t \ge T$$

Put in other words, a function that is of exponential order will grow no faster than

$$M\mathbf{e}^{\alpha t}$$

for some *M* and α and all sufficiently large *t*. One way to check whether a function is of exponential order or not is to compute the following limit.

$$\lim_{t\to\infty}\frac{\left| f\left(t\right)\right|}{\mathbf{e}^{\alpha t}}$$

If this limit is finite for some α then the function will be of exponential order α. Likewise, if the limit is infinite for every α then the function is not of exponential order.

Almost all of the functions that you are liable to deal with in a first course in differential equations are of exponential order. A good example of a function that is not of exponential order is

$$f\left(t\right) = \mathbf{e}^{t^3}$$

We can check this by computing the above limit.

$$\lim_{t\to\infty}\frac{\mathbf{e}^{t^3}}{\mathbf{e}^{\alpha t}} = \lim_{t\to\infty}\mathbf{e}^{t^3-\alpha t} = \lim_{t\to\infty}\mathbf{e}^{t\left(t^2-\alpha\right)} = \infty$$

This is true for any value of α and so the function is not of exponential order.

Do not worry too much about this exponential order stuff. This fact is occasionally needed in using Laplace transforms with non constant coefficients.

So, let's take a look at an example.

Example 1 Solve the following IVP.

$$y'' + 3ty' - 6y = 2, \qquad\qquad y(0) = 0 \quad y'(0) = 0$$

Solution

So, for this one we will need to recall that #30 in our table of Laplace transforms tells us that,

$$\begin{aligned}\mathfrak{L}\{ty'\} &= -\frac{d}{ds}\left(\mathfrak{L}\{y'\}\right)\\ &= -\frac{d}{ds}\left(sY(s) - y(0)\right)\\ &= -sY'(s) - Y(s)\end{aligned}$$

So, upon taking the Laplace transforms of everything and plugging in the initial conditions we get,

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 3\left(-sY'(s) - Y(s)\right) - 6Y(s) &= \frac{2}{s}\\ -3sY'(s) + \left(s^2 - 9\right)Y(s) &= \frac{2}{s}\\ Y'(s) + \left(\frac{3}{s} - \frac{s}{3}\right)Y(s) &= -\frac{2}{3s^2}\end{aligned}$$

Unlike the examples in the previous section where we ended up with a transform for the solution, here we get a linear first order differential equation that must be solved in order to get a transform for the solution.

The integrating factor for this differential equation is,

$$\mu(s) = \mathbf{e}^{\int\left(\frac{3}{s} - \frac{s}{3}\right)ds} = \mathbf{e}^{\ln\left(s^3\right) - \frac{s^2}{6}} = s^3\mathbf{e}^{-\frac{s^2}{6}}$$

Multiplying through, integrating and solving for *Y(s)* gives,

$$\begin{aligned}\int\left(s^3\mathbf{e}^{-\frac{s^2}{6}}Y(s)\right)'ds &= \int -\frac{2}{3}s\mathbf{e}^{-\frac{s^2}{6}}\,ds\\ s^3\mathbf{e}^{-\frac{s^2}{6}}Y(s) &= 2\mathbf{e}^{-\frac{s^2}{6}} + c\\ Y(s) &= \frac{2}{s^3} + c\frac{\mathbf{e}^{\frac{s^2}{6}}}{s^3}\end{aligned}$$

Now, we have a transform for the solution. However that second term looks unlike anything we've seen to this point. This is where the fact about the transforms of exponential order functions comes into play. We are going to assume that whatever our solution is, it is of exponential order. This means that

$$\lim_{s\to\infty}\left(\frac{2}{s^3} + \frac{c\mathbf{e}^{\frac{s^2}{6}}}{s^3}\right) = 0$$

The first term does go to zero in the limit. The second term however, will only go to zero if c = 0. Therefore, we must have $c = 0$ in order for this to be the transform of our solution.

So, the transform of our solution, as well as the solution is,

$$Y(s) = \frac{2}{s^3} \qquad\qquad y(t) = t^2$$

I'll leave it to you to verify that this is in fact a solution if you'd like to.

Now, not all nonconstant differential equations need to use (1). So, let's take a look at one more example.

Example 2 Solve the following IVP.

$$ty'' - ty' + y = 2, \qquad\qquad y(0) = 2 \quad y'(0) = -4$$

Solution

From the first example we have,

$$\mathfrak{L}\{ty'\} = -sY'(s) - Y(s)$$

We'll also need,

$$\begin{aligned}\mathfrak{L}\{ty''\} &= -\frac{d}{ds}\left(\mathfrak{L}\{y''\}\right)\\ &= -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right)\\ &= -s^2Y'(s) - 2sY(s) + y(0)\end{aligned}$$

Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$\begin{aligned}-s^2Y'(s) - 2sY(s) + y(0) - \left(-sY'(s) - Y(s)\right) + Y(s) &= \frac{2}{s}\\ \left(s - s^2\right)Y'(s) + (2 - 2s)Y(s) + 2 &= \frac{2}{s}\\ s(1-s)Y'(s) + 2(1-s)Y(s) &= \frac{2(1-s)}{s}\\ Y'(s) + \frac{2}{s}Y(s) &= \frac{2}{s^2}\end{aligned}$$

Once again we have a linear first order differential equation that we must solve in order to get a transform for the solution. Notice as well that we never used the second initial condition in this work. That is okay, we will use it eventually.

Since this linear differential equation is much easier to solve compared to the first one, we'll leave the details to you. Upon solving the differential equation we get,

$$Y(s) = \frac{2}{s} + \frac{c}{s^2}$$

Now, this transform goes to zero for all values of *c* and we can take the inverse transform of the second term. Therefore, we won't need to use (1) to get rid of the second term as did in the previous example.

Taking the inverse transform gives,

$$y(t) = 2 + ct$$

Now, is where we will use the second initial condition. Upon differentiating and plugging in the second initial condition we can see that $c = -4$.

So, the solution to this IVP is,

$$y(t) = 2 - 4t$$

So, we've seen how to use Laplace transforms to solve some nonconstant coefficient differential equations. Notice however that all we did was add in an occasional *t* to the coefficients. We couldn't get too complicated with the coefficients. If we had we would not have been able to easily use Laplace transforms to solve them.

Sometimes Laplace transforms can be used to solve nonconstant differential equations, however, in general, nonconstant differential equations are still very difficult to solve.

IVP's With Step Functions

In this section we will use Laplace transforms to solve IVP's which contain Heaviside functions in the forcing function. This is where Laplace transform really starts to come into its own as a solution method.

To work these problems we'll just need to remember the following two formulas,

$$\mathfrak{L}\left\{u_c(t)\,f(t-c)\right\} = \mathbf{e}^{-cs}F(s) \qquad \text{where } F(s) = \mathfrak{L}\left\{f(t)\right\}$$

$$\mathfrak{L}^{-1}\left\{\mathbf{e}^{-cs}F(s)\right\} = u_c(t)\,f(t-c) \qquad \text{where } f(t) = \mathfrak{L}^{-1}\left\{F(s)\right\}$$

In other words, we will always need to remember that in order to take the transform of a function that involves a Heaviside we've got to make sure the function has been properly shifted.

Let's work an example.

Example 1 Solve the following IVP.

$$y'' - y' + 5y = 4 + u_2(t)\mathbf{e}^{4-2t}, \qquad y(0) = 2 \quad y'(0) = -1$$

Solution

First let's rewrite the forcing function to make sure that it's being shifted correctly and to identify the function that is actually being shifted.

$$y'' - y' + 5y = 4 + u_2(t)\mathbf{e}^{-2(t-2)}$$

So, it is being shifted correctly and the function that is being shifted is $\mathbf{e}^{-2t}$. Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$s^2Y(s) - sy(0) - y'(0) - \left(sY(s) - y(0)\right) + 5Y(s) = \frac{4}{s} + \frac{\mathbf{e}^{-2s}}{s+2}$$

$$\left(s^2 - s + 5\right)Y(s) - 2s + 3 = \frac{4}{s} + \frac{\mathbf{e}^{-2s}}{s+2}$$

Now solve for *Y(s)*.

$$\left(s^2 - s + 5\right)Y(s) = \frac{4}{s} + \frac{\mathbf{e}^{-2s}}{s+2} + 2s - 3$$

$$\left(s^2 - s + 5\right)Y(s) = \frac{2s^2 - 3s + 4}{s} + \frac{\mathbf{e}^{-2s}}{s+2}$$

$$Y(s) = \frac{2s^2 - 3s + 4}{s\left(s^2 - s + 5\right)} + \mathbf{e}^{-2s}\frac{1}{(s+2)\left(s^2 - s + 5\right)}$$

$$Y(s) = F(s) + \mathbf{e}^{-2s}G(s)$$

Notice that we combined a couple of terms to simplify things a little. Now we need to partial fraction *F(s)* and *G(s)*. We'll leave it to you to check the details of the partial fractions.

$$F\left(s\right)=\frac{2s^2-3s+4}{s\left(s^2-s+5\right)}=\frac{1}{5}\left(\frac{4}{s}+\frac{6s-11}{s^2-s+5}\right)$$

$$G\left(s\right)=\frac{1}{\left(s+2\right)\left(s^2-s+5\right)}=\frac{1}{11}\left(\frac{1}{s+2}-\frac{s-3}{s^2-s+5}\right)$$

We now need to do the inverse transforms on each of these. We'll start with *F(s)*.

$$\begin{aligned}F\left(s\right)&=\frac{1}{5}\left(\frac{4}{s}+\frac{6\left(s-\frac{1}{2}+\frac{1}{2}\right)-11}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)\\&=\frac{1}{5}\left(\frac{4}{s}+\frac{6\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}-\frac{8\frac{\sqrt{19}}{2}\frac{2}{\sqrt{19}}}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)\\f\left(t\right)&=\frac{1}{5}\left(4+6\mathbf{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{19}}{2}t\right)-\frac{16}{\sqrt{19}}\mathbf{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{19}}{2}t\right)\right)\end{aligned}$$

Now *G(s)*.

$$\begin{aligned}G\left(s\right)&=\frac{1}{11}\left(\frac{1}{s+2}-\frac{s-\frac{1}{2}+\frac{1}{2}-3}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)\\&=\frac{1}{11}\left(\frac{1}{s+2}-\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}+\frac{\frac{5}{2}\frac{\sqrt{19}}{\sqrt{19}}}{\left(s-\frac{1}{2}\right)^2+\frac{19}{4}}\right)\\g\left(t\right)&=\frac{1}{11}\left(\mathbf{e}^{-2t}-\mathbf{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{19}}{2}t\right)+\frac{5}{\sqrt{19}}\mathbf{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{19}}{2}t\right)\right)\end{aligned}$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$\begin{aligned}Y\left(s\right)&=F\left(s\right)+\mathbf{e}^{-2s}G\left(s\right)\\y\left(t\right)&=f\left(t\right)+u_2\left(t\right)g\left(t-2\right)\end{aligned}$$

where *f(t)* and *g(t)* are the functions shown above.

There is can be a fair amount of work involved in solving differential equations that involve Heaviside functions.

Let's take a look at another example or two.

Example 2 Solve the following IVP.

$$y'' - y' = \cos(2t) + \cos(2t - 12)u_6(t) \qquad y(0) = -4,\ y'(0) = 0$$

Solution

Let's rewrite the differential equation so we can identify the function that is actually being shifted.

$$y'' - y' = \cos(2t) + \cos\left(2(t-6)\right)u_6(t)$$

So, the function that is being shifted is $\cos(2t)$ and it is being shifted correctly. Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$s^2Y(s) - sy(0) - y'(0) - \left(sY(s) - y(0)\right) = \frac{s}{s^2+4} + \frac{s\mathbf{e}^{-6s}}{s^2+4}$$

$$\left(s^2 - s\right)Y(s) + 4s - 4 = \frac{s}{s^2+4} + \frac{s\mathbf{e}^{-6s}}{s^2+4}$$

Now solve for *Y(s)*.

$$\left(s^2 - s\right)Y(s) = \frac{s + s\mathbf{e}^{-6s}}{s^2+4} - 4s + 4$$

$$Y(s) = \frac{s\left(1+\mathbf{e}^{-6s}\right)}{s(s-1)\left(s^2+4\right)} - 4\frac{s-1}{s(s-1)}$$

$$= \frac{1+\mathbf{e}^{-6s}}{(s-1)\left(s^2+4\right)} - \frac{4}{s}$$

$$Y(s) = \left(1+\mathbf{e}^{-6s}\right)F(s) - \frac{4}{s}$$

Notice that we combined the first two terms to simplify things a little. Also there was some canceling going on in this one. Do not expect that to happen on a regular basis. We now need to partial fraction *F(s)*. We'll leave the details to you to check.

$$F(s) = \frac{1}{(s-1)\left(s^2+4\right)} = \frac{1}{5}\left(\frac{1}{s-1} - \frac{s+1}{s^2+4}\right)$$

$$f(t) = \frac{1}{5}\left(\mathbf{e}^{t} - \cos(2t) - \frac{1}{2}\sin(2t)\right)$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y(s) = F(s) + F(s)\mathbf{e}^{-6s} - \frac{4}{s}$$

$$y(t) = f(t) + u_6(t)\,f(t-6) - 4$$

where *f(t)* is given above.

Example 3 Solve the following IVP.

$$y'' - 5y' - 14y = 9 + u_3(t) + 4(t-1)u_1(t) \qquad y(0) = 0,\ y'(0) = 10$$

Solution

Let's take the Laplace transform of everything and note that in the third term we are shifting 4*t*.

$$s^2Y(s) - sy(0) - y'(0) - 5\left(sY(s) - y(0)\right) - 14Y(s) = \frac{9}{s} + \frac{\mathbf{e}^{-3s}}{s} + 4\frac{\mathbf{e}^{-s}}{s^2}$$

$$\left(s^2 - 5s - 14\right)Y(s) - 10 = \frac{9 + \mathbf{e}^{-3s}}{s} + 4\frac{\mathbf{e}^{-s}}{s^2}$$

Now solve for *Y(s)*.

$$\left(s^2 - 5s - 14\right)Y(s) - 10 = \frac{9 + \mathbf{e}^{-3s}}{s} + 4\frac{\mathbf{e}^{-s}}{s^2}$$

$$Y(s) = \frac{9 + \mathbf{e}^{-3s}}{s(s-7)(s+2)} + \frac{4\mathbf{e}^{-s}}{s^2(s-7)(s+2)} + \frac{10}{(s-7)(s+2)}$$

$$Y(s) = \left(9 + \mathbf{e}^{-3s}\right)F(s) + 4\mathbf{e}^{-s}G(s) + H(s)$$

So, we have three functions that we'll need to partial fraction for this problem. I'll leave it to you to check the details.

$$F(s) = \frac{1}{s(s-7)(s+2)} = -\frac{1}{14}\frac{1}{s} + \frac{1}{63}\frac{1}{s-7} + \frac{1}{18}\frac{1}{s+2}$$

$$f(t) = -\frac{1}{14} + \frac{1}{63}\mathbf{e}^{7t} + \frac{1}{18}\mathbf{e}^{-2t}$$

$$G(s) = \frac{1}{s^2(s-7)(s+2)} = \frac{5}{196}\frac{1}{s} - \frac{1}{14}\frac{1}{s^2} + \frac{1}{441}\frac{1}{s-7} - \frac{1}{36}\frac{1}{s+2}$$

$$g(t) = \frac{5}{196} - \frac{1}{14}t + \frac{1}{441}\mathbf{e}^{7t} - \frac{1}{36}\mathbf{e}^{-2t}$$

$$H(s) = \frac{10}{(s-7)(s+2)} = \frac{10}{9}\frac{1}{s-7} - \frac{10}{9}\frac{1}{s+2}$$

$$h(t) = \frac{10}{9}\mathbf{e}^{7t} - \frac{10}{9}\mathbf{e}^{-2t}$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y(s) = 9F(s) + \mathbf{e}^{-3s}F(s) + 4\mathbf{e}^{-s}G(s) + H(s)$$

$$y(t) = 9f(t) + u_3(t)\,f(t-3) + 4u_1(t)\,g(t-1) + h(t)$$

where *f(t)*, *g(t)* and *h(t)* are given above.

Let's work one more example.

Example 4 Solve the following IVP.

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 0 \quad y'(0) = -2$$

where,

$$g(t) = \begin{cases} 2 & t < 6 \\ t & 6 \le t < 10 \\ 4 & t \ge 10 \end{cases}$$

Solution

The first step is to get *g(t)* written in terms of Heaviside functions so that we can take the transform.

$$g(t) = 2 + (t-2)u_6(t) + (4-t)u_{10}(t)$$

Now, while this is *g(t)* written in terms of Heaviside functions it is not yet in proper form for us to take the transform. Remember that each function must be shifted by a proper amount. So, getting things set up for the proper shifts gives us,

$$g(t) = 2 + (t-6+6-2)u_6(t) + \left(4-(t-10+10)\right)u_{10}(t)$$

$$g(t) = 2 + (t-6+4)u_6(t) + \left(-6-(t-10)\right)u_{10}(t)$$

So, for the first Heaviside it looks like $f(t) = t+4$ is the function that is being shifted and for the second Heaviside it looks like $f(t) = -6-t$ is being shifted.

Now take the Laplace transform of everything and plug in the initial conditions.

$$s^2Y(s) - sy(0) - y'(0) + 3\left(sY(s) - y(0)\right) + 2Y(s) = \frac{2}{s} + \mathbf{e}^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - \mathbf{e}^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

$$\left(s^2+3s+2\right)Y(s) + 2 = \frac{2}{s} + \mathbf{e}^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - \mathbf{e}^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

Solve for *Y(s)*.

$$\left(s^2+3s+2\right)Y(s) = \frac{2}{s} + \mathbf{e}^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right) - \mathbf{e}^{-10s}\left(\frac{1}{s^2} + \frac{6}{s}\right) - 2$$

$$\left(s^2+3s+2\right)Y(s) = \frac{2+4\mathbf{e}^{-6s} - 6\mathbf{e}^{-10s}}{s} + \frac{\mathbf{e}^{-6s} - \mathbf{e}^{-10s}}{s^2} - 2$$

$$Y(s) = \frac{2+4\mathbf{e}^{-6s} - 6\mathbf{e}^{-10s}}{s(s+1)(s+2)} + \frac{\mathbf{e}^{-6s} - \mathbf{e}^{-10s}}{s^2(s+1)(s+2)} - \frac{2}{(s+1)(s+2)}$$

$$Y(s) = \left(2+4\mathbf{e}^{-6s} - 6\mathbf{e}^{-10s}\right)F(s) + \left(\mathbf{e}^{-6s} - \mathbf{e}^{-10s}\right)G(s) - H(s)$$

Now, in the solving process we simplified things into as few terms as possible. Even doing this, it looks like we'll still need to do three partial fractions.

I'll leave the details of the partial fractioning to you to verify. The partial fraction form and inverse transform of each of these are.

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

$$f(t) = \frac{1}{2} - \mathbf{e}^{-t} + \frac{1}{2}\mathbf{e}^{-2t}$$

$$G(s) = \frac{1}{s^2(s+1)(s+2)} = -\frac{\frac{3}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{1}{s+1} - \frac{\frac{1}{4}}{s+2}$$

$$g(t) = -\frac{3}{4} + \frac{1}{2}t + \mathbf{e}^{-t} - \frac{1}{4}\mathbf{e}^{-2t}$$

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

$$h(t) = 2\mathbf{e}^{-t} - 2\mathbf{e}^{-2t}$$

Putting this all back together is going to be a little messy. First rewrite the transform a little to make the inverse transform process possible.

$$Y(s) = 2F(s) + \mathbf{e}^{-6s}\left(4F(s) + G(s)\right) - \mathbf{e}^{-10s}\left(6F(s) + G(s)\right) - H(s)$$

Now, taking the inverse transform of all the pieces gives us the final solution to the IVP.

$$y(t) = 2f(t) - h(t) + u_6(t)\left(4f(t-6) + g(t-6)\right) - u_{10}(t)\left(6f(t-10) + g(t-10)\right)$$

where *f(t)*, *g(t)*, and *h(t)* are defined above.

So, the answer to this example is a little messy to write down, but overall the work here wasn't too terribly bad.

Before proceeding with the next section let's see how we would have had to solve this IVP if we hadn't had Laplace transforms. To solve this IVP we would have had to solve three separate IVP's. One for each portion of *g(t)*. Here is a list of the IVP's that we would have had to solve.

1. $0 < t < 6$

$$y'' + 3y' + 2y = 2, \qquad y(0) = 0 \qquad y'(0) = -2$$

The solution to this IVP, with some work, can be made to look like,

$$y_1(t) = 2f(t) - h(t)$$

2. $6 \le t < 10$

$$y'' + 3y' + 2y = t, \qquad y(6) = y_1(6) \qquad y'(6) = y_1'(6)$$

where, $y_1(t)$ is the solution to the first IVP. The solution to this IVP, with some work, can be made to look like,

$$y_2(t) = 2f(t) - h(t) + 4f(t-6) + g(t-6)$$

3. $t \geq 10$

$$y'' + 3y' + 2y = 4, \qquad\qquad y(10) = y_2(10) \qquad y'(10) = y_2'(10)$$

where, $y_2(t)$ is the solution to the second IVP. The solution to this IVP, with some work, can be made to look like,

$$y_3(t) = 2f(t) - h(t) + 4f(t-6) + g(t-6) - 6f(t-10) - g(t-10)$$

There is a considerable amount of work required to solve all three of these and in each of these the forcing function is not that complicated. Using Laplace transforms saved us a fair amount of work.

Dirac Delta Function

When we first introduced Heaviside functions we noted that we could think of them as switches changing the forcing function, *g(t)*, at specified times. However, Heaviside functions are really not suited to forcing functions that exert a "large" force over a "small" time frame.

Examples of this kind of forcing function would be a hammer striking an object or a short in an electrical system. In both of these cases a large force (or voltage) would be exerted on the system over a very short time frame. The Dirac Delta function is used to deal with these kinds of forcing function.

Dirac Delta Function
There are many ways to actually define the Dirac Delta function. To see some of these definitions visit Wolframs MathWorld. There are three main properties of the Dirac Delta function that we need to be aware of. These are,

1. $\delta(t-a)=0, \quad t \neq a$

2. $\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)\,dt = 1, \qquad \varepsilon > 0$

3. $\int_{a-\varepsilon}^{a+\varepsilon} f(t)\,\delta(t-a)\,dt = f(a), \qquad \varepsilon > 0$

At $t = a$ the Dirac Delta function is sometimes thought of has having an "infinite" value. So, the Dirac Delta function is a function that is zero everywhere except one point and at that point it can be thought of as either undefined or as having an "infinite" value.

Note that the integrals in the second and third property are actually true for any interval containing $t = a$, provided it's not one of the endpoints. The limits given here are needed to prove the properties and so they are also given in the properties. We will however use the fact that they are true provided we are integrating over an interval containing $t = a$.

This is a very strange function. It is zero everywhere except one point and yet the integral of any interval containing that one point has a value of 1. The Dirac Delta function is not a real function as we think of them. It is instead an example of something called a **generalized function** or **distribution**.

Despite the strangeness of this "function" it does a very nice job of modeling sudden shocks or large forces to a system.

Before solving an IVP we will need the transform of the Dirac Delta function. We can use the third property above to get this.

$$\mathfrak{L}\{\delta(t-a)\} = \int_0^\infty \mathbf{e}^{-st}\delta(t-a)\,dt = \mathbf{e}^{-as} \qquad \text{provided } a > 0$$

Note that often the second and third properties are given with limits of infinity and negative infinity, but they are valid for any interval in which $t = a$ is in the interior of the interval.

With this we can now solve an IVP that involves a Dirac Delta function.

Example 1 Solve the following IVP.

$$y'' + 2y' - 15y = 6\delta(t-9), \qquad y(0) = -5 \quad y'(0) = 7$$

Solution

As with all previous problems we'll first take the Laplace transform of everything in the differential equation and apply the initial conditions.

$$s^2 Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) - 15Y(s) = 6\mathbf{e}^{-9s}$$

$$\left(s^2 + 2s - 15\right)Y(s) + 5s + 3 = 6\mathbf{e}^{-9s}$$

Now solve for *Y(s)*.

$$\begin{aligned} Y(s) &= \frac{6\mathbf{e}^{-9s}}{(s+5)(s-3)} - \frac{5s+3}{(s+5)(s-3)} \\ &= 6\mathbf{e}^{-9s} F(s) - G(s) \end{aligned}$$

We'll leave it to you to verify the partial fractions and their inverse transforms are,

$$F(s) = \frac{1}{(s+5)(s-3)} = \frac{\frac{1}{8}}{s-3} - \frac{\frac{1}{8}}{s+5}$$

$$f(t) = \frac{1}{8}\mathbf{e}^{3t} - \frac{1}{8}\mathbf{e}^{-5t}$$

$$G(s) = \frac{5s+3}{(s+5)(s-3)} = \frac{\frac{9}{4}}{s-3} + \frac{\frac{11}{4}}{s+5}$$

$$g(t) = \frac{9}{4}\mathbf{e}^{3t} + \frac{11}{4}\mathbf{e}^{-5t}$$

The solution is then,

$$Y(s) = 6\mathbf{e}^{-9s} F(s) - G(s)$$

$$y(t) = 6u_9(t)\, f(t-9) - g(t)$$

where, *f(t)* and *g(t)* are defined above.

Example 2 Solve the following IVP.

$$2y'' + 10y = 3u_{12}(t) - 5\delta(t-4), \qquad y(0) = -1 \quad y'(0) = -2$$

Solution

Take the Laplace transform of everything in the differential equation and apply the initial conditions.

$$2\left(s^2 Y(s) - sy(0) - y'(0)\right) + 10Y(s) = \frac{3\mathbf{e}^{-12s}}{s} - 5\mathbf{e}^{-4s}$$

$$\left(2s^2 + 10\right)Y(s) + 2s + 4 = \frac{3\mathbf{e}^{-12s}}{s} - 5\mathbf{e}^{-4s}$$

Now solve for *Y(s)*.

$$Y(s) = \frac{3\mathbf{e}^{-12s}}{s(2s^2+10)} - \frac{5\mathbf{e}^{-4s}}{2s^2+10} - \frac{2s+4}{2s^2+10}$$
$$= 3\mathbf{e}^{-12s}F(s) - 5\mathbf{e}^{-4s}G(s) - H(s)$$

We'll need to partial fraction the first function. The remaining two will just need a little work and they'll be ready. I'll leave the details to you to check.

$$F(s) = \frac{1}{s(2s^2+10)} = \frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s}{s^2+5}$$
$$f(t) = \frac{1}{10} - \frac{1}{10}\cos\left(\sqrt{5}\,t\right)$$
$$g(t) = \frac{1}{2\sqrt{5}}\sin\left(\sqrt{5}\,t\right)$$
$$h(t) = \cos\left(\sqrt{5}\,t\right) + \frac{2}{\sqrt{5}}\sin\left(\sqrt{5}\,t\right)$$

The solution is then,

$$Y(s) = 3\mathbf{e}^{-12s}F(s) - 5\mathbf{e}^{-4s}G(s) - H(s)$$
$$y(t) = 3u_{12}(t)\,f(t-12) - 5u_4(t)\,g(t-4) - h(t)$$

where, *f(t)*,*g(t)* and *h(t)* are defined above.

So, with the exception of the new function these work the same way that all the problems that we've seen to this point work. Note as well that the exponential was introduced into the transform by the Dirac Delta function, but once in the transform it doesn't matter where it came from. In other words, when we went to the inverse transforms it came back out as a Heaviside function.

Before proceeding to the next section let's take a quick side trip and note that we can relate the Heaviside function and the Dirac Delta function. Start with the following integral.

$$\int_{-\infty}^{t} \delta(u-a)\,du = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

However, this is precisely the definition of the Heaviside function. So,

$$\int_{-\infty}^{t} \delta(u-a)\,du = u_a(t)$$

Now, recalling the Fundamental Theorem of Calculus, we get,

$$u_a'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-a)\,du\right) = \delta(t-a)$$

So, the derivative of the Heaviside function is the Dirac Delta function.

Convolution Integrals

On occasion we will run across transforms of the form,

$$H(s) = F(s)G(s)$$

that can't be dealt with easily using partial fractions. We would like a way to take the inverse transform of such a transform. We can use a convolution integral to do this.

Convolution Integral

If *f(t)* and *g(t)* are piecewise continuous function on $[0, \infty)$ then the convolution integral of *f(t)* and *g(t)* is,

$$(f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

A nice property of convolution integrals is.

$$(f * g)(t) = (g * f)(t)$$

Or,

$$\int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

The following fact will allow us to take the inverse transforms of a product of transforms.

Fact

$$\mathfrak{L}\{f * g\} = F(s)G(s) \qquad\qquad \mathfrak{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Let's work a quick example to see how this can be used.

Example 1 Use a convolution integral to find the inverse transform of the following transform.

$$H(s) = \frac{1}{\left(s^2 + a^2\right)^2}$$

Solution

First note that we could use #11 from out table to do this one so that will be a nice check against our work here.

Now, since we are going to use a convolution integral here we will need to write it as a product whose terms are easy to find the inverse transforms of. This is easy to do in this case.

$$H(s) = \left(\frac{1}{s^2 + a^2}\right)\left(\frac{1}{s^2 + a^2}\right)$$

So, in this case we have,

$$F(s) = G(s) = \frac{1}{s^2 + a^2} \qquad \Rightarrow \qquad f(t) = g(t) = \frac{1}{a}\sin(at)$$

Using a convolution integral *h(t)* is,

$$\begin{aligned} h(t) &= (f * g)(t) \\ &= \frac{1}{a^2}\int_0^t \sin(at - a\tau)\sin(a\tau)\,d\tau \\ &= \frac{1}{2a^3}\left(\sin(at) - at\cos(at)\right) \end{aligned}$$

This is exactly what we would have gotten by using #11 from the table.

Convolution integrals are very useful in the following kinds of problems.

Example 2 Solve the following IVP

$$4y'' + y = g(t), \qquad y(0) = 3 \qquad y'(0) = -7$$

Solution

First, notice that the forcing function in this case has not been specified. Prior to this section we would not have been able to get a solution to this IVP. With convolution integrals we will be able to get a solution to this kind of IVP. The solution will be in terms of *g(t)* but it will be a solution.

Take the Laplace transform of all the terms and plug in the initial conditions.

$$\begin{aligned} 4\left(s^2Y(s) - sy(0) - y'(0)\right) + Y(s) &= G(s) \\ \left(4s^2 + 1\right)Y(s) - 12s + 28 &= G(s) \end{aligned}$$

Notice here that all we could do for the forcing function was to write down *G(s)* for its transform. Now, solve for *Y(s)*.

$$\begin{aligned} \left(4s^2 + 1\right)Y(s) &= G(s) + 12s - 28 \\ Y(s) &= \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \end{aligned}$$

We factored out a 4 from the denominator in preparation for the inverse transform process. To take inverse transforms we'll need to split up the first term and we'll also rewrite the second term a little.

$$\begin{aligned} Y(s) &= \frac{12s - 28}{4\left(s^2 + \frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2 + \frac{1}{4}\right)} \\ &= \frac{3s}{s^2 + \frac{1}{4}} - \frac{7\frac{2}{2}}{s^2 + \frac{1}{4}} + \frac{1}{4}G(s)\frac{\frac{2}{2}}{s^2 + \frac{1}{4}} \end{aligned}$$

Now, the first two terms are easy to inverse transform. We'll need to use a convolution integral on the last term. The two functions that we will be using are,

$$g(t) \qquad\qquad f(t) = 2\sin\left(\frac{t}{2}\right)$$

We can shift either of the two functions in the convolution integral. We'll shift *g(t)* in our solution. Taking the inverse transform gives us,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{\tau}{2}\right) g(t - \tau)\,d\tau$$

So, once we decide on a *g(t)* all we need to do is to an integral and we'll have the solution.

As this last example has shown, using convolution integrals will allow us to solve IVP's with general forcing functions. This could be very convenient in cases where we have a variety of possible forcing functions and don't which one we're going to use. With a convolution integral all that we need to do in these cases is solve the IVP once then go back and evaluate an integral for each possible *g(t)*. This will save us the work of having to solve the IVP for each and every *g(t)*.

Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	$\mathbf{e}^{at}$	$\dfrac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\dfrac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\dfrac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\dfrac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\dfrac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\dfrac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\dfrac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\dfrac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\dfrac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.

2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{\mathbf{e}^{t} + \mathbf{e}^{-t}}{2} \qquad \sinh(t) = \frac{\mathbf{e}^{t} - \mathbf{e}^{-t}}{2}$$

3. Be careful when using "normal" trig function vs. hyperbolic functions. The only difference in the formulas is the "+ a^2" for the "normal" trig functions becomes a "- a^2" for the hyperbolic functions!

4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty \mathbf{e}^{-x} x^{t-1}\, dx$$

If n is a positive integer then,

$$\Gamma(n+1) = n!$$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Systems of Differential Equations

Introduction

To this point we've only looked as solving single differential equations. However, many "real life" situations are governed by a system of differential equations. Consider the population problems that we looked at back in the modeling section of the first order differential equations chapter. In these problems we looked only at a population of one species, yet the problem also contained some information about predators of the species. We assumed that any predation would be constant in these cases. However, in most cases the level of predation would also be dependent upon the population of the predator. So, to be more realistic we should also have a second differential equation that would give the population of the predators. Also note that the population of the predator would be, in some way, dependent upon the population of the prey as well. In other words, we would need to know something about one population to find the other population. So to find the population of either the prey or the predator we would need to solve a system of at least two differential equations.

The next topic of discussion is then how to solve systems of differential equations. However, before doing this we will first need to do a quick review of Linear Algebra. Much of what we will be doing in this chapter will be dependent upon topics from linear algebra. This review is not intended to completely teach you the subject of linear algebra, as that is a topic for a complete class. The quick review is intended to get you familiar enough with some of the basic topics that you will be able to do the work required once we get around to solving systems of differential equations.

Here is a brief listing of the topics covered in this chapter.

Complex Eigenvalues – Solving systems of differential equations with complex eigenvalues.

Repeated Eigenvalues – Solving systems of differential equations with repeated eigenvalues.

Nonhomogeneous Systems – Solving nonhomogeneous systems of differential equations using undetermined coefficients and variation of parameters.

Laplace Transforms – A very brief look at how Laplace transforms can be used to solve a system of differential equations.

Modeling – In this section we'll take a quick look at some extensions of some of the modeling we did in previous chapters that lead to systems of equations.

Review : Systems of Equations

Because we are going to be working almost exclusively with systems of equations in which the number of unknowns equals the number of equations we will restrict our review to these kinds of systems.

All of what we will be doing here can be easily extended to systems with more unknowns than equations or more equations than unknowns if need be.

Let's start with the following system of *n* equations with the *n* unknowns, $x_1, x_2, \ldots, x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad \textbf{(1)}$$

Note that in the subscripts on the coefficients in this system, a_{ij}, the i corresponds to the equation that the coefficient is in and the j corresponds to the unknown that is multiplied by the coefficient.

To use linear algebra to solve this system we will first write down the **augmented matrix** for this system. An augmented matrix is really just all the coefficients of the system and the numbers for the right side of the system written in matrix form. Here is the augmented matrix for this system.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix}$$

To solve this system we will use elementary row operations (which we'll define these in a bit) to rewrite the augmented matrix in triangular form. The matrix will be in triangular form if all the entries below the main diagonal (the diagonal containing $a_{11}, a_{22}, \ldots, a_{nn}$) are zeroes.

Once this is done we can recall that each row in the augmented matrix corresponds to an equation. We will then convert our new augmented matrix back to equations and at this point solving the system will become very easy.

Before working an example let's first define the elementary row operations. There are three of them.

1. Interchange two rows. This is exactly what it says. We will interchange row *i* with row *j*. The notation that we'll use to denote this operation is : $R_i \leftrightarrow R_j$
2. Multiply row *i* by a constant, *c*. This means that every entry in row *i* will get multiplied by the constant *c*. The notation for this operation is : cR_i
3. Add a multiply of row *i* to row *j*. In our heads we will multiply row *i* by an appropriate constant and then add the results to row *j* and put the new row back into row *j* leaving row *i* in the matrix unchanged. The notation for this operation is : $cR_i + R_j$

It's always a little easier to understand these operations if we see them in action. So, let's solve a couple of systems.

Example 1 Solve the following system of equations.

$$\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 + x_3 &= -1 \end{aligned}$$

Solution

The first step is to write down the augmented matrix for this system. Don't forget that coefficients of terms that aren't present are zero.

$$\begin{pmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{pmatrix}$$

Now, we want the entries below the main diagonal to be zero. The main diagonal has been colored red so we can keep track of it during this first example. For reasons that will be apparent eventually we would prefer to get the main diagonal entries to all be ones as well.

We can get a one in the upper most spot by noticing that if we interchange the first and second row we will get a one in the uppermost spot for free. So let's do that.

$$\begin{pmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{pmatrix}$$

Now we need to get the last two entries (the -2 and 3) in the first column to be zero. We can do this using the third row operation. Note that if we take 2 times the first row and add it to the second row we will get a zero in the second entry in the first column and if we take -3 times the first row to the third row we will get the 3 to be a zero. We can do both of these operations at the same time so let's do that.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} 2R_1 + R_2 \\ -3R_1 + R_3 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{pmatrix}$$

Before proceeding with the next step, let's make sure that you followed what we just did. Let's take a look at the first operation that we performed. This operation says to multiply an entry in row 1 by 2 and add this to the corresponding entry in row 2 then replace the old entry in row 2 with this new entry. The following are the four individual operations that we performed to do this.

$$\begin{aligned} 2(1) + (-2) &= 0 \\ 2(2) + 1 &= 5 \\ 2(3) + (-1) &= 5 \\ 2(13) + 4 &= 30 \end{aligned}$$

Okay, the next step optional, but again is convenient to do. Technically, the 5 in the second column is okay to leave. However, it will make our life easier down the road if it is a 1. We can use the second row operation to take care of this. We can divide the whole row by 5. Doing this gives,

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{pmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{pmatrix}$$

The next step is to then use the third row operation to make the -6 in the second column into a zero.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{pmatrix} \xrightarrow{6R_2+R_3} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

Now, officially we are done, but again it's somewhat convenient to get all ones on the main diagonal so we'll do one last step.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

We can now convert back to equations.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

At this point the solving is quite easy. We get x_3 for free and once we get that we can plug this into the second equation and get x_2. We can then use the first equation to get x_1. Note as well that having 1's along the main diagonal helped somewhat with this process.

The solution to this system of equation is

$$x_1 = -1 \qquad x_2 = 4 \qquad x_3 = 2$$

The process used in this example is called **Gaussian Elimination**. Let's take a look at another example.

Example 2 Solve the following system of equations.

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= -2 \\
-x_1 + x_2 - 2x_3 &= 3 \\
2x_1 - x_2 + 3x_3 &= 1
\end{aligned}
$$

Solution

First write down the augmented matrix.

$$
\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix}
$$

We won't put down as many words in working this example. Here's the work for this augmented matrix.

$$
\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix}
\begin{array}{c} R_1 + R_2 \\ -2R_1 + R_3 \\ \rightarrow \end{array}
\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{pmatrix}
$$

$$
\begin{array}{c} -R_2 \\ \rightarrow \end{array}
\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{pmatrix}
\begin{array}{c} -3R_2 + R_3 \\ \rightarrow \end{array}
\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix}
$$

We won't go any farther in this example. Let's go back to equations to see why.

$$
\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix}
\Rightarrow
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= -2 \\
x_2 - x_3 &= -1 \\
0 &= 8
\end{aligned}
$$

The last equation should cause some concern. There's one of three options here. First, we've somehow managed to prove that 0 equals 8 and we know that's not possible. Second, we've made a mistake, but after going back over our work it doesn't appear that we have made a mistake.

This leaves the third option. When we get something like the third equation that simply doesn't make sense we immediately know that there is no solution. In other words, there is no set of three numbers that will make all three of the equations true at the same time.

Let's work another example. We are going to get the system for this new example by making a very small change to the system from the previous example.

Example 3 Solve the following system of equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

Solution

So, the only difference between this system and the system from the second example is we changed the 1 on the right side of the equal sign in the third equation to a -7.

Now write down the augmented matrix for this system.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & -7 \end{pmatrix}$$

The steps for this problem are identical to the steps for the second problem so we won't write them all down. Upon performing the same steps we arrive at the following matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This time the last equation reduces to

$$0 = 0$$

and unlike the second example this is not a problem. Zero does in fact equal zero!

We could stop here and go back to equations to get a solution and there is a solution in this case. However, if we go one more step and get a zero above the one in the second column as well as below it our life will be a little simpler. Doing this gives,

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\Rightarrow}{2R_2 + R_1} \begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If we now go back to equation we get the following two equations.

$$\begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} x_1 + x_3 &= -4 \\ x_2 - x_3 &= -1 \end{aligned}$$

We have two equations and three unknowns. This means that we can solve for two of the variables in terms of the remaining variable. Since x_3 is in both equations we will solve in terms of that.

$$\begin{aligned} x_1 &= -x_3 - 4 \\ x_2 &= x_3 - 1 \end{aligned}$$

What this solution means is that we can pick the value of x_3 to be anything that we'd like and then find values of x_1 and x_2. In these cases we typically write the solution as follows,

$$
\begin{aligned}
x_1 &= -t-4 \\
x_2 &= t-1 \qquad\qquad t = \text{ any real number} \\
x_3 &= t
\end{aligned}
$$

In this way we get an infinite number of solutions, one for each and every value of *t*.

These three examples lead us to a nice fact about systems of equations.

Fact

Given a system of equations, (1), we will have one of the three possibilities for the number of solutions.

1. No solution.
2. Exactly one solution.
3. Infinitely many solutions.

Before moving on to the next section we need to take a look at one more situation. The system of equations in (1) is called a nonhomogeneous system if at least one of the b_i's is not zero. If however all of the b_i's are zero we call the system homogeneous and the system will be,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned} \tag{2}
$$

Now, notice that in the homogeneous case we are guaranteed to have the following solution.

$$x_1 = x_2 = \cdots = x_n = 0$$

This solution is often called the **trivial solution**.

For homogeneous systems the fact above can be modified to the following.

Fact

Given a homogeneous system of equations, (2), we will have one of the two possibilities for the number of solutions.

1. Exactly one solution, the trivial solution
2. Infinitely many non-zero solutions in addition to the trivial solution.

In the second possibility we can say non-zero solution because if there are going to be infinitely many solutions and we know that one of them is the trivial solution then all the rest must have at least one of the x_i's be non-zero and hence we get a non-zero solution.

Review : Matrices and Vectors

This section is intended to be a catch all for many of the basic concepts that are used occasionally in working with systems of differential equations. There will not be a lot of details in this section, nor will we be working large numbers of examples. Also, in many cases we will not be looking at the general case since we won't need the general cases in our differential equations work.

Let's start with some of the basic notation for matrices. An n x m (this is often called the **size** or **dimension** of the matrix) matrix is a matrix with n rows and m columns and the entry in the i^{th} row and j^{th} column is denoted by a_{ij}. A short hand method of writing a general n x m matrix is the following.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}_{n \text{ x } m} = \left(a_{ij}\right)_{n \text{ x } m}$$

The size or dimension of a matrix is subscripted as shown if required. If it's not required or clear from the problem the subscripted size is often dropped from the matrix.

Special Matrices

There are a few "special" matrices out there that we may use on occasion. The first special matrix is the **square matrix**. A square matrix is any matrix whose size (or dimension) is n x n. In other words it has the same number of rows as columns. In a square matrix the diagonal that starts in the upper left and ends in the lower right is often called the **main diagonal**.

The next two special matrices that we want to look at are the zero matrix and the identity matrix. The **zero matrix**, denoted $0_{n \text{ x } m}$, is a matrix all of whose entries are zeroes. The **identity matrix** is a square n x n matrix, denoted I_n, whose main diagonals are all 1's and all the other elements are zero. Here are the general zero and identity matrices.

$$0_{n \text{ x } m} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n \text{ x } m} \qquad\qquad I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{n \text{ x } n}$$

In matrix arithmetic these two matrices will act in matrix work like zero and one act in the real number system.

The last two special matrices that we'll look at here are the **column matrix** and the **row matrix**. These are matrices that consist of a single column or a single row. In general they are,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \text{ x } 1} \qquad\qquad y = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \end{pmatrix}_{1 \text{ x } m}$$

We will often refer to these as **vectors**.

Arithmetic
We next need to take a look at arithmetic involving matrices. We'll start with **addition** and **subtraction** of two matrices. So, suppose that we have two n x m matrices, A and B. The sum (or difference) of these two matrices is then,

$$A_{n \times m} \pm B_{n \times m} = \left(a_{ij}\right)_{n \times m} \pm \left(b_{ij}\right)_{n \times m} = \left(a_{ij} \pm b_{ij}\right)_{n \times m}$$

The sum or difference of two matrices of the same size is a new matrix of identical size whose entries are the sum or difference of the corresponding entries from the original two matrices. Note that we can't add or subtract entries with different sizes.

Next, let's look at **scalar multiplication**. In scalar multiplication we are going to multiply a matrix A by a constant (sometimes called a scalar) α. In this case we get a new matrix whose entries have all been multiplied by the constant, α.

$$\alpha A_{n \times m} = \alpha\left(a_{ij}\right)_{n \times m} = \left(\alpha\, a_{ij}\right)_{n \times m}$$

Example 1 Given the following two matrices,

$$A = \begin{pmatrix} 3 & -2 \\ -9 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} -4 & 1 \\ 0 & -5 \end{pmatrix}$$

compute A-5B.

Solution
There isn't much to do here other than the work.

$$\begin{aligned} A - 5B &= \begin{pmatrix} 3 & -2 \\ -9 & 1 \end{pmatrix} - 5\begin{pmatrix} -4 & 1 \\ 0 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ -9 & 1 \end{pmatrix} - \begin{pmatrix} -20 & 5 \\ 0 & -25 \end{pmatrix} \\ &= \begin{pmatrix} 23 & -7 \\ -9 & 26 \end{pmatrix} \end{aligned}$$

We first multiplied all the entries of B by 5 then subtracted corresponding entries to get the entries in the new matrix.

The final matrix operation that we'll take a look at is **matrix multiplication**. Here we will start with two matrices, $A_{n \times p}$ and $B_{p \times m}$. Note that A must have the same number of columns as B has rows. If this isn't true then we can't perform the multiplication. If it is true then we can perform the following multiplication.

$$A_{n \times p} B_{p \times m} = \left(c_{ij}\right)_{n \times m}$$

The new matrix will have size n x m and the entry in the i^{th} row and j^{th} column, c_{ij}, is found by multiplying row i of matrix A by column j of matrix B. This doesn't always make sense in words so let's look at an example.

Example 2 Given

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 6 & 1 \end{pmatrix}_{2 \text{ x } 3} \qquad B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -4 & 3 & 1 & 0 \\ 0 & 3 & 0 & -2 \end{pmatrix}_{3 \text{ x } 4}$$

compute *AB*.

Solution

The new matrix will have size 2 x 4. The entry in row 1 and column 1 of the new matrix will be found by multiplying row 1 of *A* by column 1 of *B*. This means that we multiply corresponding entries from the row of *A* and the column of *B* and then add the results up. Here are a couple of the entries computed all the way out.

$$\begin{aligned} c_{11} &= (2)(1) + (-1)(-4) + (0)(0) = 6 \\ c_{13} &= (2)(-1) + (-1)(1) + (0)(0) = -3 \\ c_{24} &= (-3)(2) + (6)(0) + (1)(-2) = -8 \end{aligned}$$

Here's the complete solution.

$$C = \begin{pmatrix} 6 & -3 & -3 & 4 \\ -27 & 21 & 9 & -8 \end{pmatrix}$$

In this last example notice that we could not have done the product *BA* since the number of columns of *B* does not match the number of row of *A*. It is important to note that just because we can compute *AB* doesn't mean that we can compute *BA*. Likewise, even if we can compute both *AB* and *BA* they may or may not be the same matrix.

Determinant

The next topic that we need to take a look at is the **determinant** of a matrix. The determinant is actually a function that takes a square matrix and converts it into a number. The actual formula for the function is somewhat complex and definitely beyond the scope of this review.

The main method for computing determinants of any square matrix is called the method of cofactors. Since we are going to be dealing almost exclusively with 2 x 2 matrices and the occasional 3 x 3 matrix we won't go into the method here. We can give simple formulas for each of these cases. The standard notation for the determinant of the matrix *A* is.

$$\det(A) = |A|$$

Here are the formulas for the determinant of 2 x 2 and 3 x 3 matrices.

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 3 Find the determinant of each of the following matrices.

$$A = \begin{pmatrix} -9 & -18 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & 1 \\ -1 & -6 & 7 \\ 4 & 5 & -1 \end{pmatrix}$$

Solution

For the 2 x 2 there isn't much to do other than to plug it into the formula.

$$\det(A) = \begin{vmatrix} -9 & -18 \\ 2 & 4 \end{vmatrix} = (-9)(4) - (-18)(2) = 0$$

For the 3 x 3 we could plug it into the formula, however unlike the 2 x 2 case this is not an easy formula to remember. There is an easier way to get the same result. A quicker way of getting the same result is to do the following. First write down the matrix and tack a copy of the first two columns onto the end as follows.

$$\det(B) = \left|\begin{array}{ccc|cc} 2 & 3 & 1 & 2 & 3 \\ -1 & -6 & 7 & -1 & -6 \\ 4 & 5 & -1 & 4 & 5 \end{array}\right.$$

Now, notice that there are three diagonals that run from left to right and three diagonals that run from right to left. What we do is multiply the entries on each diagonal up and the if the diagonal runs from left to right we add them up and if the diagonal runs from right to left we subtract them.

Here is the work for this matrix.

$$\begin{aligned} \det(B) &= \left|\begin{array}{ccc|cc} 2 & 3 & 1 & 2 & 3 \\ -1 & -6 & 7 & -1 & -6 \\ 4 & 5 & -1 & 4 & 5 \end{array}\right. \\ &= (2)(-6)(-1) + (3)(7)(4) + (1)(-1)(5) - \\ &\qquad (3)(-1)(-1) - (2)(7)(5) - (1)(-6)(4) \\ &= 42 \end{aligned}$$

You can either use the formula or the short cut to get the determinant of a 3 x 3.

If the determinant of a matrix is zero we call that matrix **singular** and if the determinant of a matrix isn't zero we call the matrix **nonsingular**. The 2 x 2 matrix in the above example was singular while the 3 x 3 matrix is nonsingular.

Matrix Inverse

Next we need to take a look at the **inverse** of a matrix. Given a square matrix, A, of size n x n if we can find another matrix of the same size, B such that,

$$AB = BA = I_n$$

then we call B the **inverse** of A and denote it by $B=A^{-1}$.

Computing the inverse of a matrix, A, is fairly simple. First we form a new matrix,

$$\begin{pmatrix} A & I_n \end{pmatrix}$$

and then use the row operations from the previous section and try to convert this matrix into the form,

$$\left(I_n \quad B \right)$$

If we can then B is the inverse of A. If we can't then there is no inverse of the matrix A.

Example 4 Find the inverse of the following matrix, if it exists.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -5 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

Solution

We first form the new matrix by tacking on the 3 x 3 identity matrix to this matrix. This is

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ -5 & -3 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

We will now use row operations to try and convert the first three columns to the 3 x 3 identity. In other words we want a 1 on the diagonal that starts at the upper left corner and zeroes in all the other entries in the first three columns.

If you think about it, this process is very similar to the process we used in the last section to solve systems, it just goes a little farther. Here is the work for this problem.

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ -5 & -3 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_3 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ -5 & -3 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} R_2 + 5R_1 \\ R_3 - 2R_1 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & -5 & 0 & 1 & 5 \\ 0 & -1 & 3 & 1 & 0 & -2 \end{pmatrix} \begin{matrix} \frac{1}{2}R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & \frac{-5}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & -1 & 3 & 1 & 0 & -2 \end{pmatrix} \begin{matrix} R_3 + R_2 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & \frac{-5}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{matrix} 2R_3 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & \frac{-5}{2} & 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 + \frac{5}{2}R_3 \\ R_1 + R_3 \\ \Rightarrow \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 5 & 3 & 5 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 - R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 0 & 0 & -3 & -2 & -3 \\ 0 & 1 & 0 & 5 & 3 & 5 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{pmatrix}$$

So, we were able to convert the first three columns into the 3 x 3 identity matrix therefore the inverse exists and it is,

$$A^{-1} = \begin{pmatrix} -3 & -2 & -3 \\ 5 & 3 & 5 \\ 2 & 1 & 1 \end{pmatrix}$$

So, there was an example in which the inverse did exist. Let's take a look at an example in which the inverse doesn't exist.

Example 5 Find the inverse of the following matrix, provided it exists.

$$B = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

Solution

In this case we will tack on the 2 x 2 identity to get the new matrix and then try to convert the first two columns to the 2 x 2 identity matrix.

$$\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{array}\right) \overset{2R_1 + R_2}{\Rightarrow} \left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

And we don't need to go any farther. In order for the 2 x 2 identity to be in the first two columns we must have a 1 in the second entry of the second column and a 0 in the second entry of the first column. However, there is no way to get a 1 in the second entry of the second column that will keep a 0 in the second entry in the first column. Therefore, we can't get the 2 x 2 identity in the first two columns and hence the inverse of *B* doesn't exist.

We will leave off this discussion of inverses with the following fact.

Fact

Given a square matrix *A*.

1. If *A* is nonsingular then A^{-1} will exist.
2. If *A* is singular then A^{-1} will NOT exist.

I'll leave it to you to verify this fact for the previous two examples.

Systems of Equations Revisited

We need to do a quick revisit of systems of equations. Let's start with a general system of equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad \textbf{(1)}$$

Now, covert each side into a vector to get,

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The left side of this equation can be thought of as a matrix multiplication.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Simplifying up the notation a little gives,

$$A\vec{x} = \vec{b} \qquad \textbf{(2)}$$

where, $\vec{x}$ is a vector whose components are the unknowns in the original system of equations. We call (2) the matrix form of the system of equations (1) and solving (2) is equivalent to solving (1). The solving process is identical. The augmented matrix for (2) is

$$\left(A \;\; \vec{b} \right)$$

Once we have the augmented matrix we proceed as we did with a system that hasn't been wrote in matrix form.

We also have the following fact about solutions to (2).

Fact

Given the system of equation (2) we have one of the following three possibilities for solutions.

1. There will be no solutions.
2. There will be exactly one solution.
3. There will be infinitely many solutions.

In fact we can go a little farther now. Since we are assuming that we've got the same number of equations as unknowns the matrix *A* in (2) is a square matrix and so we can compute its determinant. This gives the following fact.

Fact

Given the system of equations in (2) we have the following.

1. If *A* is nonsingular then there will be exactly one solution to the system.
2. If *A* is singular then there will either be no solution or infinitely many solutions to the system.

The matrix form of a homogeneous system is

$$A\vec{x} = \vec{0} \qquad \textbf{(3)}$$

where $\vec{0}$ is the vector of all zeroes. In the homogeneous system we are guaranteed to have a solution, $\vec{x} = \vec{0}$. The fact above for homogeneous systems is then,

Fact

Given the homogeneous system (3) we have the following.

1. If *A* is nonsingular then the only solution will be $\vec{x} = \vec{0}$.
2. If *A* is singular then there will be infinitely many nonzero solutions to the system.

Linear Independence/Linear Dependence

This is not the first time that we've seen this topic. We also saw linear independence and linear dependence back when we were looking at second order differential equations. In that section we were dealing with functions, but the concept is essentially the same here. If we start with n vectors,

$$\vec{x}_1,\ \vec{x}_2,\ \ldots,\ \vec{x}_n$$

If we can find constants, $c_1, c_2, \ldots, c_n$ with at least two nonzero such that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n = \vec{0} \qquad \textbf{(4)}$$

then we call the vectors linearly dependent. If the only constants that work in (4) are $c_1=0$, $c_2=0$, …, $c_n=0$ then we call the vectors linearly independent.

If we further make the assumption that each of the n vectors has n components, *i.e.* each of the vectors look like,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

we can get a very simple test for linear independence and linear dependence. Note that this does not have to be the case, but in all of our work we will be working with n vectors each of which has n components.

Fact

Given the n vectors each with n components,

$$\vec{x}_1,\ \vec{x}_2,\ \ldots,\ \vec{x}_n$$

form the matrix,

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{pmatrix}$$

So, the matrix X is a matrix whose i^{th} column is the i^{th} vector, $\vec{x}_i$. Then,

1. If X is nonsingular (*i.e.* $\det(X)$ is not zero) then the n vectors are linearly independent, and
2. if X is singular (*i.e.* $\det(X) = 0$) then the n vectors are linearly dependent and the constants that make (4) true can be found by solving the system

   $$X\,\vec{c} = \vec{0}$$

   where $\vec{c}$ is a vector containing the constants in (4).

Example 6 Determine if the following set of vectors are linearly independent or linearly dependent. If they are linearly dependent find the relationship between them.

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}, \quad \vec{x}^{(2)} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}, \quad \vec{x}^{(3)} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$$

Solution

So, the first thing to do is to form X and compute its determinant.

$$X = \begin{pmatrix} 1 & -2 & 6 \\ -3 & 1 & -2 \\ 5 & 4 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \det(X) = -79$$

This matrix is non singular and so the vectors are linearly independent.

Example 7 Determine if the following set of vectors are linearly independent or linearly dependent. If they are linearly dependent find the relationship between them.

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \vec{x}^{(2)} = \begin{pmatrix} -4 \\ 1 \\ -6 \end{pmatrix}, \quad \vec{x}^{(3)} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Solution

As with the last example first form X and compute its determinant.

$$X = \begin{pmatrix} 1 & -4 & 2 \\ -1 & 1 & -1 \\ 3 & -6 & 4 \end{pmatrix} \qquad \Rightarrow \qquad \det(X) = 0$$

So, these vectors are linearly dependent. We now need to find the relationship between the vectors. This means that we need to find constants that will make (4) true.

So we need to solve the system

$$X\,\vec{c} = \vec{0}$$

Here is the augmented matrix and the solution work for this system.

$$\left(\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & -6 & 4 & 0 \end{array}\right) \begin{array}{c} R_2 + R_1 \\ R_3 - 3R_1 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 6 & -2 & 0 \end{array}\right) \begin{array}{c} R_3 + 2R_2 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} -\frac{1}{3}R_2 \\ \Rightarrow \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} R_1 + 4R_2 \\ \Rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{c} c_1 + \frac{2}{3}c_3 = 0 \\ c_2 - \frac{1}{3}c_3 = 0 \\ 0 = 0 \end{array} \Rightarrow \begin{array}{c} c_1 = -\frac{2}{3}c_3 \\ c_2 = \frac{1}{3}c_3 \end{array}$$

Now, we would like actual values for the constants so, if use $c_3 = 3$ we get the following solution $c_1 = -2$, $c_2 = 1$, and $c_3 = 3$. The relationship is then.

$$-2\vec{x}^{(1)} + \vec{x}^{(2)} + 3\vec{x}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Calculus with Matrices

There really isn't a whole lot to this other than to just make sure that we can deal with calculus with matrices.

First, to this point we've only looked at matrices with numbers as entries, but the entries in a matrix can be functions as well. So we can look at matrices in the following form,

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{pmatrix}$$

Now we can talk about differentiating and integrating a matrix of this form. To differentiate or integrate a matrix of this form all we do is differentiate or integrate the individual entries.

$$A'(t)=\begin{pmatrix} a'_{11}(t) & a'_{12}(t) & \cdots & a'_{1n}(t) \\ a'_{21}(t) & a'_{22}(t) & \cdots & a'_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a'_{m1}(t) & a'_{m2}(t) & \cdots & a'_{mn}(t) \end{pmatrix}$$

$$\int A(t)\,dt=\begin{pmatrix} \int a_{11}(t)\,dt & \int a_{12}(t)\,dt & \cdots & \int a_{1n}(t)\,dt \\ \int a_{21}(t)\,dt & \int a_{22}(t)\,dt & \cdots & \int a_{2n}(t)\,dt \\ \vdots & \vdots & & \vdots \\ \int a_{m1}(t)\,dt & \int a_{m2}(t)\,dt & \cdots & \int a_{mn}(t)\,dt \end{pmatrix}$$

So when we run across this kind of thing don't get excited about it. Just differentiate or integrate as we normally would.

In this section we saw a very condensed set of topics from linear algebra. When we get back to differential equations many of these topics will show up occasionally and you will at least need to know what the words mean.

The main topic from linear algebra that you must know however if you are going to be able to solve systems of differential equations is the topic of the next section.

Review : Eigenvalues and Eigenvectors

If you get nothing out of this quick review of linear algebra you must get this section. Without this section you will not be able to do any of the differential equations work that is in this chapter.

So let's start with the following. If we multiply an n x n matrix by an n x 1 vector we will get a new n x 1 vector back. In other words,

$$A\vec{\eta} = \vec{y}$$

What we want to know is if it is possible for the following to happen. Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \textbf{(1)}$$

In other words is it possible, at least for certain λ and $\vec{\eta}$, to have matrix multiplication be the same as just multiplying the vector by a constant? Of course, we probably wouldn't be talking about this if the answer was no. So, it is possible for this to happen, however, it won't happen for just any value of λ or $\vec{\eta}$. If we do happen to have a λ and $\vec{\eta}$ for which this works (and they will always come in pairs) then we call λ an **eigenvalue** of A and $\vec{\eta}$ an **eigenvector** of A.

So, how do we go about find the eigenvalues and eigenvectors for a matrix? Well first notice that if $\vec{\eta} = \vec{0}$ then (1) is going to be true for any value of λ and so we are going to make the assumption that $\vec{\eta} \neq \vec{0}$. With that out of the way let's rewrite (1) a little.

$$\begin{aligned} A\vec{\eta} - \lambda\vec{\eta} &= \vec{0} \\ A\vec{\eta} - \lambda I_n\vec{\eta} &= \vec{0} \\ \left(A - \lambda I_n\right)\vec{\eta} &= \vec{0} \end{aligned}$$

Notice that before we factored out the $\vec{\eta}$ we added in the appropriately sized identity matrix. This is equivalent to multiplying things by a one and so doesn't change the value of anything. We needed to do this because without it we would have had the difference of a matrix, A, and a constant, λ, and this can't be done. We now have the difference of two matrices of the same size which can be done.

So, with this rewrite we see that

$$\left(A - \lambda I_n\right)\vec{\eta} = \vec{0} \qquad \textbf{(2)}$$

is equivalent to (1). In order to find the eigenvectors for a matrix we will need to solve a homogeneous system. Recall the fact from the previous section that we know that we will either have exactly one solution ($\vec{\eta} = \vec{0}$) or we will have infinitely many nonzero solutions. Since we've already said that don't want $\vec{\eta} = \vec{0}$ this means that we want the second case.

Knowing this will allow us to find the eigenvalues for a matrix. Recall from this fact that we will get the second case only if the matrix in the system is singular. Therefore we will need to determine the values of λ for which we get,

$$\det\left(A - \lambda I\right) = 0$$

Once we have the eigenvalues we can then go back and determine the eigenvectors for each eigenvalue. Let's take a look at a couple of quick facts about eigenvalues and eigenvectors.

Fact

If A is an n x n matrix then $\det\left(A-\lambda I\right)=0$ is an n^{th} degree polynomial. This polynomial is called the **characteristic polynomial**.

To find eigenvalues of a matrix all we need to do is solve a polynomial. That's generally not too bad provided we keep n small. Likewise this fact also tells us that for an n x n matrix, A, we will have n eigenvalues if we include all repeated eigenvalues.

Fact

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the complete list of eigenvalues for A (including all repeated eigenvalues) then,

1. If λ occurs only once in the list then we call λ **simple**.
2. If λ occurs $k>1$ times in the list then we say that λ has **multiplicity *k***.
3. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \le n)$ are the simple eigenvalues in the list with corresponding eigenvectors $\vec{\eta}^{(1)}$, $\vec{\eta}^{(2)}$, …, $\vec{\eta}^{(k)}$ then the eigenvectors are all linearly independent.
4. If λ is an eigenvalue of multiplicity $k>1$ then λ will have anywhere from 1 to k linearly independent eigenvectors.

The usefulness of these facts will become apparent when we get back into differential equations since in that work we will want linearly independent solutions.

Let's work a couple of examples now to see how we actually go about finding eigenvalues and eigenvectors.

Example 1 Find the eigenvalues and eigenvectors of the following matrix.

$$A=\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$$

Solution

The first thing that we need to do is find the eigenvalues. That means we need the following matrix,

$$A-\lambda I=\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}$$

In particular we need to determine where the determinant of this matrix is zero.

$$\det\left(A-\lambda I\right)=\left(2-\lambda\right)\left(-6-\lambda\right)+7=\lambda^2+4\lambda-5=\left(\lambda+5\right)\left(\lambda-1\right)$$

So, it looks like we will have two simple eigenvalues for this matrix, $\lambda_1=-5$ and $\lambda_2=1$. We will now need to find the eigenvectors for each of these. Also note that according to the fact above, the two eigenvectors should be linearly independent.

To find the eigenvectors we simply plug in each eigenvalue into (2) and solve. So, let's do that.

$\lambda_1 = -5$:

In this case we need to solve the following system.

$$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Recall that officially to solve this system we use the following augmented matrix.

$$\begin{pmatrix} 7 & 7 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{matrix} \frac{1}{7}R_1 + R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 7 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Upon reducing down we see that we get a single equation

$$7\eta_1 + 7\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

that will yield an infinite number of solutions. This is expected behavior. Recall that we picked the eigenvalues so that the matrix would be singular and so we would get infinitely many solutions.

Notice as well that we could have identified this from the original system. This won't always be the case, but in the 2 x 2 case we can see from the system that one row will be a multiple of the other and so we will get infinite solutions. From this point on we won't be actually solving systems in these cases. We will just go straight to the equation and we can use either of the two rows for this equation.

Now, let's get back to the eigenvector, since that is what we were after. In general then the eigenvector will be any vector that satisfies the following,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad ,\eta_2 \neq 0$$

To get this we used the solution to the equation that we found above.

We really don't want a general eigenvector however so we will pick a value for η_2 to get a specific eigenvector. We can choose anything (except $\eta_2 = 0$), so pick something that will make the eigenvector "nice". Note as well that since we've already assumed that the eigenvector is not zero we must choose a value that will not give us zero, which is why we want to avoid $\eta_2 = 0$ in this case. Here's the eigenvector for this eigenvalue.

$$\vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \text{using } \eta_2 = 1$$

Now we get to do this all over again for the second eigenvalue.

$\lambda_2 = 1$:

We'll do much less work with this part than we did with the previous part. We will need to solve the following system.

$$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly both rows are multiples of each other and so we will get infinitely many solutions. We can choose to work with either row. We'll run with the first because to avoid having too many minus signs floating around. Doing this gives us,

$$\eta_1 + 7\eta_2 = 0 \qquad\qquad \eta_1 = -7\eta_2$$

Note that we can solve this for either of the two variables. However, with an eye towards working with these later on let's try to avoid as many fractions as possible. The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -7\eta_2 \\ \eta_2 \end{pmatrix} \qquad ,\eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \qquad\qquad \text{using } \eta_2 = 1$$

Summarizing we have,

$$\lambda_1 = -5 \qquad\qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \qquad\qquad \vec{\eta}^{(1)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

Note that the two eigenvectors are linearly independent as predicted.

Example 2 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 1 & -1 \\ \frac{4}{9} & -\frac{1}{3} \end{pmatrix}$$

Solution

This matrix has fractions in it. That's life so don't get excited about it. First we need the eigenvalues.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -1 \\ \frac{4}{9} & -\frac{1}{3}-\lambda \end{vmatrix} \\ &= (1-\lambda)\left(-\frac{1}{3}-\lambda\right) + \frac{4}{9} \\ &= \lambda^2 - \frac{2}{3}\lambda + \frac{1}{9} \\ &= \left(\lambda - \frac{1}{3}\right)^2 \qquad\qquad \Rightarrow \qquad \lambda_{1,2} = \frac{1}{3} \end{aligned}$$

So, it looks like we've got an eigenvalue of multiplicity 2 here. Remember that the power on the term will be the multiplicity.

Now, let's find the eigenvector(s). This one is going to be a little different from the first example. There is only one eigenvalue so let's do the work for that one. We will need to solve the following system,

$$\begin{pmatrix} \frac{2}{3} & -1 \\ \frac{4}{9} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad R_1 = \frac{3}{2}R_2$$

So, the rows are multiples of each other. We'll work with the first equation in this example to find the eigenvector.

$$\frac{2}{3}\eta_1 - \eta_2 = 0 \qquad\qquad \eta_2 = \frac{2}{3}\eta_1$$

Recall in the last example we decided that we wanted to make these as "nice" as possible and so should avoid fractions if we can. Sometimes, as in this case, we simply can't so we'll have to deal with it. In this case the eigenvector will be,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \frac{2}{3}\eta_1 \end{pmatrix}, \qquad\qquad \eta_1 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad\qquad \eta_1 = 3$$

Note that by careful choice of the variable in this case we were able to get rid of the fraction that we had. This is something that in general doesn't much matter if we do or not. However, when we get back to differential equations it will be easier on us if we don't have any fractions so we will usually try to eliminate them at this step.

Also in this case we are only going to get a single (linearly independent) eigenvector. We can get other eigenvectors, by choosing different values of η_1. However, each of these will be linearly dependent with the first eigenvector. If you're not convinced of this try it. Pick some values for η_1 and get a different vector and check to see if the two are linearly dependent.

Recall from the fact above that an eigenvalue of multiplicity *k* will have anywhere from 1 to *k* linearly independent eigenvectors. In this case we got one. For most of the 2 x 2 matrices that we'll be working with this will be the case, although it doesn't have to be. We can, on occasion, get two.

Example 3 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}$$

Solution

So, we'll start with the eigenvalues.

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} -4-\lambda & -17 \\ 2 & 2-\lambda \end{vmatrix} \\ &= \left(-4-\lambda\right)\left(2-\lambda\right) + 34 \\ &= \lambda^2 + 2\lambda + 26 \end{aligned}$$

This doesn't factor, so upon using the quadratic formula we arrive at,

$$\lambda_{1,2} = -1 \pm 5i$$

In this case we get complex eigenvalues which are definitely a fact of life with eigenvalue/eigenvector problems so get used to them.

Finding eigenvectors for complex eigenvalues is identical to the previous two examples, but it will be somewhat messier. So, let's do that.

$\lambda_1 = -1+5\,i$:
The system that we need to solve this time is

$$\begin{pmatrix} -4-(-1+5i) & -17 \\ 2 & 2-(-1+5i) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3-5i & -17 \\ 2 & 3-5i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, it's not super clear that the rows are multiples of each other, but they are. In this case we have,

$$R_1 = -\frac{1}{2}(3+5i)\,R_2$$

This is not something that you need to worry about, we just wanted to make the point. For the work that we'll be doing later on with differential equations we will just assume that we've done everything correctly and we've got two rows that are multiples of each other. Therefore, all that we need to do here is pick one of the rows and work with it.

We'll work with the second row this time.

$$2\eta_1 + (3-5i)\,\eta_2 = 0$$

Now we can solve for either of the two variables. However, again looking forward to differential equations, we are going to need the "*i*" in the numerator so solve the equation in such a way as this will happen. Doing this gives,

$$2\eta_1 = -(3-5i)\,\eta_2$$

$$\eta_1 = -\frac{1}{2}(3-5i)\,\eta_2$$

So, the eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3-5i)\,\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{\,(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

As with the previous example we choose the value of the variable to clear out the fraction.

Now, the work for the second eigenvector is almost identical and so we'll not dwell on that too much.

$\lambda_2 = -1-5i$:

The system that we need to solve here is

$$\begin{pmatrix} -4-(-1-5i) & -17 \\ 2 & 2-(-1-5i) \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+5i & -17 \\ 2 & 3+5i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Working with the second row again gives,

$$2\eta_1 + (3+5i)\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\frac{1}{2}(3+5i)\eta_2$$

The eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3+5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

Summarizing,

$$\lambda_1 = -1+5i \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1-5i \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}$$

There is a nice fact that we can use to simplify the work when we get complex eigenvalues. We need a bit of terminology first however.

If we start with a complex number,

$$z = a + bi$$

then the **complex conjugate** of z is

$$\bar{z} = a - bi$$

To compute the complex conjugate of a complex number we simply change the sign on the term that contains the "*i*". The complex conjugate of a vector is just the conjugate of each of the vector's components.

We now have the following fact about complex eigenvalues and eigenvectors.

Fact

If A is an n x n matrix with only real numbers and if $\lambda_1 = a + bi$ is an eigenvalue with eigenvector $\vec{\eta}^{(1)}$. Then $\lambda_2 = \overline{\lambda_1} = a - bi$ is also an eigenvalue and its eigenvector is the conjugate of $\vec{\eta}^{(1)}$.

This fact is something that you should feel free to use as you need to in our work.

Now, we need to work one final eigenvalue/eigenvector problem. To this point we've only worked with 2 x 2 matrices and we should work at least one that isn't 2 x 2. Also, we need to work one in which we get an eigenvalue of multiplicity greater than one that has more than one linearly independent eigenvector.

Example 4 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution

Despite the fact that this is a 3 x 3 matrix, it still works the same as the 2 x 2 matrices that we've been working with. So, start with the eigenvalues

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= -\lambda^3 + 3\lambda + 2 \\ &= \left(\lambda - 2\right)\left(\lambda + 1\right)^2 \qquad \lambda_1 = 2, \ \lambda_{2,3} = -1 \end{aligned}$$

So, we've got a simple eigenvalue and an eigenvalue of multiplicity 2. Note that we used the same method of computing the determinant of a 3 x 3 matrix that we used in the previous section. We just didn't show the work.

Let's now get the eigenvectors. We'll start with the simple eigenvector.

$\lambda_1 = 2$:

Here we'll need to solve,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This time, unlike the 2 x 2 cases we worked earlier, we actually need to solve the system. So let's do that.

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 - R_1 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

$$\begin{matrix} -\frac{1}{3}R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix} \begin{matrix} R_3 - 3R_2 \\ R_1 + 2R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Going back to equations gives,

$$\begin{aligned} \eta_1 - \eta_3 = 0 \qquad &\Rightarrow \qquad \eta_1 = \eta_3 \\ \eta_2 - \eta_3 = 0 \qquad &\Rightarrow \qquad \eta_2 = \eta_3 \end{aligned}$$

So, again we get infinitely many solutions as we should for eigenvectors. The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_3 \\ \eta_3 \\ \eta_3 \end{pmatrix}, \qquad\qquad \eta_3 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad\qquad \eta_3 = 1$$

Now, let's do the other eigenvalue.

$\lambda_2 = -1$:
Here we'll need to solve,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Okay, in this case is clear that all three rows are the same and so there isn't any reason to actually solve the system since we can clear out the bottom two rows to all zeroes in one step. The equation that we get then is,

$$\eta_1 + \eta_2 + \eta_3 = 0 \Rightarrow \qquad \eta_1 = -\eta_2 - \eta_3$$

So, in this case we get to pick two of the values for free and will still get infinitely many solutions. Here is the general eigenvector for this case,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} -\eta_2 - \eta_3 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \qquad \eta_2 \neq 0 \text{ and } \eta_3 \neq 0 \text{ at the same time}$$

Notice the restriction this time. Recall that we only require that the eigenvector not be the zero vector. This means that we can allow one or the other of the two variables to be zero, we just can't allow both of them to be zero at the same time!

What this means for us is that we are going to get two linearly independent eigenvectors this time. Here they are.

$$\vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \eta_2 = 0 \text{ and } \eta_3 = 1$$

$$\vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \eta_2 = 1 \text{ and } \eta_3 = 0$$

Now when we talked about linear independent vectors in the last section we only looked at *n* vectors each with *n* components. We can still talk about linear independence in this case however. Recall back with we did linear independence for functions we saw at the time that if two functions were linearly dependent then they were multiples of each other. Well the same thing holds true for vectors. Two vectors will be linearly dependent if they are multiples of each other. In this case there is no way to get $\vec{\eta}^{(2)}$ by multiplying $\vec{\eta}^{(3)}$ by a constant. Therefore, these two vectors must be linearly independent.

So, summarizing up, here are the eigenvalues and eigenvectors for this matrix

$$\lambda_1 = 2 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1 \qquad \vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Systems of Differential Equations

In the introduction to this section we briefly discussed how a system of differential equations can arise from a population problem in which we keep track of the population of both the prey and the predator. It makes sense that the number of prey present will affect the number of the predator present. Likewise, the number of predator present will affect the number of prey present. Therefore the differential equation that governs the population of either the prey or the predator should in some way depend on the population of the other. This will lead to two differential equations that must be solved simultaneously in order to determine the population of the prey and the predator.

The whole point of this is to notice that systems of differential equations can arise quite easily from naturally occurring situations. Developing an effective predator-prey system of differential equations is not the subject of this chapter. However, systems can arise from n^{th} order linear differential equations as well. Before we get into this however, let's write down a system and get some terminology out of the way.

We are going to be looking at first order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions. Here is an example of a system of first order, linear differential equations.

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}$$

We call this kind of system a **coupled** system since knowledge of x_2 is required in order to find x_1 and likewise knowledge of x_1 is required to find x_2. We will worry about how to go about solving these later. At this point we are only interested in becoming familiar with some of the basics of systems.

Now, as mentioned earlier, we can write an n^{th} order linear differential equation as a system. Let's see how that can be done.

Example 1 Write the following 2^{nd} order differential equation as a system of first order, linear differential equations.

$$2y'' - 5y' + y = 0 \qquad y(3) = 6 \qquad y'(3) = -1$$

Solution

We can write higher order differential equations as a system with a very simple change of variable. We'll start by defining the following two new functions.

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{aligned}$$

Now notice that if we differentiate both sides of these we get,

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \end{aligned}$$

Note the use of the differential equation in the second equation. We can also convert the initial conditions over to the new functions.

$$x_1(3) = y(3) = 6$$
$$x_2(3) = y'(3) = -1$$

Putting all of this together gives the following system of differential equations.

$$\begin{aligned} x_1' &= x_2 & x_1(3) &= 6 \\ x_2' &= -\frac{1}{2}x_1 + \frac{5}{2}x_2 & x_2(3) &= -1 \end{aligned}$$

We will call the system in the above example an **Initial Value Problem** just as we did for differential equations with initial conditions.

Let's take a look at another example.

Example 2 Write the following 4th order differential equation as a system of first order, linear differential equations.

$$y^{(4)} + 3y'' - \sin(t)\,y' + 8y = t^2 \qquad y(0) = 1 \quad y'(0) = 2 \quad y''(0) = 3 \quad y'''(0) = 4$$

Solution

Just as we did in the last example we'll need to define some new functions. This time we'll need 4 new functions.

$$\begin{aligned} x_1 &= y & &\Rightarrow & x_1' &= y' = x_2 \\ x_2 &= y' & &\Rightarrow & x_2' &= y'' = x_3 \\ x_3 &= y'' & &\Rightarrow & x_3' &= y''' = x_4 \\ x_4 &= y''' & &\Rightarrow & x_4' &= y^{(4)} = -8y + \sin(t)\,y' - 3y'' + t^2 = -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \end{aligned}$$

The system along with the initial conditions is then,

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= 1 \\ x_2' &= x_3 & x_2(0) &= 2 \\ x_3' &= x_4 & x_3(0) &= 3 \\ x_4' &= -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 & x_4(0) &= 4 \end{aligned}$$

Now, when we finally get around to solving these we will see that we generally don't solve systems in the form that we've given them in this section. Systems of differential equations can be converted to **matrix form** and this is the form that we usually use in solving systems.

Example 3 Convert the following system to matrix from.

$$x_1' = 4x_1 + 7x_2$$
$$x_2' = -2x_1 - 5x_2$$

Solution
First write the system so that each side is a vector.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4x_1 + 7x_2 \\ -2x_1 - 5x_2 \end{pmatrix}$$

Now the right side can be written as a matrix multiplication,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now, if we define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

then,

$$\vec{x}\,' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

The system can then be wrote in the matrix form,

$$\vec{x}\,' = \begin{pmatrix} 4 & 7 \\ -2 & -5 \end{pmatrix} \vec{x}$$

Example 4 Convert the systems from Examples 1 and 2 into matrix form.

Solution
We'll start with the system from Example 1.

$$x_1' = x_2 \qquad\qquad x_1(3) = 6$$
$$x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 \qquad\qquad x_2(3) = -1$$

First define,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The system is then,

$$\vec{x}\,' = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \vec{x} \qquad\qquad \vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Now, let's do the system from Example 2.

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= 1 \\ x_2' &= x_3 & x_2(0) &= 2 \\ x_3' &= x_4 & x_3(0) &= 3 \\ x_4' &= -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 & x_4(0) &= 4 \end{aligned}$$

In this case we need to be careful with the t^2 in the last equation. We'll start by writing the system as a vector again and then break it up into two vectors, one vector that contains the unknown functions and the other that contains any known functions.

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 + t^2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -8x_1 + \sin(t)\,x_2 - 3x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix}$$

Now, the first vector can now be written as a matrix multiplication and we'll leave the second vector alone.

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ t^2 \end{pmatrix} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}$$

Note that occasionally for "large" systems such as this we will go one step farther and write the system as,

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

The last thing that we need to do in this section is get a bit of terminology out of the way. Starting with

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

we say that the system is **homogeneous** if $\vec{g}(t) = \vec{0}$ and we say the system is **nonhomogeneous** if $\vec{g}(t) \neq \vec{0}$.

Solutions to Systems

Now that we've got some of the basics out of the way for systems of differential equations it's time to start thinking about how to solve a system of differential equations. We will start with the homogeneous system written in matrix form,

$$\vec{x}' = A\,\vec{x} \tag{1}$$

where, A is an n x n matrix and $\vec{x}$ is a vector whose components are the unknown functions in the system.

Now, if we start with $n = 1$ then the system reduces to a fairly simple linear (or separable) first order differential equation.

$$x' = ax$$

and this has the following solution,

$$x(t) = c\mathbf{e}^{at}$$

So, let's use this as a guide and for a general n let's see if

$$\vec{x}(t) = \vec{\eta}\,\mathbf{e}^{rt} \tag{2}$$

will be a solution. Note that the only real difference here is that we let the constant in front of the exponential be a vector. All we need to do then is plug this into the differential equation and see what we get. First notice that the derivative is,

$$\vec{x}'(t) = r\vec{\eta}\,\mathbf{e}^{rt}$$

So upon plugging the guess into the differential equation we get,

$$\begin{aligned} r\vec{\eta}\mathbf{e}^{rt} &= A\vec{\eta}\mathbf{e}^{rt} \\ \left(A\vec{\eta} - r\vec{\eta}\right)\mathbf{e}^{rt} &= \vec{0} \\ \left(A - rI\right)\vec{\eta}\mathbf{e}^{rt} &= \vec{0} \end{aligned}$$

Now, since we know that exponentials are not zero we can drop that portion and we then see that in order for (2) to be a solution to (1) then we must have

$$\left(A - rI\right)\vec{\eta} = \vec{0}$$

Or, in order for (2) to be a solution to (1), r and $\vec{\eta}$ must be an eigenvalue and eigenvector for the matrix A.

Therefore, in order to solve (1) we first find the eigenvalues and eigenvectors of the matrix A and then we can form solutions using (2). There are going to be three cases that we'll need to look at. The cases are real, distinct eigenvalues, complex eigenvalues and repeated eigenvalues.

None of this tells us how to completely solve a system of differential equations. We'll need the following couple of facts to do this.

Fact

1. If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions to a homogeneous system, (1), then
$$c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$$
is also a solution to the system.
2. Suppose that A is an n x n matrix and suppose that $\vec{x}_1(t)$, $\vec{x}_2(t)$, …, $\vec{x}_n(t)$ are solutions to a homogeneous system, (1). Define,
$$X=\left(\vec{x}_1\ \vec{x}_2\ \cdots\ \vec{x}_n\right)$$
In other words, X is a matrix whose *ith* column is the i^{th} solution. Now define,
$$W=\det(X)$$
We call W the **Wronskian**. If $W\neq 0$ then the solutions form a **fundamental set of solutions** and the general solution to the system is,
$$\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+\cdots+c_n\vec{x}_n(t)$$

Note that if we have a fundamental set of solutions then the solutions are also going to be linearly independent. Likewise, if we have a set of linearly independent solutions then they will also be a fundamental set of solutions since the Wronskian will not be zero.

Phase Plane

Before proceeding with actually solving systems of differential equations there's one topic that we need to take a look at. This is a topic that's not always taught in a differential equations class but in case you're in a course where it is taught we should cover it so that you are prepared for it.

Let's start with a general homogeneous system,

$$\vec{x}' = A\vec{x} \tag{1}$$

Notice that

$$\vec{x} = \vec{0}$$

is a solution to the system of differential equations. What we'd like to ask is, do the other solutions to the system approach this solution as t increases or do they move away from this solution? We did something similar to this when we classified equilibrium solutions in a previous section. In fact, what we're doing here is simply an extension of this idea to systems of differential equations.

The solution $\vec{x} = \vec{0}$ is called an **equilibrium solution** for the system. As with the single differential equations case, equilibrium solutions are those solutions for which

$$A\vec{x} = \vec{0}$$

We are going to assume that A is a nonsingular matrix and hence will have only one solution,

$$\vec{x} = \vec{0}$$

and so we will have only one equilibrium solution.

Back in the single differential equation case recall that we started by choosing values of y and plugging these into the function $f(y)$ to determine values of y'. We then used these values to sketch tangents to the solution at that particular value of y. From this we could sketch in some solutions and use this information to classify the equilibrium solutions.

We are going to do something similar here, but it will be slightly different as well. First we are going to restrict ourselves down to the 2 x 2 case. So, we'll be looking at systems of the form,

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}$$

Solutions to this system will be of the form,

$$\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and our single equilibrium solution will be,

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the single differential equation case we were able to sketch the solution, $y(t)$ in the y-t plane and see actual solutions. However, this would somewhat difficult in this case since our solutions are actually vectors. What we're going to do here is think of the solutions to the system as points in the x_1-x_2 plane and plot these points. Our equilibrium solution will correspond to the origin of x_1-x_2 plane and the x_1-x_2 plane is called the **phase plane**.

To sketch a solution in the phase plane we can pick values of t and plug these into the solution. This gives us a point in the x_1-x_2 or phase plane that we can plot. Doing this for many values of t will then give us a sketch of what the solution will be doing in the phase plane. A sketch of a particular solution in the phase plane is called the **trajectory** of the solution. Once we have the trajectory of a solution sketched we can then ask whether or not the solution will approach the equilibrium solution as t increases.

We would like to be able to sketch trajectories without actually having solutions in hand. There are a couple of ways to do this. We'll look at one of those here and we'll look at the other in the next couple of sections.

One way to get a sketch of trajectories is to do something similar to what we did the first time we looked at equilibrium solutions. We can choose values of $\vec{x}$ (note that these will be points in the phase plane) and compute $A\vec{x}$. This will give a vector that represents $\vec{x}'$ at that particular solution. As with the single differential equation case this vector will be tangent to the trajectory at that point. We can sketch a bunch of the tangent vectors and then sketch in the trajectories.

This is a fairly work intensive way of doing these and isn't the way to do them in general. However, it is a way to get trajectories without doing any solution work. All we need is the system of differential equations. Let's take a quick look at an example.

Example 1 Sketch some trajectories for the system,

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}$$

Solution

So, what we need to do is pick some points in the phase plane, plug them into the right side of the system. We'll do this for a couple of points.

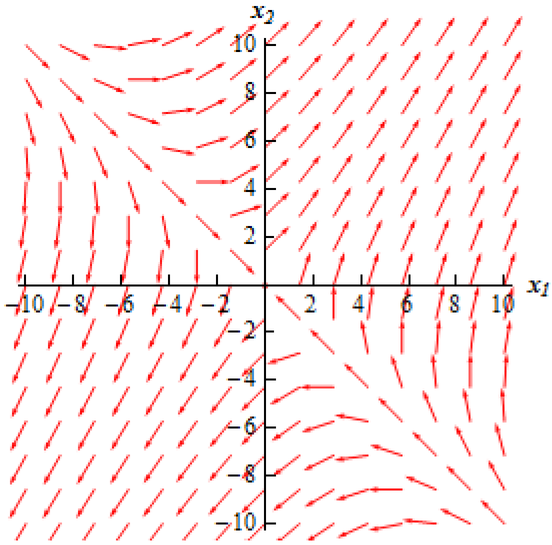
$$\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 \\ -13 \end{pmatrix}$$

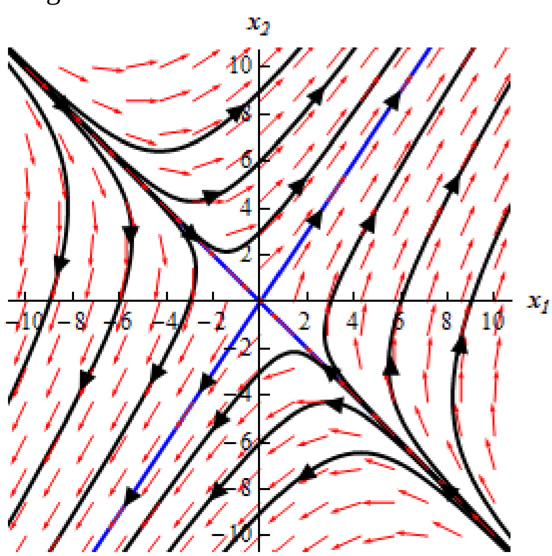
So, what does this tell us? Well at the point (-1, 1) in the phase plane there will be a vector pointing in the direction $\langle 1, -1 \rangle$. At the point (2,0) there will be a vector pointing in the direction $\langle 2, 6 \rangle$. At the point (-3,-2) there will be a vector pointing in the direction $\langle -7, -13 \rangle$.

Doing this for a large number of points in the phase plane will give the following sketch of vectors.



Now all we need to do is sketch in some trajectories. To do this all we need to do is remember that the vectors in the sketch above are tangent to the trajectories. Also the direction of the vectors give the direction of the trajectory as *t* increases so we can show the time dependence of the solution by adding in arrows to the trajectories.

Doing this gives the following sketch.



This sketch is called the **phase portrait**. Usually phase portraits only include the trajectories of the solutions and not any vectors. All of our phase portraits form this point on will only include the trajectories.

In this case it looks like most of the solutions will start away from the equilibrium solution then as *t* starts to increase they move in towards the equilibrium solution and then eventually start moving away from the equilibrium solution again.

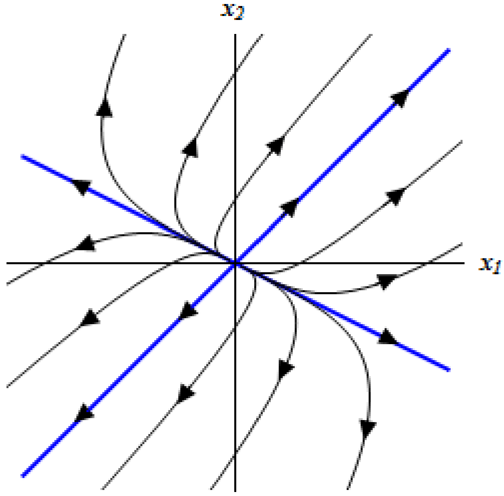
There seem to be four solutions that have slightly different behaviors. It looks like two of the solutions will start at (or near at least) the equilibrium solution and them move straight away from it while two other solution start away from the equilibrium solution and then move straight in towards the equilibrium solution.

In these kinds of cases we call the equilibrium point a **saddle point** and we call the equilibrium point in this case **unstable** since all but two of the solutions are moving away from it as t increases.

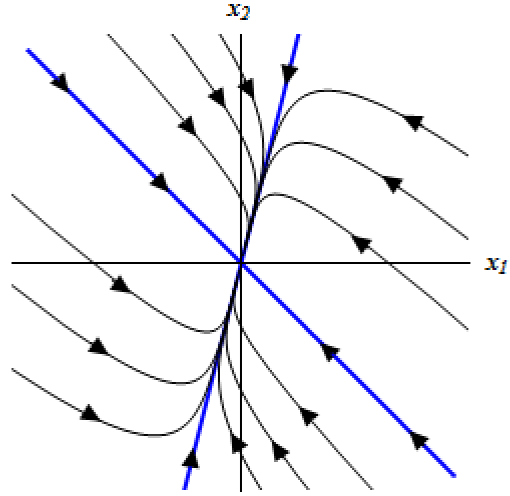
As we noted earlier this is not generally the way that we will sketch trajectories. All we really need to get the trajectories are the eigenvalues and eigenvectors of the matrix A. We will see how to do this over the next couple of sections as we solve the systems.

Here are a few more phase portraits so you can see some more possible examples. We'll actually be generating several of these throughout the course of the next couple of sections.

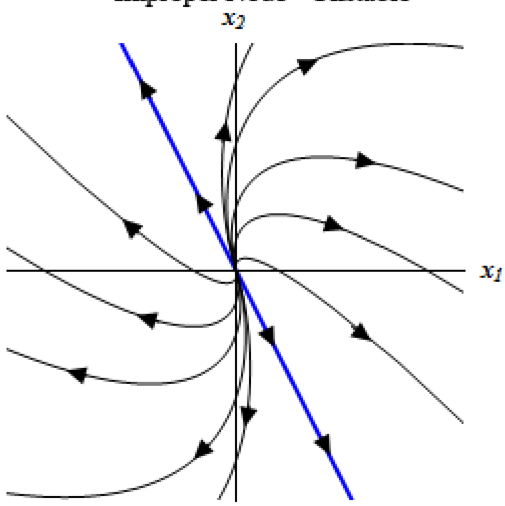
Node - Unstable



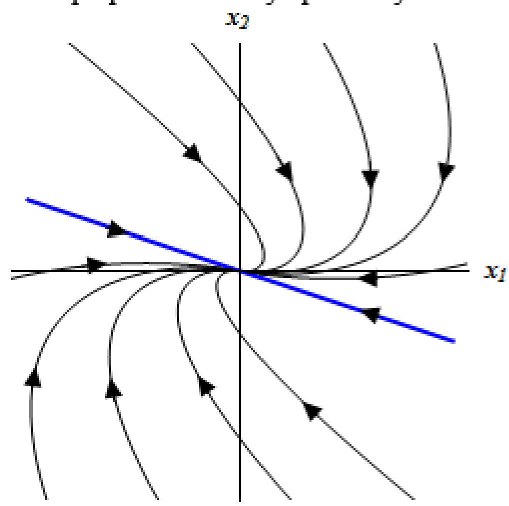
Node - Asymptotically Stable

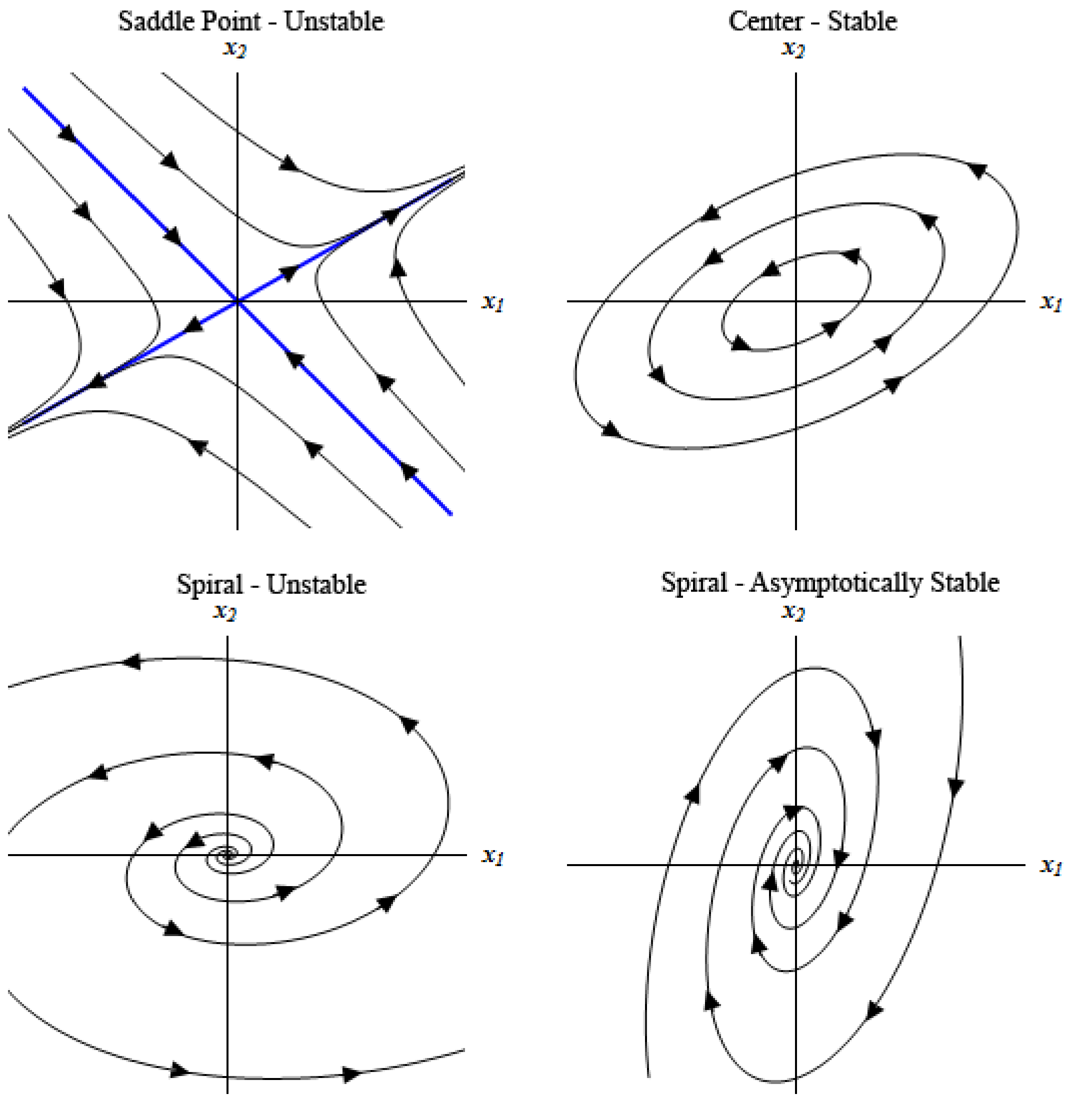


Improper Node - Unstable



Improper Node - Asymptotically Stable





Not all possible phase portraits have been shown here. These are here to show you some of the possibilities. Make sure to notice that several kinds can be either asymptotically stable or unstable depending upon the direction of the arrows.

Notice the difference between **stable** and **asymptotically stable**. In an asymptotically stable node or spiral all the trajectories will move in towards the equilibrium point as t increases, whereas a center (which is always stable) trajectory will just move around the equilibrium point but never actually move in towards it.

Real, Distinct Eigenvalues

It's now time to start solving systems of differential equations. We've seen that solutions to the system,

$$\vec{x}' = A\vec{x}$$

will be of the form

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

where λ and $\vec{\eta}$ are eigenvalues and eigenvectors of the matrix A. We will be working with 2 x 2 systems so this means that we are going to be looking for two solutions, $\vec{x}_1(t)$ and $\vec{x}_2(t)$, where the determinant of the matrix,

$$X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 \end{pmatrix}$$

is nonzero.

We are going to start by looking at the case where our two eigenvalues, λ_1 and λ_2 are real and distinct. In other words they will be real, simple eigenvalues. Recall as well that the eigenvectors for simple eigenvalues are linearly independent. This means that the solutions we get from these will also be linearly independent. If the solutions are linearly independent the matrix X must be nonsingular and hence these two solutions will be a fundamental set of solutions. The general solution in this case will then be,

$$\vec{x}(t) = c_1\mathbf{e}^{\lambda_1 t}\vec{\eta}^{(1)} + c_2\mathbf{e}^{\lambda_2 t}\vec{\eta}^{(2)}$$

Note that each of our examples will actually be broken into two examples. The first example will be solving the system and the second example will be sketching the phase portrait for the system. Phase portraits are not always taught in a differential equations course and so we'll strip those out of the solution process so that if you haven't covered them in your class you can ignore the phase portrait example for the system.

Example 1 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Solution

So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda+1)(\lambda-4) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = 4 \end{aligned}$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:

We'll need to solve,

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

$\lambda_2 = 4$:

We'll need to solve,

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad -3\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = \frac{2}{3}\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \frac{2}{3}\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \eta_2 = 3$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

All we need to do now is multiply the constants through and we then get two equations (one for each row) that we can solve for the constants. This gives,

$$\left.\begin{aligned} -c_1 + 2c_2 &= 0 \\ c_1 + 3c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{8}{5}, \; c_2 = -\frac{4}{5}$$

The solution is then,

$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, let's take a look at the phase portrait for the system.

Example 2 Sketch the phase portrait for the following system.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\vec{x}$$

Solution

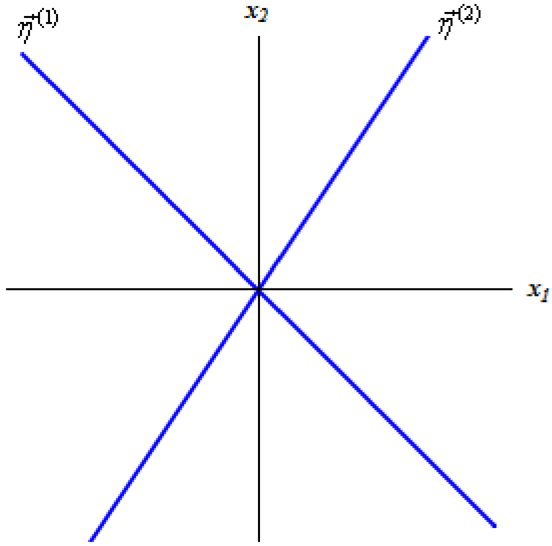
From the last example we know that the eigenvalues and eigenvectors for this system are,

$$\lambda_1 = -1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 4 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

It turns out that this is all the information that we will need to sketch the direction field. We will relate things back to our solution however so that we can see that things are going correctly.

We'll start by sketching lines that follow the direction of the two eigenvectors. This gives,



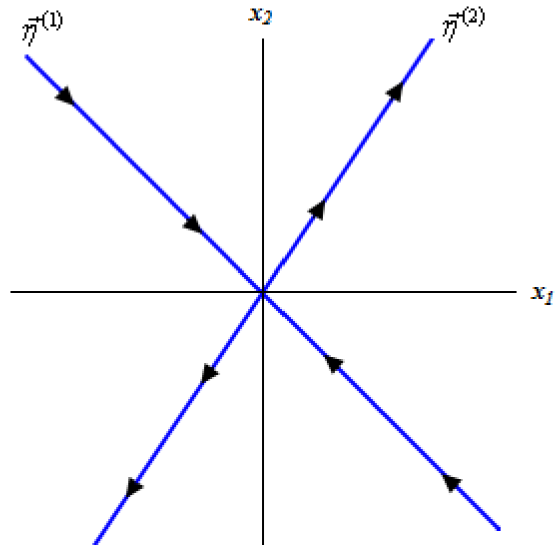
Now, from the first example our general solution is

$$\vec{x}(t) = c_1 \mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

If we have $c_2 = 0$ then the solution is an exponential times a vector and all that the exponential does is affect the magnitude of the vector and the constant c_1 will affect both the sign and the magnitude of the vector. In other words, the trajectory in this case will be a straight line that is parallel to the vector, $\vec{\eta}^{(1)}$. Also notice that as t increases the exponential will get smaller and smaller and hence the trajectory will be moving in towards the origin. If $c_1>0$ the trajectory will be in Quadrant II and if $c_1<0$ the trajectory will be in Quadrant IV.

So the line in the graph above marked with $\vec{\eta}^{(1)}$ will be a sketch of the trajectory corresponding to $c_2 = 0$ and this trajectory will approach the origin as t increases.

If we now turn things around and look at the solution corresponding to having $c_1 = 0$ we will have a trajectory that is parallel to $\vec{\eta}^{(2)}$. Also, since the exponential will increase as t increases and so in this case the trajectory will now move away from the origin as t increases. We will denote this with arrows on the lines in the graph above.

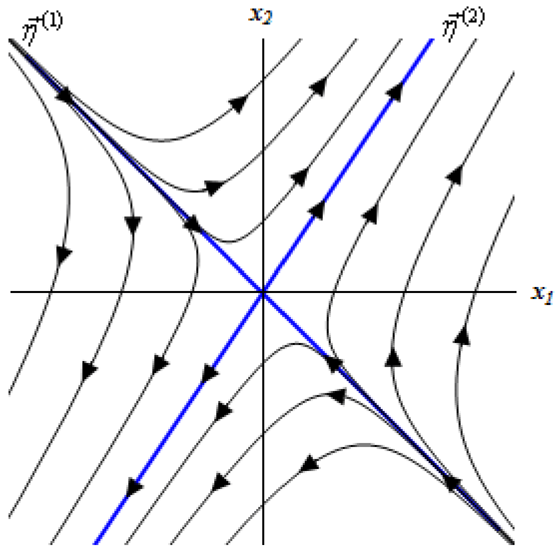


Notice that we could have gotten this information with actually going to the solution. All we really need to do is look at the eigenvalues. Eigenvalues that are negative will correspond to solutions that will move towards the origin as *t* increases in a direction that is parallel to its eigenvector. Likewise, eigenvalues that are positive move away from the origin as *t* increases in a direction that will be parallel to its eigenvector.

If both constants are in the solution we will have a combination of these behaviors. For large negative *t*'s the solution will be dominated by the portion that has the negative eigenvalue since in these cases the exponent will be large and positive. Trajectories for large negative *t*'s will be parallel to $\vec{\eta}^{(1)}$ and moving in the same direction.

Solutions for large positive *t*'s will be dominated by the portion with the positive eigenvalue. Trajectories in this case will be parallel to $\vec{\eta}^{(2)}$ and moving in the same direction.

In general, it looks like trajectories will start "near" $\vec{\eta}^{(1)}$, move in towards the origin and then as they get closer to the origin they will start moving towards $\vec{\eta}^{(2)}$ and then continue up along this vector. Sketching some of these in will give the following phase portrait. Here is a sketch of this with the trajectories corresponding to the eigenvectors marked in blue.



In this case the equilibrium solution (0,0) is called a **saddle point** and is unstable. In this case unstable means that solutions move away from it as t increases.

So, we've solved a system in matrix form, but remember that we started out without the systems in matrix form. Now let's take a quick look at an example of a system that isn't in matrix form initially.

Example 3 Find the solution to the following system.

$$x_1' = x_1 + 2x_2 \qquad x_1(0) = 0$$
$$x_2' = 3x_1 + 2x_2 \qquad x_2(0) = -4$$

Solution
We first need to convert this into matrix form. This is easy enough. Here is the matrix form of the system.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

This is just the system from the first example and so we've already got the solution to this system. Here it is.

$$\vec{x}(t) = -\frac{8}{5}\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5}\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, since we want the solution to the system not in matrix form let's go one step farther here. Let's multiply the constants and exponentials into the vectors and then add up the two vectors.

$$\vec{x}(t) = \begin{pmatrix} \frac{8}{5}\mathbf{e}^{-t} \\ -\frac{8}{5}\mathbf{e}^{-t} \end{pmatrix} - \begin{pmatrix} \frac{8}{5}\mathbf{e}^{4t} \\ \frac{12}{5}\mathbf{e}^{4t} \end{pmatrix} = \begin{pmatrix} \frac{8}{5}\mathbf{e}^{-t} - \frac{8}{5}\mathbf{e}^{4t} \\ -\frac{8}{5}\mathbf{e}^{-t} - \frac{12}{5}\mathbf{e}^{4t} \end{pmatrix}$$

Now, recall,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

So, the solution to the system is then,

$$x_1(t) = \frac{8}{5}\mathbf{e}^{-t} - \frac{8}{5}\mathbf{e}^{4t}$$

$$x_2(t) = -\frac{8}{5}\mathbf{e}^{-t} - \frac{12}{5}\mathbf{e}^{4t}$$

Let's work another example.

Example 4 Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution

So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= \lambda^2 + 7\lambda + 6 \\ &= (\lambda+1)(\lambda+6) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = -6 \end{aligned}$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:

We'll need to solve,

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -4\eta_1 + \eta_2 = 0 \quad \Rightarrow \quad \eta_2 = 4\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ 4\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = -6$:

We'll need to solve,

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now solve the system for the constants.

$$\left.\begin{aligned} c_1 - c_2 &= 1 \\ 4c_1 + c_2 &= 2 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{3}{5}, \ c_2 = -\frac{2}{5}$$

The solution is then,

$$\vec{x}(t) = \frac{3}{5} \mathbf{e}^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{2}{5} \mathbf{e}^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now let's find the phase portrait for this system.

Example 5 Sketch the phase portrait for the following system.

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}$$

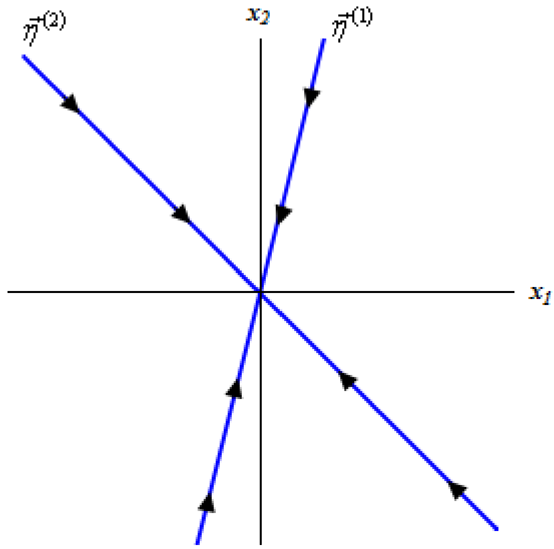
Solution

From the last example we know that the eigenvalues and eigenvectors for this system are,

$$\lambda_1 = -1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\lambda_2 = -6 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

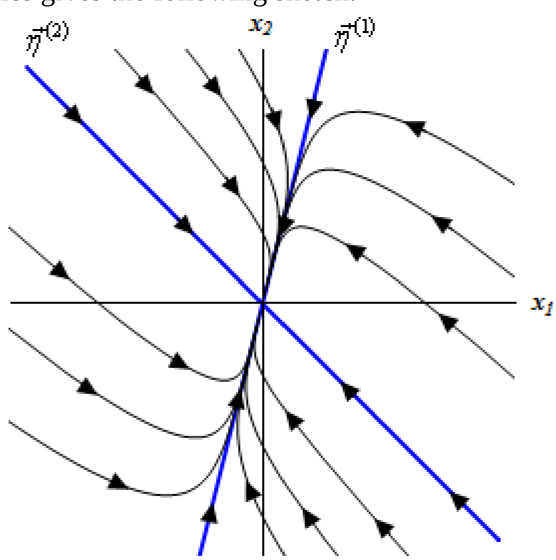
This one is a little different from the first one. However it starts in the same way. We'll first sketch the trajectories corresponding to the eigenvectors. Notice as well that both of the eigenvalues are negative and so trajectories for these will move in towards the origin as t increases. When we sketch the trajectories we'll add in arrows to denote the direction they take as t increases. Here is the sketch of these trajectories.



Now, here is where the slight difference from the first phase portrait comes up. All of the trajectories will move in towards the origin as t increases since both of the eigenvalues are negative. The issue that we need to decide upon is just how they do this. This is actually easier than it might appear to be at first.

The second eigenvalue is larger than the first. For large and positive t's this means that the solution for this eigenvalue will be smaller than the solution for the first eigenvalue. Therefore, as t increases the trajectory will move in towards the origin and do so parallel to $\vec{\eta}^{(1)}$. Likewise, since the second eigenvalue is larger than the first this solution will dominate for large and negative t's. Therefore, as we decrease t the trajectory will move away from the origin and do so parallel to $\vec{\eta}^{(2)}$.

Adding in some trajectories gives the following sketch.



In these cases we call the equilibrium solution (0,0) a **node** and it is asymptotically stable. Equilibrium solutions are asymptotically stable if all the trajectories move in towards it as t increases.

Note that nodes can also be unstable. In the last example if both of the eigenvalues had been positive all the trajectories would have moved away from the origin and in this case the equilibrium solution would have been unstable.

Before moving on to the next section we need to do one more example. When we first started talking about systems it was mentioned that we can convert a higher order differential equation into a system. We need to do an example like this so we can see how to solve higher order differential equations using systems.

Example 6 Convert the following differential equation into a system, solve the system and use this solution to get the solution to the original differential equation.

$$2y'' + 5y' - 3y = 0, \qquad\qquad y(0) = -4 \quad y'(0) = 9$$

Solution

So, we first need to convert this into a system. Here's the change of variables,

$$\begin{aligned} x_1 &= y & x_1' &= y' = x_2 \\ x_2 &= y' & x_2' &= y'' = \frac{3}{2}y - \frac{5}{2}y' = \frac{3}{2}x_1 - \frac{5}{2}x_2 \end{aligned}$$

The system is then,

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} -4 \\ 9 \end{pmatrix}$$

where,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

Now we need to find the eigenvalues for the matrix.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ \frac{3}{2} & -\frac{5}{2} - \lambda \end{vmatrix} \\ &= \lambda^2 + \tfrac{5}{2}\lambda - \tfrac{3}{2} \\ &= \tfrac{1}{2}(\lambda + 3)(2\lambda - 1) \qquad\qquad \lambda_1 = -3,\ \lambda_2 = \frac{1}{2} \end{aligned}$$

Now let's find the eigenvectors.

$\lambda_1 = -3$:

We'll need to solve,

$$\begin{pmatrix} 3 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 3\eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_2 = -3\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -3\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad \eta_1 = 1$$

$\lambda_2 = \frac{1}{2}$:

We'll need to solve,

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \qquad -\frac{1}{2}\eta_1 + \eta_2 = 0 \qquad \Rightarrow \qquad \eta_2 = \frac{1}{2}\eta_1$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \frac{1}{2}\eta_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \eta_1 = 2$$

The general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \mathbf{e}^{\frac{t}{2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Apply the initial condition.

$$\begin{pmatrix} -4 \\ 9 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

This gives the system of equations that we can solve for the constants.

$$\left.\begin{aligned} c_1 + 2c_2 &= -4 \\ -3c_1 + c_2 &= 9 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{22}{7}, \; c_2 = -\frac{3}{7}$$

The actual solution to the system is then,

$$\vec{x}(t) = -\frac{22}{7}\mathbf{e}^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \frac{3}{7}\mathbf{e}^{\frac{t}{2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now recalling that,

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

we can see that the solution to the original differential equation is just the top row of the solution to the matrix system. The solution to the original differential equation is then,

$$y(t) = -\frac{22}{7}\mathbf{e}^{-3t} - \frac{6}{7}\mathbf{e}^{\frac{t}{2}}$$

Notice that as a check, in this case, the bottom row should be the derivative of the top row.

Complex Eigenvalues

In this section we will look at solutions to

$$\vec{x}' = A\vec{x}$$

where the eigenvalues of the matrix *A* are complex. With complex eigenvalues we are going to have the same problem that we had back when we were looking at second order differential equations. We want our solutions to only have real numbers in them, however since our solutions to systems are of the form,

$$\vec{x} = \vec{\eta}\mathbf{e}^{\lambda t}$$

we are going to have complex numbers come into our solution from both the eigenvalue and the eigenvector. Getting rid of the complex numbers here will be similar to how we did it back in the second order differential equation case, but will involve a little more work this time around. It's easiest to see how to do this in an example.

Example 1 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution

We first need the eigenvalues and eigenvectors for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -9 \\ 4 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 27 \qquad \lambda_{1,2} = \pm 3\sqrt{3}\,i$$

So, now that we have the eigenvalues recall that we only need to get the eigenvector for one of the eigenvalues since we can get the second eigenvector for free from the first eigenvector.

$\lambda_1 = 3\sqrt{3}\,i$:

We need to solve the following system.

$$\begin{pmatrix} 3-3\sqrt{3}\,i & -9 \\ 4 & -3-3\sqrt{3}\,i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the first equation we get,

$$\left(3-3\sqrt{3}\,i\right)\eta_1 - 9\eta_2 = 0$$

$$\eta_2 = \frac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \frac{1}{3}\left(1-\sqrt{3}\,i\right)\eta_1 \end{pmatrix}$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix} \qquad \eta_1 = 3$$

When finding the eigenvectors in these cases make sure that the complex number appears in the numerator of any fractions since we'll need it in the numerator later on. Also try to clear out any fractions by appropriately picking the constant. This will make our life easier down the road.

Now, the second eigenvector is,

$$\vec{\eta}^{(2)} = \begin{pmatrix} 3 \\ 1+\sqrt{3}\, i \end{pmatrix}$$

However, as we will see we won't need this eigenvector.

The solution that we get from the first eigenvalue and eigenvector is,

$$\vec{x}_1(t) = \mathbf{e}^{3\sqrt{3}\, i\, t} \begin{pmatrix} 3 \\ 1-\sqrt{3}\, i \end{pmatrix}$$

So, as we can see there are complex numbers in both the exponential and vector that we will need to get rid of in order to use this as a solution. Recall from the complex roots section of the second order differential equation chapter that we can use Euler's formula to get the complex number out of the exponential. Doing this gives us,

$$\vec{x}_1(t) = \left(\cos\left(3\sqrt{3}t\right) + i\sin\left(3\sqrt{3}t\right)\right) \begin{pmatrix} 3 \\ 1-\sqrt{3}\, i \end{pmatrix}$$

The next step is to multiply the cosines and sines into the vector.

$$\vec{x}_1(t) = \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) + 3i\sin\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + i\sin\left(3\sqrt{3}t\right) - \sqrt{3}\, i\cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix}$$

Now combine the terms with an "*i*" in them and split these terms off from those terms that don't contain an "*i*". Also factor the "*i*" out of this vector.

$$\begin{aligned} \vec{x}_1(t) &= \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + i \begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sin\left(3\sqrt{3}t\right) - \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix} \\ &= \vec{u}(t) + i\,\vec{v}(t) \end{aligned}$$

Now, it can be shown (we'll leave the details to you) that $\vec{u}(t)$ and $\vec{v}(t)$ are two linearly independent solutions to the system of differential equations. This means that we can use them to form a general solution and they are both real solutions.

So, the general solution to a system with complex roots is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

where $\vec{u}(t)$ and $\vec{v}(t)$ are found by writing the first solution as

$$\vec{x}(t) = \vec{u}(t) + i\,\vec{v}(t)$$

For our system then, the general solution is,

$$\vec{x}(t) = c_1 \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + c_2 \begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sin\left(3\sqrt{3}t\right) - \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

We now need to apply the initial condition to this to find the constants.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$$

This leads to the following system of equations to be solved,

$$\left.\begin{aligned} 3c_1 &= 2 \\ c_1 - \sqrt{3}c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{3}, \;\; c_2 = \frac{14}{3\sqrt{3}}$$

The actual solution is then,

$$\vec{x}(t) = \frac{2}{3} \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ \cos\left(3\sqrt{3}t\right) + \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + \frac{14}{3\sqrt{3}} \begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sin\left(3\sqrt{3}t\right) - \sqrt{3}\cos\left(3\sqrt{3}t\right) \end{pmatrix}$$

As we did in the last section we'll do the phase portraits separately from the solution of the system in case phase portraits haven't been taught in your class.

Example 2 Sketch the phase portrait for the system.

$$\vec{x}\,' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \vec{x}$$

Solution

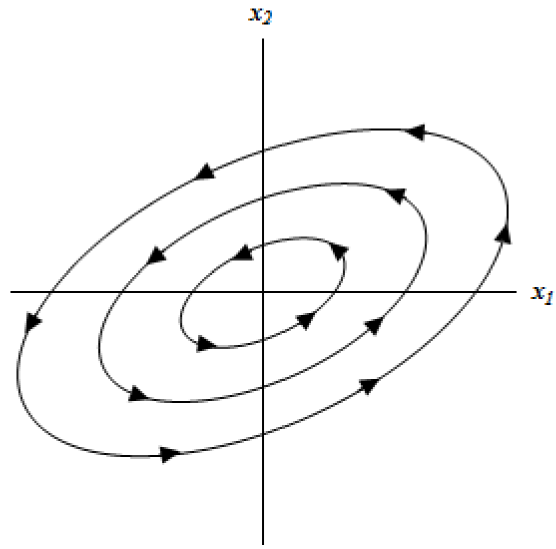
When the eigenvalues of a matrix A are purely complex, as they are in this case, the trajectories of the solutions will be circles or ellipses that are centered at the origin. The only thing that we really need to concern ourselves with here are whether they are rotating in a clockwise or counterclockwise direction.

This is easy enough to do. Recall when we first looked at these phase portraits a couple of sections ago that if we pick a value of $\vec{x}(t)$ and plug it into our system we will get a vector that will be tangent to the trajectory at that point and pointing in the direction that the trajectory is traveling.. So, let's pick the following point and see what we get.

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \vec{x}\,' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Therefore at the point (1,0) in the phase plane the trajectory will be pointing in a upwards direction. The only way that this can be is if the trajectories are traveling in a counterclockwise direction.

Here is the sketch of some of the trajectories for this problem.



The equilibrium solution in the case is called a **center** and is stable.

Note in this last example that the equilibrium solution is stable and not asymptotically stable. Asymptotically stable refers to the fact that the trajectories are moving in toward the equilibrium solution as *t* increases. In this example the trajectories are simply revolving around the equilibrium solution and not moving in towards it. The trajectories are also not moving away from the equilibrium solution and so they aren't unstable. Therefore we call the equilibrium solution stable.

Not all complex eigenvalues will result in centers so let's take a look at an example where we get something different.

Example 3 Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\vec{x} \qquad\qquad \vec{x}\left(0\right) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

Solution

Let's get the eigenvalues and eigenvectors for the matrix.

$$\begin{aligned} \det\left(A - \lambda I\right) &= \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} \\ &= \lambda^2 - 4\lambda + 68 \qquad\qquad \lambda_{1,2} = 2 \pm 8i \end{aligned}$$

Now get the eigenvector for the first eigenvalue.

$\lambda_1 = 2 + 8i$:

We need to solve the following system.

$$\begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the second equation we get,

$$5\eta_1 + \left(-1-8i\right)\eta_2 = 0$$
$$\eta_1 = \frac{1}{5}\left(1+8i\right)\eta_2$$

So, the first eigenvector is,

$$\vec{\eta} = \begin{pmatrix} \frac{1}{5}\left(1+8i\right)\eta_2 \\ \eta_2 \end{pmatrix}$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix} \qquad\qquad \eta_2 = 5$$

The solution corresponding the this eigenvalue and eigenvector is

$$\begin{aligned}\vec{x}_1\left(t\right) &= \mathbf{e}^{(2+8i)t}\begin{pmatrix} 1+8i \\ 5 \end{pmatrix} \\ &= \mathbf{e}^{2t}\mathbf{e}^{8it}\begin{pmatrix} 1+8i \\ 5 \end{pmatrix} \\ &= \mathbf{e}^{2t}\left(\cos\left(8t\right)+i\sin\left(8t\right)\right)\begin{pmatrix} 1+8i \\ 5 \end{pmatrix}\end{aligned}$$

As with the first example multiply cosines and sines into the vector and split it up. Don't forget about the exponential that is in the solution this time.

$$\begin{aligned}\vec{x}_1\left(t\right) &= \mathbf{e}^{2t}\begin{pmatrix} \cos\left(8t\right)-8\sin\left(8t\right) \\ 5\cos\left(8t\right) \end{pmatrix} + i\mathbf{e}^{2t}\begin{pmatrix} 8\cos\left(8t\right)+\sin\left(8t\right) \\ 5\sin\left(8t\right) \end{pmatrix} \\ &= \vec{u}\left(t\right) + i\,\vec{v}\left(t\right)\end{aligned}$$

The general solution to this system then,

$$\vec{x}\left(t\right) = c_1\mathbf{e}^{2t}\begin{pmatrix} \cos\left(8t\right)-8\sin\left(8t\right) \\ 5\cos\left(8t\right) \end{pmatrix} + c_2\mathbf{e}^{2t}\begin{pmatrix} 8\cos\left(8t\right)+\sin\left(8t\right) \\ 5\sin\left(8t\right) \end{pmatrix}$$

Now apply the initial condition and find the constants.

$$\begin{pmatrix} 3 \\ -10 \end{pmatrix} = \vec{x}\left(0\right) = c_1\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$
$$\left.\begin{aligned} c_1 + 8c_2 &= 3 \\ 5c_1 &= -10 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -2,\ \ c_2 = \frac{5}{8}$$

The actual solution is then,

$$\vec{x}\left(t\right) = -2\mathbf{e}^{2t}\begin{pmatrix} \cos\left(8t\right)-8\sin\left(8t\right) \\ 5\cos\left(8t\right) \end{pmatrix} + \frac{5}{8}\mathbf{e}^{2t}\begin{pmatrix} 8\cos\left(8t\right)+\sin\left(8t\right) \\ 5\sin\left(8t\right) \end{pmatrix}$$

Let's take a look at the phase portrait for this problem.

Example 4 Sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x}$$

Solution

When the eigenvalues of a system are complex with a real part the trajectories will spiral into or out of the origin. We can determine which one it will be by looking at the real portion. Since the real portion will end up being the exponent of an exponential function (as we saw in the solution to this system) if the real part is positive the solution will grow very large as t increases. Likewise, if the real part is negative the solution will die out as t increases.

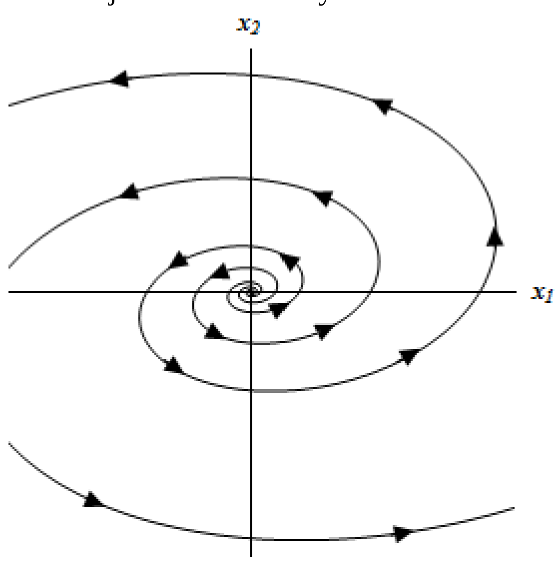
So, if the real part is positive the trajectories will spiral out from the origin and if the real part is negative they will spiral into the origin. We determine the direction of rotation (clockwise vs. counterclockwise) in the same way that we did for the center.

In our case the trajectories will spiral out from the origin since the real part is positive and

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

will rotate in the counterclockwise direction as the last example did.

Here is a sketch of some of the trajectories for this system.



Here we call the equilibrium solution a **spiral** (oddly enough…) and in this case it's unstable since the trajectories move away from the origin.

If the real part of the eigenvalue is negative the trajectories will spiral into the origin and in this case the equilibrium solution will be asymptotically stable.

Repeated Eigenvalues

This is the final case that we need to take a look at. In this section we are going to look at solutions to the system,

$$\vec{x}\,' = A\vec{x}$$

where the eigenvalues are repeated eigenvalues. Since we are going to be working with systems in which A is a 2 x 2 matrix we will make that assumption from the start. So the system will have a double eigenvalue, λ.

This presents us with a problem. We want two linearly independent solutions so that we can form a general solution. However, with a double eigenvalue we will have only one,

$$\vec{x}_1 = \vec{\eta}\,\mathbf{e}^{\lambda t}$$

So, we need to come up with a second solution. Recall that when we looked at the double root case with the second order differential equations we ran into a similar problem. In that section we simply added a t to the solution and were able to get a second solution. Let’s see if the same thing will work in this case as well. We’ll see if

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta}$$

will also be a solution.

To check all we need to do is plug into the system. Don’t forget to product rule the proposed solution when you differentiate!

$$\vec{\eta}\,\mathbf{e}^{\lambda t} + \lambda\vec{\eta}\,t\,\mathbf{e}^{\lambda t} = A\vec{\eta}\,t\,\mathbf{e}^{\lambda t}$$

Now, we got two functions here on the left side, an exponential by itself and an exponential times a t. So, in order for our guess to be a solution we will need to require,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \Rightarrow \qquad \left(A - \lambda I\right)\vec{\eta} = \vec{0}$$
$$\vec{\eta} = \vec{0}$$

The first requirement isn’t a problem since this just says that λ is an eigenvalue and it’s eigenvector is $\vec{\eta}$. We already knew this however so there’s nothing new there. The second however is a problem. Since $\vec{\eta}$ is an eigenvector we know that it can’t be zero, yet in order to satisfy the second condition it would have to be.

So, our guess was incorrect. The problem seems to be that there is a lone term with just an exponential in it so let’s see if we can’t fix up our guess to correct that. Let’s try the following guess.

$$\vec{x} = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

where $\vec{\rho}$ is an unknown vector that we’ll need to determine.

As with the first guess let’s plug this into the system and see what we get.

$$\vec{\eta}\,\mathbf{e}^{\lambda t} + \lambda\vec{\eta}\,t\,\mathbf{e}^{\lambda t} + \lambda\vec{\rho}\,\mathbf{e}^{\lambda t} = A\left(\vec{\eta}\,t\,\mathbf{e}^{\lambda t} + \vec{\rho}\,\mathbf{e}^{\lambda t}\right)$$
$$\left(\vec{\eta} + \lambda\vec{\rho}\right)\mathbf{e}^{\lambda t} + \lambda\vec{\eta}\,t\,\mathbf{e}^{\lambda t} = A\vec{\eta}\,t\,\mathbf{e}^{\lambda t} + A\vec{\rho}\,\mathbf{e}^{\lambda t}$$

Now set coefficients equal again,

$$\lambda\vec{\eta} = A\vec{\eta} \qquad \Rightarrow \qquad (A-\lambda I)\vec{\eta} = \vec{0}$$
$$\vec{\eta} + \lambda\vec{\rho} = A\vec{\rho} \qquad \Rightarrow \qquad (A-\lambda I)\vec{\rho} = \vec{\eta}$$

As with our first guess the first equation tells us nothing that we didn't already know. This time the second equation is not a problem. All the second equation tells us is that $\vec{\rho}$ must be a solution to this equation.

It looks like our second guess worked. Therefore,

$$\vec{x}_2 = t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}$$

will be a solution to the system provided $\vec{\rho}$ is a solution to

$$(A-\lambda I)\vec{\rho} = \vec{\eta}$$

Also this solution and the first solution are linearly independent and so they form a fundamental set of solutions and so the general solution in the double eigenvalue case is,

$$\vec{x} = c_1\mathbf{e}^{\lambda t}\vec{\eta} + c_2\left(t\,\mathbf{e}^{\lambda t}\vec{\eta} + \mathbf{e}^{\lambda t}\vec{\rho}\right)$$

Let's work an example.

Example 1 Solve the following IVP.

$$\vec{x}\,' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x} \qquad\qquad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Solution

First find the eigenvalues for the system.

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} \\ &= \lambda^2 - 10\lambda + 25 \\ &= (\lambda-5)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = 5 \end{aligned}$$

So, we got a double eigenvalue. Of course that shouldn't be too surprising given the section that we're in. Let's find the eigenvector for this eigenvalue.

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + \eta_2 = 0 \qquad \eta_2 = -2\eta_1$$

The eigenvector is then,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ -2\eta_1 \end{pmatrix} \qquad\qquad \eta_1 \neq 0$$
$$\vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad\qquad \eta_1 = 1$$

The next step is find $\vec{\rho}$. To do this we'll need to solve,

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \Rightarrow \qquad 2\rho_1 + \rho_2 = 1 \qquad \rho_2 = 1 - 2\rho_1$$

Note that this is almost identical to the system that we solve to find the eigenvalue. The only difference is the right hand side. The most general possible $\vec{\rho}$ is

$$\vec{\rho} = \begin{pmatrix} \rho_1 \\ 1 - 2\rho_1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{if } \rho_1 = 0$$

In this case, unlike the eigenvector system we can choose the constant to be anything we want, so we might as well pick it to make our life easier. This usually means picking it to be zero.

We can now write down the general solution to the system.

$$\vec{x}(t) = c_1 \mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\left(\mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Applying the initial condition to find the constants gives us,

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} c_1 &= 2 \\ -2c_1 + c_2 &= -5 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = 2, \; c_2 = -1$$

The actual solution is then,

$$\begin{aligned} \vec{x}(t) &= 2\mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(t\mathbf{e}^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \mathbf{e}^{5t}\begin{pmatrix} 2 \\ -4 \end{pmatrix} - \mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \mathbf{e}^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \mathbf{e}^{5t}\begin{pmatrix} 2 \\ -5 \end{pmatrix} - \mathbf{e}^{5t} t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

Note that we did a little combining here to simplify the solution up a little.

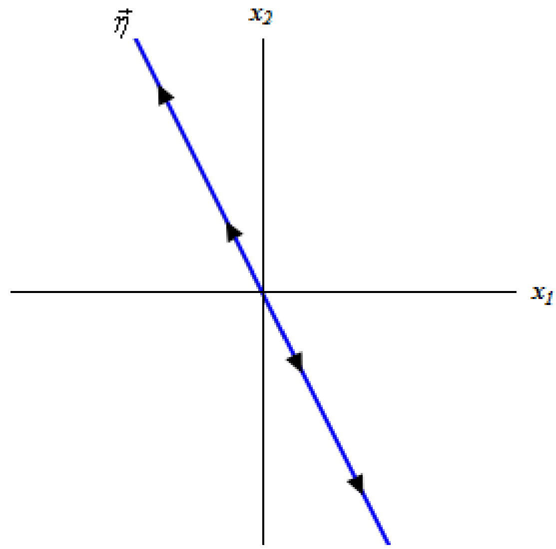
So, the next example will be to sketch the phase portrait for this system.

Example 2 Sketch the phase portrait for the system.

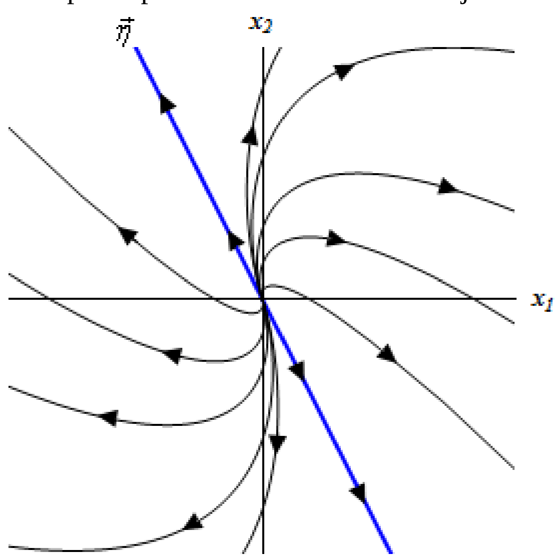
$$\vec{x}\,' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\vec{x}$$

Solution

These will start in the same way that real, distinct eigenvalue phase portraits start. We'll first sketch in a trajectory that is parallel to the eigenvector and note that since the eigenvalue is positive the trajectory will be moving away from the origin.



Now, it will be easier to explain the remainder of the phase portrait if we actually have one in front of us. So here is the full phase portrait with some more trajectories sketched in.



Trajectories in these cases always emerge from (or move into) the origin in a direction that is parallel to the eigenvector. Likewise they will start in one direction before turning around and moving off into the other direction. The directions in which they move are opposite depending on which side of the trajectory corresponding to the eigenvector we are on. Also, as the trajectories move away from the origin it should start becoming parallel to the trajectory corresponding to the eigenvector.

So, how do we determine the direction? We can do the same thing that we did in the complex case. We'll plug in (1,0) into the system and see which direction the trajectories are moving at that point. Since this point is directly to the right of the origin the trajectory at that point must have already turned around and so this will give the direction that it will traveling after turning around.

Doing that for this problem to check our phase portrait gives,

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

This vector will point down into the fourth quadrant and so the trajectory must be moving into the fourth quadrant as well. This does match up with our phase portrait.

In these cases the equilibrium is called a **node** and is unstable in this case. Note that sometimes you will hear nodes for the repeated eigenvalue case called **degenerate nodes** or **improper nodes**.

Let's work one more example.

Example 3 Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix}\vec{x} \qquad\qquad \vec{x}(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

First the eigenvalue for the system.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix} \\ &= \lambda^2 + 3\lambda + \frac{9}{4} \\ &= \left(\lambda + \frac{3}{2}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = -\frac{3}{2} \end{aligned}$$

Now let's get the eigenvector.

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\eta_1 + \frac{3}{2}\eta_2 = 0 \qquad \eta_1 = -3\eta_2$$

$$\vec{\eta} = \begin{pmatrix} -3\eta_2 \\ \eta_2 \end{pmatrix} \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \eta_2 = 1$$

Now find $\vec{\rho}$,

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \frac{1}{2}\rho_1 + \frac{3}{2}\rho_2 = -3 \qquad \rho_1 = -6 - 3\rho_2$$

$$\vec{\rho} = \begin{pmatrix} -6-3\rho_2 \\ \rho_2 \end{pmatrix} \qquad \Rightarrow \qquad \vec{\rho} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \qquad \text{if } \rho_2 = 0$$

The general solution for the system is then,

$$\vec{x}\left(t\right)=c_1\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix}-3\\1\end{pmatrix}+c_2\left(t\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix}-3\\1\end{pmatrix}+\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix}-6\\0\end{pmatrix}\right)$$

Applying the initial condition gives,

$$\begin{pmatrix}1\\0\end{pmatrix}=\vec{x}\left(2\right)=c_1\mathbf{e}^{-3}\begin{pmatrix}-3\\1\end{pmatrix}+c_2\left(2\mathbf{e}^{-3}\begin{pmatrix}-3\\1\end{pmatrix}+\mathbf{e}^{-3}\begin{pmatrix}-6\\0\end{pmatrix}\right)$$

Note that we didn't use *t=0* this time! We now need to solve the following system,

$$\left.\begin{aligned}-3\mathbf{e}^{-3}c_1-12\mathbf{e}^{-3}c_2&=1\\ \mathbf{e}^{-3}c_1+2\mathbf{e}^{-3}c_2&=0\end{aligned}\right\}\qquad\Rightarrow\qquad c_1=\frac{\mathbf{e}^3}{3},\;\;c_2=-\frac{\mathbf{e}^3}{6}$$

The actual solution is then,

$$\begin{aligned}\vec{x}\left(t\right)&=\frac{\mathbf{e}^3}{3}\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix}-3\\1\end{pmatrix}-\frac{\mathbf{e}^3}{6}\left(t\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix}-3\\1\end{pmatrix}+\mathbf{e}^{-\frac{3t}{2}}\begin{pmatrix}-6\\0\end{pmatrix}\right)\\&=\mathbf{e}^{-\frac{3t}{2}+3}\begin{pmatrix}0\\\frac{1}{3}\end{pmatrix}+t\mathbf{e}^{-\frac{3t}{2}+3}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{6}\end{pmatrix}\end{aligned}$$

And just to be consistent with all the other problems that we've done let's sketch the phase portrait.

Example 4 Sketch the phase portrait for the system.

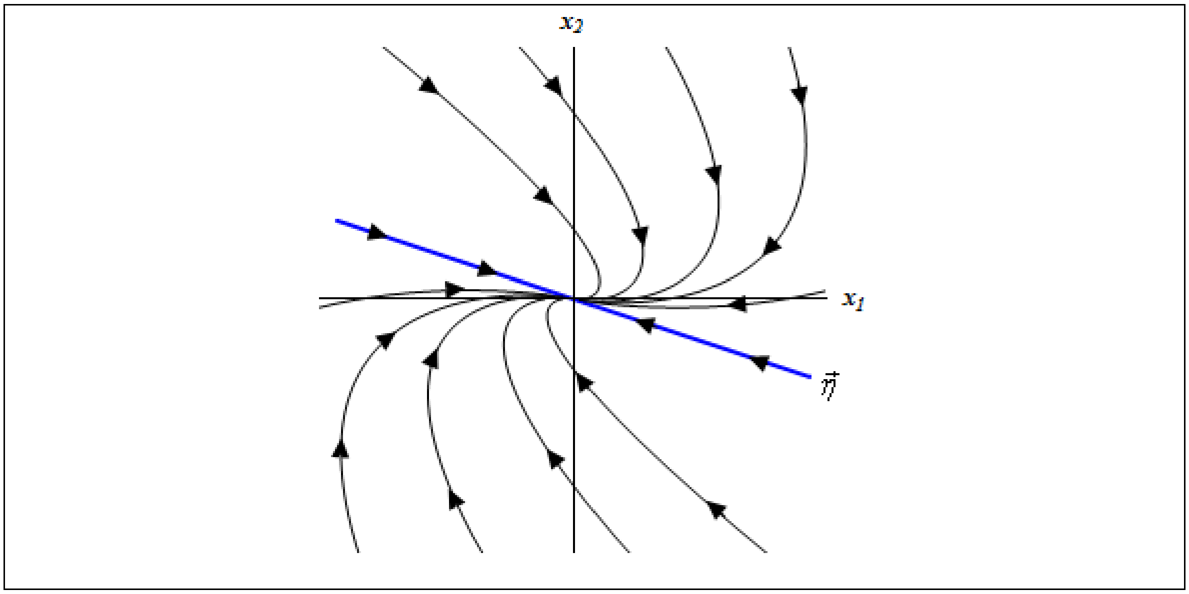
$$\vec{x}'=\begin{pmatrix}-1&\frac{3}{2}\\-\frac{1}{6}&-2\end{pmatrix}\vec{x}$$

Solution

Let's first notice that since the eigenvalue is negative in this case the trajectories should all move in towards the origin. Let's check the direction of the trajectories at (1,0)

$$\begin{pmatrix}-1&\frac{3}{2}\\-\frac{1}{6}&-2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-1\\-\frac{1}{6}\end{pmatrix}$$

So it looks like the trajectories should be pointing into the third quadrant at (1,0). This gives the following phase portrait.



x_2
x_1
$\vec{\eta}$

Nonhomogeneous Systems

We now need to address nonhomogeneous systems briefly. Both of the methods that we looked at back in the second order differential equations chapter can also be used here. As we will see Undetermined Coefficients is almost identical when used on systems while Variation of Parameters will need to have a new formula derived, but will actually be slightly easier when applied to systems.

Undetermined Coefficients

The method of Undetermined Coefficients for systems is pretty much identical to the second order differential equation case. The only difference is that the coefficients will need to be vectors now.

Let's take a quick look at an example.

Example 1 Find the general solution to the following system.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution

We already have the complementary solution as we solved that part back in the real eigenvalue section. It is,

$$\vec{x}_c(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Guessing the form of the particular solution will work in exactly the same way it did back when we first looked at this method. We have a linear polynomial and so our guess will need to be a linear polynomial. The only difference is that the "coefficients" will need to be vectors instead of constants. The particular solution will have the form,

$$\vec{x}_P = t\vec{a} + \vec{b} = t \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

So, we need to differentiate the guess

$$\vec{x}_P' = \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Before plugging into the system let's simplify the notation a little to help with our work. We'll write the system as,

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix} = A\vec{x} + t\vec{g}$$

This will make the following work a little easier. Now, let's plug things into the system.

$$\vec{a} = A\left(t\vec{a} + \vec{b}\right) + t\vec{g}$$
$$\vec{a} = tA\vec{a} + A\vec{b} + t\vec{g}$$
$$\vec{0} = t\left(A\vec{a} + \vec{g}\right) + \left(A\vec{b} - \vec{a}\right)$$

Now we need to set the coefficients equal. Doing this gives,

$$t^1: \qquad A\vec{a} + \vec{g} = \vec{0} \qquad\qquad A\vec{a} = -\vec{g}$$
$$t^0: \qquad A\vec{b} - \vec{a} = \vec{0} \qquad\qquad A\vec{b} = \vec{a}$$

Now only $\vec{a}$ is unknown in the first equation so we can use Gaussian elimination to solve the system. We'll leave this work to you to check.

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\begin{pmatrix} 2 \\ -4 \end{pmatrix} \qquad \Rightarrow \qquad \vec{a} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix}$$

Now that we know $\vec{a}$ we can solve the second equation for $\vec{b}$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} \qquad \Rightarrow \qquad \vec{b} = \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

So, since we were able to solve both equations, the particular solution is then,

$$\vec{x}_P = t\begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

The general solution is then,

$$\vec{x}\left(t\right) = c_1\mathbf{e}^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\mathbf{e}^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -\frac{5}{2} \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix}$$

So, as you can see undetermined coefficients is nearly the same as the first time we saw it. The work in solving for the "constants" is a little messier however.

Variation of Parameters

In this case we will need to derive a new formula for variation of parameters for systems. The derivation this time will be much simpler than the when we first saw variation of parameters.

First let *X(t)* be a matrix whose i^{th} column is the i^{th} linearly independent solution to the system,

$$\vec{x}' = A\vec{x}$$

Now it can be shown that *X(t)* will be a solution to the following differential equation.

$$X' = AX \qquad \textbf{(1)}$$

This is nothing more than the original system with the matrix in place of the original vector.

We are going to try and find a particular solution to

$$\vec{x}' = A\vec{x} + \vec{g}\left(t\right)$$

We will assume that we can find a solution of the form,

$$\vec{x}_P = X(t)\vec{v}(t)$$

where we will need to determine the vector $\vec{v}(t)$. To do this we will need to plug this into the nonhomogeneous system. Don't forget to product rule the particular solution when plugging the guess into the system.

$$X'\vec{v} + X\vec{v}' = A X \vec{v} + \vec{g}$$

Note that we dropped the "*(t)*" part of things to simplify the notation a little. Now using (1) we can rewrite this a little.

$$\begin{aligned} X'\vec{v} + X\vec{v}' &= X'\vec{v} + \vec{g} \\ X\vec{v}' &= \vec{g} \end{aligned}$$

Because we formed *X* using linearly independent solutions we know that det(*X*) must be nonzero and this in turn means that we can find the inverse of *X*. So, multiply both sides by the inverse of *X*.

$$\vec{v}' = X^{-1}\vec{g}$$

Now all that we need to do is integrate both sides to get $\vec{v}(t)$.

$$\vec{v}(t) = \int X^{-1}\vec{g}\,dt$$

As with the second order differential equation case we can ignore any constants of integration. The particular solution is then,

$$\vec{x}_P = X\int X^{-1}\vec{g}\,dt \qquad \textbf{(2)}$$

Let's work a quick example using this.

Example 2 Find the general solution to the following system.

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\vec{x} + \mathbf{e}^{2t}\begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

Solution

We found the complementary solution to this system in the real eigenvalue section. It is,

$$\vec{x}_c(t) = c_1\mathbf{e}^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2\mathbf{e}^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Now the matrix *X* is,

$$X = \begin{pmatrix} \mathbf{e}^{-t} & -\mathbf{e}^{-6t} \\ 4\mathbf{e}^{-t} & \mathbf{e}^{-6t} \end{pmatrix}$$

Now, we need to find the inverse of this matrix. We saw how to find inverses of matrices back in the second linear algebra review section and the process is the same here even though we don't have constant entries. We'll leave the detail to you to check.

$$X^{-1} = \begin{pmatrix} \frac{1}{5}\mathbf{e}^{t} & \frac{1}{5}\mathbf{e}^{t} \\ -\frac{4}{5}\mathbf{e}^{6t} & \frac{1}{5}\mathbf{e}^{6t} \end{pmatrix}$$

Now do the multiplication in the integral.

$$X^{-1}\vec{g} = \begin{pmatrix} \frac{1}{5}\mathbf{e}^{t} & \frac{1}{5}\mathbf{e}^{t} \\ -\frac{4}{5}\mathbf{e}^{6t} & \frac{1}{5}\mathbf{e}^{6t} \end{pmatrix}\begin{pmatrix} 6\mathbf{e}^{2t} \\ -\mathbf{e}^{2t} \end{pmatrix} = \begin{pmatrix} \mathbf{e}^{3t} \\ -5\mathbf{e}^{8t} \end{pmatrix}$$

Now do the integral.

$$\int X^{-1}\vec{g}\,dt = \int \begin{pmatrix} \mathbf{e}^{3t} \\ -5\mathbf{e}^{8t} \end{pmatrix} dt = \begin{pmatrix} \int \mathbf{e}^{3t}\,dt \\ \int -5\mathbf{e}^{8t}\,dt \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\mathbf{e}^{3t} \\ -\frac{5}{8}\mathbf{e}^{8t} \end{pmatrix}$$

Remember that to integrate a matrix or vector you just integrate the individual entries.

We can now get the particular solution.

$$\begin{aligned} \vec{x}_P &= X \int X^{-1}\vec{g}\,dt \\ &= \begin{pmatrix} \mathbf{e}^{-t} & -\mathbf{e}^{-6t} \\ 4\mathbf{e}^{-t} & \mathbf{e}^{-6t} \end{pmatrix}\begin{pmatrix} \frac{1}{3}\mathbf{e}^{3t} \\ -\frac{5}{8}\mathbf{e}^{8t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{23}{24}\mathbf{e}^{2t} \\ \frac{17}{24}\mathbf{e}^{2t} \end{pmatrix} \\ &= \mathbf{e}^{2t}\begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix} \end{aligned}$$

The general solution is then,

$$\vec{x}(t) = c_1\mathbf{e}^{-t}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2\mathbf{e}^{-6t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mathbf{e}^{2t}\begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix}$$

So, some of the work can be a little messy, but overall not too bad.

We looked at two methods of solving nonhomogeneous differential equations here and while the work can be a little messy they aren't too bad. Of course we also kept the nonhomogeneous part fairly simple here. More complicated problems will have significant amounts of work involved.

Laplace Transforms

There's not too much to this section. We're just going to work an example to illustrate how Laplace transforms can be used to solve systems of differential equations.

Example 1 Solve the following system.

$$x_1' = 3x_1 - 3x_2 + 2 \qquad x_1(0) = 1$$
$$x_2' = -6x_1 - t \qquad x_2(0) = -1$$

Solution

First notice that the system is not given in matrix form. This is because the system won't be solved in matrix form. Also note that the system is nonhomogeneous.

We start just as we did when we used Laplace transforms to solve single differential equations. We take the transform of both differential equations.

$$sX_1(s) - x_1(0) = 3X_1(s) - 3X_2(s) + \frac{2}{s}$$
$$sX_2(s) - x_2(0) = -6X_1(s) - \frac{1}{s^2}$$

Now plug in the initial condition and simplify things a little.

$$(s-3)X_1(s) + 3X_2(s) = \frac{2}{s} + 1 = \frac{2+s}{s}$$
$$6X_1(s) + sX_2(s) = -\frac{1}{s^2} - 1 = -\frac{s^2+1}{s^2}$$

Now we need to solve this for one of the transforms. We'll do this by multiplying the top equation by s and the bottom by -3 and then adding. This gives,

$$\left(s^2 - 3s - 18\right)X_1(s) = 2 + s + \frac{3s^2+3}{s^2}$$

Solving for X_1 gives,

$$X_1(s) = \frac{s^3 + 5s^2 + 3}{s^2(s+3)(s-6)}$$

Partial fractioning gives,

$$X_1(s) = \frac{1}{108}\left(\frac{133}{s-6} - \frac{28}{s+3} + \frac{3}{s} - \frac{18}{s^2}\right)$$

Taking the inverse transform gives us the first solution,

$$x_1(t) = \frac{1}{108}\left(133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 - 18t\right)$$

Now to find the second solution we could go back up and eliminate X_1 to find the transform for X_2 and sometimes we would need to do that. However, in this case notice that the second differential equation is,

$$x_2' = -6x_1 - t \qquad \Rightarrow \qquad x_2 = \int -6x_1 - t \, dt$$

So, plugging the first solution in and integrating gives,

$$\begin{aligned} x_2(t) &= -\frac{1}{18}\int 133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3\, dt \\ &= -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t\right) + c \end{aligned}$$

Now, reapplying the second initial condition to get the constant of integration gives

$$-1 = -\frac{1}{108}(133 + 56) + c \qquad \Rightarrow \qquad c = \frac{3}{4}$$

The second solution is then,

$$x_2(t) = -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t - 81\right)$$

So, putting all this together gives the solution to the system as,

$$\begin{aligned} x_1(t) &= \frac{1}{108}\left(133\mathbf{e}^{6t} - 28\mathbf{e}^{-3t} + 3 - 18t\right) \\ x_2(t) &= -\frac{1}{108}\left(133\mathbf{e}^{6t} + 56\mathbf{e}^{-3t} + 18t - 81\right) \end{aligned}$$

Compared to the last section the work here wasn't too bad. That won't always be the case of course, but you can see that using Laplace transforms to solve systems isn't too bad in at least some cases.

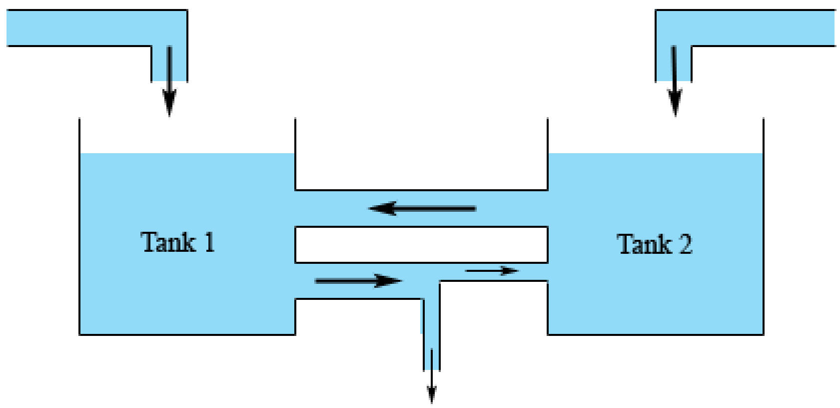
Modeling

In this section we're going to go back and revisit the idea of modeling only this time we're going to look at it in light of the fact that we now know how to solve systems of differential equations.

We're not actually going to be solving any differential equations in this section. Instead we'll just be setting up a couple of problems that are extensions of some of the work that we've done in earlier modeling sections whether it is the first order modeling or the vibrations work we did in the second order chapter. Almost all of the systems that we'll be setting up here will be nonhomogeneous systems (which we only briefly looked at), will be nonlinear (which we didn't look at) and/or will involve systems with more than two differential equations (which we didn't look at, although *most* of what we do know will still be true).

Mixing Problems

Let's start things by looking at a mixing problem. The last time we saw these was back in the first order chapter. In those problems we had a tank of liquid with some type of contaminate dissolved in it. Liquid, possibly with more contaminate dissolved in it, entered the tank and liquid left the tank. In this situation we want to extend things out to the following situation.



We'll now have two tanks that are interconnected with liquid potentially entering both and with an exit for some of the liquid if we need it (as illustrated by the lower connection). For this situation we're going to make the following assumptions.

1. The inflow and outflow from each tank are equal, or in other words the volume in each tank is constant. When we worked with a single tank we didn't need to worry about this, but here if we don't well end up with a system with nonconstant coefficients and those can be quite difficult to solve.

2. The concentration of the contaminate in each tank is the same at each point in the tank. In reality we know that this won't be true but without this assumption we'd need to deal with partial differential equations.

3. The concentration of contaminate in the outflow from tank 1 (the lower connection in the figure above) is the same as the concentration in tank 1. Likewise, the concentration of contaminate in the outflow from tank 2 (the upper connection) is the same as the concentration in tank 2.

4. The outflow from tank 1 is split and only some of the liquid exiting tank 1 actually reaches tank 2. The remainder exits the system completely. Note that if we don't want any liquid to completely exit the system we can think of the exit as having a value that is turned off. Also note that we could just as easily done the same thing for the outflow from tank 2 if we'd wanted to.

Let's take a look at a quick example.

Example 2 Two 1000 liter tanks are with salt water. Tank 1 contains 800 liters of water initially containing 20 grams of salt dissolved in it and tank 2 contains 1000 liters of water and initially has 80 grams of salt dissolved in it. Salt water with a concentration of ½ gram/liter of salt enters tank 1 at a rate of 4 liters/hour. Fresh water enters tank 2 at a rate of 7 liters/hour. Through a connecting pipe water flows from tank 2 into tank 1 at a rate of 10 liters/hour. Through a different connecting pipe 14 liters/hour flows out of tank 1 and 11 liters/hour are drained out of the pipe (and hence out of the system completely) and only 3 liters/hour flows back into tank 2. Set up the system that will give the amount of salt in each tank at any given time.

Solution

Okay, let $Q_1(t)$ and $Q_2(t)$ be the amount of salt in tank 1 and tank 2 at any time t respectively. Now all we need to do is set up a differential equation for each tank just as we did back when we had a single tank. The only difference is that we now need to deal with the fact that we've got a second inflow to each tank and the concentration of the second inflow will be the concentration of the other tank.

Recall that the basic differential equation is the rate of change of salt (Q') equals the rate at which salt enters minus the rate at salt leaves. Each entering/leaving rate is found by multiplying the flow rate times the concentration.

Here is the differential equation for tank 1.

$$\begin{aligned} Q_1' &= (4)\left(\frac{1}{2}\right) + (10)\left(\frac{Q_2}{1000}\right) - (14)\left(\frac{Q_1}{800}\right) \qquad\qquad Q_1(0) = 20 \\ &= 2 + \frac{Q_2}{100} - \frac{7Q_1}{400} \end{aligned}$$

In this differential equation the first pair of numbers is the salt entering from the external inflow. The second set of numbers is the salt that entering into the tank from the water flowing in from tank 2. The third set is the salt leaving tank as water flows out.

Here's the second differential equation.

$$\begin{aligned} Q_2' &= (7)(0) + (3)\left(\frac{Q_1}{800}\right) - (10)\left(\frac{Q_2}{1000}\right) \qquad\qquad Q_2(0) = 80 \\ &= \frac{3Q_1}{800} - \frac{Q_2}{100} \end{aligned}$$

Note that because the external inflow into tank 2 is fresh water the concentration of salt in this is zero.

In summary here is the system we'd need to solve,

$$Q_1' = 2 + \frac{Q_2}{100} - \frac{7Q_1}{400} \qquad\qquad Q_1(0) = 20$$
$$Q_2' = \frac{3Q_1}{800} - \frac{Q_2}{100} \qquad\qquad Q_2(0) = 80$$

This is a nonhomogeneous system because of the first term in the first differential equation. If we had fresh water flowing into both of these we would in fact have a homogeneous system.

Population

The next type of problem to look at is the population problem. Back in the first order modeling section we looked at some population problems. In those problems we looked at a single population and often included some form of predation. The problem in that section was we assumed that the amount of predation would be constant. This however clearly won't be the case in most situations. The amount of predation will depend upon the population of the predators and the population of the predators will depend, as least partially, upon the population of the prey.

So, in order to more accurately (well at least more accurate than what we originally did) we really need to set up a model that will cover both populations, both the predator and the prey. These types of problems are usually called **predator-prey** problems. Here are the assumptions that we'll make when we build up this model.

1. The prey will grow at a rate that is proportional to its current population if there are no predators.

2. The population of predators will decrease at a rate proportional to its current population if there is no prey.

3. The number of encounters between predator and prey will be proportional to the product of the populations.

4. Each encounter between the predator and prey will increase the population of the predator and decrease the population of the prey.

So, given these assumptions let's write down the system for this case.

Example 3 Write down the system of differential equations for the population of predators and prey using the assumptions above.

Solution

We'll start off by letting *x* represent the population of the predators and *y* represent the population of the prey.

Now, the first assumption tells us that, in the absence of predators, the prey will grow at a rate of ay where $a > 0$. Likewise the second assumption tells us that, in the absence of prey, the predators will decrease at a rate of $-bx$ where $b > 0$.

Next, the third and fourth assumptions tell us how the population is affected by encounters between predators and prey. So, with each encounter the population of the predators will increase at a rate of αxy and the population of the prey will decrease at a rate of $-\beta xy$ where $\alpha > 0$ and $\beta > 0$.

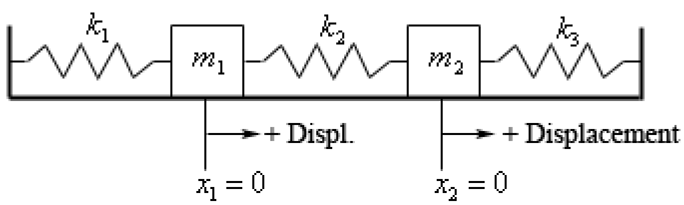
Putting all of this together we arrive at the following system.

$$\begin{aligned} x' &= -bx + \alpha xy = x(\alpha y - b) \\ y' &= ay - \beta xy = y(a - \beta x) \end{aligned}$$

Note that this is a nonlinear system and we've not (nor will we here) discuss how to solve this kind of system. We simply wanted to give a "better" model for some population problems and to point out that not all systems will be nice and simple linear systems.

Mechanical Vibrations

When we first looked at mechanical vibrations we looked at a single mass hanging on a spring with the possibility of both a damper and/or an external force acting on the mass. Here we want to look at the following situation.



In the figure above we are assuming that the system is at rest. In other words all three springs are currently at their natural lengths and are not exerting any forces on either of the two masses and that there are no currently any external forces acting on either mass.

We will use the following assumptions about this situation once we start the system in motion.

1. x_1 will measure the displacement of mass m_1 from its equilibrium (*i.e.* resting) position and x_2 will measure the displacement of mass m_2 from its equilibrium position.

2. As noted in the figure above all displacement will be assumed to be positive if it is to the right of equilibrium position and negative if to the left of the equilibrium position.

3. All forces acting to the right are positive forces and all forces acting to the left are negative forces.

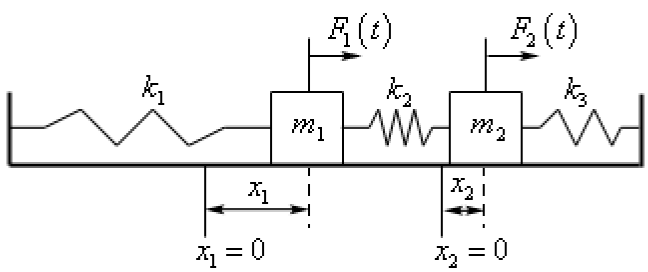
4. The spring constants, k_1, k_2, and k_3, are all positive and may or may not be the same value.

5. The surface that the system is sitting on is frictionless and so the mass of each of the objects will not affect the system in any way.

Before writing down the system for this case recall that the force exerted by the spring on the each mass is the spring constant times the amount that the spring has been compressed or stretched and we'll need to be careful with signs to make sure that the force is acting in the correct direction.

Example 4 Write down the system of differential equations for the spring and mass system above.

Solution
To help us out let's first take a quick look at a situation in which both of the masses have been moved. This is shown below.



Before proceeding let's note that this is only a representation of a typical case, but most definitely not all possible cases.

In this case we're assuming that both x_1 and x_2 are positive and that $x_2 - x_1 < 0$, or in other words, both masses have been moved to the right of their respective equilibrium points and that m_1 has been moved farther than m_2. So, under these assumption on x_1 and x_2 we know that the spring on the left (with spring constant k_1) has been stretched past it's natural length while the middle spring (spring constant k_2) and the right spring (spring constant k_3) are both under compression.

Also, we've shown the external forces, $F_1(t)$ and $F_2(t)$, as present and acting in the positive direction. They do not, in practice, need to be present in every situation in which case we will assume that $F_1(t) = 0$ and/or $F_2(t) = 0$. Likewise, if the forces are in fact acting in the negative direction we will then assume that $F_1(t) < 0$ and/or $F_2(t) < 0$.

Before proceeding we need to talk a little bit about how the middle spring will behave as the masses move. Here are all the possibilities that we can have and the affect each will have on $x_2 - x_1$. Note that in each case the amount of compression/stretch in the spring is given by $\left| x_2 - x_1 \right|$ although we won't be using the absolute value bars when we set up the differential equations.

1. If both mass move the same amount in the same direction then the middle spring will not have changed length and we'll have $x_2 - x_1 = 0$.

2. If both masses move in the positive direction then the sign of $x_2 - x_1$ will tell us which has moved more. If m_1 moves more than m_2 then the spring will be in compression and $x_2 - x_1 < 0$. Likewise, if m_2 moves more than m_1 then the spring will have been stretched and $x_2 - x_1 > 0$.

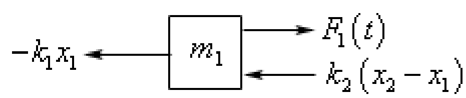
3. If both masses move in the negative direction we'll have pretty much the opposite behavior as #2. If m_1 moves more than m_2 then the spring will have been stretched and $x_2 - x_1 > 0$. Likewise, if m_2 moves more than m_1 then the spring will be in compression and $x_2 - x_1 < 0$.

4. If m_1 moves in the positive direction and m_2 moves in the negative direction then the spring will be in compression and $x_2 - x_1 < 0$.

5. Finally, if m_1 moves in the negative direction and m_2 moves in the positive direction then the spring will have been stretched and $x_2 - x_1 > 0$.

Now, we'll use the figure above to help us develop the differential equations (the figure corresponds to case 2 above…) and then make sure that they will also hold for the other cases as well.

Let's start off by getting the differential equation for the forces acting on m_1. Here is a quick sketch of the forces acting on m_1 for the figure above.



In this case $x_1 > 0$ and so the first spring has been stretched and so will exert a negative (*i.e.* to the left) force on the mass. The force from the first spring is then $-k_1 x_1$ and the "-" is needed because the force is negative but both k_1 and x_1 are positive.

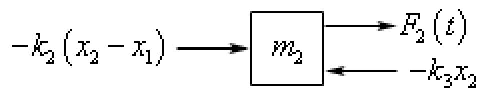
Next, because we're assuming that m_1 has moved more than m_2 and both have moved in the positive direction we also know that $x_2 - x_1 < 0$. Because m_1 has moved more than m_2 we know that the second spring will be under compression and so the force should be acting in the negative direction on m_1 and so the force will be $k_2(x_2 - x_1)$. Note that because k_2 is positive and $x_2 - x_1$ is negative this force will have the correct sign (*i.e.* negative).

The differential equation for m_1 is then,

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) + F_1(t)$$

Note that this will also hold for all the other cases. If m_1 has been moved in the negative direction the force form the spring on the right that acts on the mass will be positive and $-k_1 x_1$ will be a positive quantity in this case. Next, if the middle is has been stretched (*i.e.* $x_2 - x_1 > 0$) then the force from this spring on m_1 will be in the positive direction and $k_2\left(x_2 - x_1\right)$ will be a positive quantity in this case. Therefore, this differential equation holds for all cases not just the one we illustrated at the start of this problem.

Let's now write down the differential equation for all the forces that are acting on m_2. Here is a sketch of the forces acting on this mass for the situation sketched out in the figure above.



In this case x_2 is positive and so the spring on the right is under compression and will exert a negative force on m_2 and so this force should be $-k_3 x_2$, where the "-" is required because both k_3 and x_2 are positive. Also, the middle spring is still under compression but the force that it exerts on this mass is now a positive force, unlike in the case of m_1, and so is given by $-k_2\left(x_2 - x_1\right)$. The "-" on this force is required because $x_2 - x_1$ is negative and the force must be positive.

The differential equation for m_2 is then,

$$m_2 x_2'' = -k_3 x_2 - k_2\left(x_2 - x_1\right) + F_2\left(t\right)$$

We'll leave it to you to verify that this differential equation does in fact hold for all the other cases.

Putting all of this together and doing a little rewriting will then give the following system of differential equations for this situation.

$$\begin{aligned} m_1 x_1'' &= -\left(k_1 + k_2\right) x_1 + k_2 x_2 + F_1\left(t\right) \\ m_2 x_2'' &= k_2 x_1 - \left(k_2 + k_3\right) x_2 + F_2\left(t\right) \end{aligned}$$

This is a system to two linear second order differential equations that may or may not be nonhomogeneous depending whether there are any external forces, $F_1\left(t\right)$ and $F_2\left(t\right)$, acting on the masses.

We have not talked about how to solve systems of second order differential equations. However, it can be converted to a system of first order differential equations as the next example shows and in many cases we could solve that.

Example 5 Convert the system from the previous example to a system of 1st order differential equations.

Solution

This isn't too hard to do. Recall that we did this for single higher order differential equations earlier in the chapter when we first started to look at systems. To convert this to a system of first order differential equations we can make the following definitions.

$$u_1 = x_1 \qquad u_2 = x_1' \qquad u_3 = x_2 \qquad u_4 = x_2'$$

We can then convert each of the differential equations as we did earlier in the chapter.

$$\begin{aligned}
u_1' &= x_1' = u_2 \\
u_2' &= x_1'' = \frac{1}{m_1}\left(-\left(k_1 + k_2\right)x_1 + k_2 x_2 + F_1\left(t\right)\right) = \frac{1}{m_1}\left(-\left(k_1 + k_2\right)u_1 + k_2 u_3 + F_1\left(t\right)\right) \\
u_3' &= x_2' = u_4 \\
u_4' &= x_2'' = \frac{1}{m_2}\left(k_2 x_1 - \left(k_2 + k_3\right)x_2 + F_2\left(t\right)\right) = \frac{1}{m_2}\left(k_2 u_1 - \left(k_2 + k_3\right)u_3 + F_2\left(t\right)\right)
\end{aligned}$$

Eliminating the "middle" step we get the following system of first order differential equations.

$$\begin{aligned}
u_1' &= u_2 \\
u_2' &= \frac{1}{m_1}\left(-\left(k_1 + k_2\right)u_1 + k_2 u_3 + F_1\left(t\right)\right) \\
u_3' &= u_4 \\
u_4' &= \frac{1}{m_2}\left(k_2 u_1 - \left(k_2 + k_3\right)u_3 + F_2\left(t\right)\right)
\end{aligned}$$

The matrix form of this system would be,

$$\vec{u}\,' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-\left(k_1 + k_2\right)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & \frac{-\left(k_2 + k_3\right)}{m_2} & 0 \end{bmatrix}\vec{u} + \begin{pmatrix} 0 \\ \frac{F_1\left(t\right)}{m_1} \\ 0 \\ \frac{F_2\left(t\right)}{m_2} \end{pmatrix} \qquad \text{where, } \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

While we never discussed how to solve systems of more than two linear first order differential equations we know most of what we need to solve this.

In an earlier section we discussed briefly solving nonhomogeneous systems and all of that information is still valid here.

For the homogenous system, that we'd still need to solve for the general solution to the nonhomogeneous system, we know most of what we need to know in order to solve this. The only issues that we haven't dealt with are what to do with repeated complex eigenvalues (which are now a possibility) and what to do with eigenvalues of multiplicity greater than 2 (which are again now a possibility).

Series Solutions to Differential Equations

Introduction

In this chapter we will finally be looking at nonconstant coefficient differential equations. While we won't cover all possibilities in this chapter we will be looking at two of the more common methods for dealing with this kind of differential equation.

The first method that we'll be taking a look at, series solutions, will actually find a series representation for the solution instead of the solution itself. You first saw something like this when you looked at Taylor series in your Calculus class. As we will see however, these won't work for every differential equation.

The second method that we'll look at will only work for a special class of differential equations. This special case will cover some of the cases in which series solutions can't be used.

Here is a brief listing of the topics in this chapter.

Review : Power Series

Before looking at series solutions to a differential equation we will first need to do a cursory review of power series. A power series is a series in the form,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{1}$$

where, x_0 and a_n are numbers. We can see from this that a power series is a function of x. The function notation is not always included, but sometimes it is so we put it into the definition above.

Before proceeding with our review we should probably first recall just what series really are. Recall that series are really just summations. One way to write our power series is then,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots \end{aligned} \tag{2}$$

Notice as well that if we needed to for some reason we could always write the power series as,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} a_n (x - x_0)^n \end{aligned}$$

All that we're doing here is noticing that if we ignore the first term (corresponding to $n = 0$) the remainder is just a series that starts at $n = 1$. When we do this we say that we've stripped out the $n = 0$, or first, term. We don't need to stop at the first term either. If we strip out the first three terms we'll get,

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \sum_{n=3}^{\infty} a_n (x - x_0)^n$$

There are times when we'll want to do this so make sure that you can do it.

Now, since power series are functions of x and we know that not every series will in fact exist, it then makes sense to ask if a power series will exist for all x. This question is answered by looking at the convergence of the power series. We say that a power series **converges** for $x = c$ if the series,

$$\sum_{n=0}^{\infty} a_n (c - x_0)^n$$

converges. Recall that this series will converge if the limit of partial sums,

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (c - x_0)^n$$

exists and is finite. In other words, a power series will converge for $x=c$ if

$$\sum_{n=0}^{\infty} a_n (c - x_0)^n$$

is a finite number.

Note that a power series will always converge if $x = x_0$. In this case the power series will become

$$\sum_{n=0}^{\infty} a_n \left(x_0 - x_0\right)^n = a_0$$

With this we now know that power series are guaranteed to exist for at least one value of x. We have the following fact about the convergence of a power series.

Fact

Given a power series, (1), there will exist a number $0 \le \rho \le \infty$ so that the power series will converge for $\left|x - x_0\right| < \rho$ and diverge for $\left|x - x_0\right| > \rho$. This number is called the **radius of convergence**.

Determining the radius of convergence for most power series is usually quite simple if we use the ratio test.

Ratio Test

Given a power series compute,

$$L = \left|x - x_0\right| \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$$

then,

$$\begin{array}{lll} L < 1 & \Rightarrow & \text{the series converges} \\ L > 1 & \Rightarrow & \text{the series diverges} \\ L = 1 & \Rightarrow & \text{the series may converge or diverge} \end{array}$$

Let's take a quick look at how this can be used to determine the radius of convergence.

Example 1 Determine the radius of convergence for the following power series.

$$\sum_{n=0}^{\infty} \frac{\left(-3\right)^n}{n\,7^{n+1}} \left(x - 5\right)^n$$

Solution

So, in this case we have,

$$a_n = \frac{\left(-3\right)^n}{n\,7^{n+1}} \qquad\qquad a_{n+1} = \frac{\left(-3\right)^{n+1}}{\left(n+1\right)7^{n+2}}$$

Remember that to compute a_{n+1} all we do is replace all the n's in a_n with $n+1$. Using the ratio test then gives,

$$
\begin{aligned}
L &= |x-5| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= |x-5| \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(n+1)\,7^{n+2}} \, \frac{n\,7^{n+1}}{(-3)^n} \right| \\
&= |x-5| \lim_{n\to\infty} \left| \frac{-3}{(n+1)\,7} \, \frac{n}{1} \right| \\
&= \frac{3}{7}|x-5|
\end{aligned}
$$

Now we know that the series will converge if,

$$\frac{3}{7}|x-5| < 1 \qquad \Rightarrow \qquad |x-5| < \frac{7}{3}$$

and the series will diverge if,

$$\frac{3}{7}|x-5| > 1 \qquad \Rightarrow \qquad |x-5| > \frac{7}{3}$$

In other words, the radius of the convergence for this series is,

$$\rho = \frac{7}{3}$$

As this last example has shown, the radius of convergence is found almost immediately upon using the ratio test.

So, why are we worried about the convergence of power series? Well in order for a series solution to a differential equation to exist at a particular x it will need to be convergent at that x. If it's not convergent at a given x then the series solution won't exist at that x. So, the convergence of power series is fairly important.

Next we need to do a quick review of some of the basics of manipulating series. We'll start with addition and subtraction.

There really isn't a whole lot to addition and subtraction. All that we need to worry about is that the two series start at the same place and both have the same exponent of the x-x_0. If they do then we can perform addition and/or subtraction as follows,

$$\sum_{n=n_0}^{\infty} a_n (x-x_0)^n \pm \sum_{n=n_0}^{\infty} b_n (x-x_0)^n = \sum_{n=n_0}^{\infty} (a_n \pm b_n)(x-x_0)^n$$

In other words all we do is add or subtract the coefficients and we get the new series.

One of the rules that we're going to have when we get around to finding series solutions to differential equations is that the only x that we want in a series is the x that sits in $(x-x_0)^n$. This means that we will need to be able to deal with series of the form,

$$\left(x-x_0\right)^c \sum_{n=0}^{\infty} a_n \left(x-x_0\right)^n$$

where c is some constant. These are actually quite easy to deal with.

$$\begin{aligned}\left(x-x_0\right)^c \sum_{n=0}^{\infty} a_n \left(x-x_0\right)^n &= \left(x-x_0\right)^c \left(a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + \cdots\right) \\ &= a_0\left(x-x_0\right)^c + a_1\left(x-x_0\right)^{1+c} + a_2\left(x-x_0\right)^{2+c} + \cdots \\ &\sum_{n=0}^{\infty} a_n \left(x-x_0\right)^{n+c}\end{aligned}$$

So, all we need to do is to multiply the term in front into the series and add exponents. Also note that in order to do this both the coefficient in front of the series and the term inside the series must be in the form x-x_0. If they are not the same we can't do this, we will eventually see how to deal with terms that aren't in this form.

Next we need to talk about differentiation of a power series. By looking at (2) it should be fairly easy to see how we will differentiate a power series. Since a series is just a giant summation all we need to do is differentiate the individual terms. The derivative of a power series will be,

$$\begin{aligned}f'(x) &= a_1 + 2a_2\left(x-x_0\right) + 3a_3\left(x-x_0\right)^2 + \cdots \\ &= \sum_{n=1}^{\infty} n a_n \left(x-x_0\right)^{n-1} \\ &= \sum_{n=0}^{\infty} n a_n \left(x-x_0\right)^{n-1}\end{aligned}$$

So, all we need to do is just differentiate the term inside the series and we're done. Notice as well that there are in fact two forms of the derivative. Since the *n=0* term of the derivative is zero it won't change the value of the series and so we can include it or not as we need to. In our work we will usually want the derivative to start at *n=1*, however there will be the occasional problem were it would be more convenient to start it at *n=0*.

Following how we found the first derivative it should make sense that the second derivative is,

$$\begin{aligned}f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n \left(x-x_0\right)^{n-2} \\ &= \sum_{n=1}^{\infty} n(n-1) a_n \left(x-x_0\right)^{n-2} \\ &= \sum_{n=0}^{\infty} n(n-1) a_n \left(x-x_0\right)^{n-2}\end{aligned}$$

In this case since the *n=0* and *n=1* terms are both zero we can start at any of three possible starting points as determined by the problem that we're working.

Next we need to talk about **index shifts**. As we will see eventually we are going to want our power series written in terms of $\left(x-x_0\right)^n$ and they often won't, initially at least, be in that form. To get them into the form we need we will need to perform an index shift.

Index shifts themselves really aren't concerned with the exponent on the *x* term, they instead are concerned with where the series starts as the following example shows.

Example 2 Write the following as a series that starts at *n=0* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2}$$

Solution

An index shift is a fairly simple manipulation to perform. First we will notice that if we define *i=n-3* then when *n=3* we will have *i=0*. So what we'll do is rewrite the series in terms of *i* instead of *n*. We can do this by noting that *n=i+3*. So, everywhere we see an *n* in the actual series term we will replace it with an *i+3*. Doing this gives,

$$\begin{aligned}\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} &= \sum_{i=0}^{\infty}\left(i+3\right)^2 a_{i+3-1}\left(x+4\right)^{i+3+2}\\ &= \sum_{i=0}^{\infty}\left(i+3\right)^2 a_{i+2}\left(x+4\right)^{i+5}\end{aligned}$$

The upper limit won't change in this process since infinity minus three is still infinity.

The final step is to realize that the letter we use for the index doesn't matter and so we can just switch back to *n*'s.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} = \sum_{n=0}^{\infty}\left(n+3\right)^2 a_{n+2}\left(x+4\right)^{n+5}$$

Now, we usually don't go through this process to do an index shift. All we do is notice that we dropped the starting point in the series by 3 and everywhere else we saw an *n* in the series we increased it by 3. In other words, all the *n*'s in the series move in the opposite direction that we moved the starting point.

Example 3 Write the following as a series that starts at *n=5* instead of *n=3*.

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2}$$

Solution

To start the series to start at *n=5* all we need to do is notice that this means we will increase the starting point by 2 and so all the other *n*'s will need to decrease by 2. Doing this for the series in the previous example would give,

$$\sum_{n=3}^{\infty} n^2 a_{n-1}\left(x+4\right)^{n+2} = \sum_{n=5}^{\infty}\left(n-2\right)^2 a_{n-3}\left(x+4\right)^{n}$$

Now, as we noted when we started this discussion about index shift the whole point is to get our series into terms of $\left(x-x_0\right)^n$. We can see in the previous example that we did exactly that with an index shift. The original exponent on the *(x+4)* was *n+2*. To get this down to an *n* we needed to decrease the exponent by 2. This can be done with an index that increases the starting point by 2.

Let's take a look at a couple of more examples of this.

Example 4 Write each of the following as a single series in terms of $\left(x-x_0\right)^n$.

(a) $\left(x+2\right)^2\sum_{n=3}^{\infty} na_n\left(x+2\right)^{n-4}-\sum_{n=1}^{\infty} na_n\left(x+2\right)^{n+1}$ [Solution]

(b) $x\sum_{n=0}^{\infty}\left(n-5\right)^2 b_{n+1}\left(x-3\right)^{n+3}$ [Solution]

Solution

(a) $\left(x+2\right)^2\sum_{n=3}^{\infty} na_n\left(x+2\right)^{n-4}-\sum_{n=1}^{\infty} na_n\left(x+2\right)^{n+1}$

First, notice that there are two series here and the instructions clearly ask for only a single series. So, we will need to subtract the two series at some point in time. The vast majority of our work will be to get the two series prepared for the subtraction. This means that the two series can't have any coefficients in front of them (other than one of course…), they will need to start at the same value of *n* and they will need the same exponent on the x-x_0.

We'll almost always want to take care of any coefficients first. So, we have one in front of the first series so let's multiply that into the first series. Doing this gives,

$$\sum_{n=3}^{\infty} na_n\left(x+2\right)^{n-2}-\sum_{n=1}^{\infty} na_n\left(x+2\right)^{n+1}$$

Now, the instructions specify that the new series must be in terms of $\left(x-x_0\right)^n$, so that's the next thing that we've got to take care of. We will do this by an index shift on each of the series. The exponent on the first series needs to go up by two so we'll shift the first series down by 2. On the second series will need to shift up by 1 to get the exponent to move down by 1. Performing the index shifts gives us the following,

$$\sum_{n=1}^{\infty}\left(n+2\right)a_{n+2}\left(x+2\right)^n-\sum_{n=2}^{\infty}\left(n-1\right)a_{n-1}\left(x+2\right)^n$$

Finally, in order to subtract the two series we'll need to get them to start at the same value of *n*. Depending on the series in the problem we can do this in a variety of ways. In this case let's notice that since there is an *n-1* in the second series we can in fact start the second series at *n=1* without changing its value. Also note that in doing so we will get both of the series to start at *n=1* and so we can do the subtraction. Our final answer is then,

$$\sum_{n=1}^{\infty}\left(n+2\right)a_{n+2}\left(x+2\right)^n-\sum_{n=1}^{\infty}\left(n-1\right)a_{n-1}\left(x+2\right)^n=\sum_{n=1}^{\infty}\left[\left(n+2\right)a_{n+2}-\left(n-1\right)a_{n-1}\right]\left(x+2\right)^n$$

[Return to Problems]

(b) $x\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3}$

In this part the main issue is the fact that we can't just multiply the coefficient into the series this time since the coefficient doesn't have the same form as the term inside the series. Therefore, the first thing that we'll need to do is correct the coefficient so that we can bring it into the series. We do this as follows,

$$\begin{aligned} x\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} &= (x-3+3)\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} \\ &= (x-3)\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} + 3\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+3} \end{aligned}$$

We can now move the coefficient into the series, but in the process of we managed to pick up a second series. This will happen so get used to it. Moving the coefficients of both series in gives,

$$\sum_{n=0}^{\infty}(n-5)^2 b_{n+1}(x-3)^{n+4} + \sum_{n=0}^{\infty}3(n-5)^2 b_{n+1}(x-3)^{n+3}$$

We now need to get the exponent in both series to be an n. This will mean shifting the first series up by 4 and the second series up by 3. Doing this gives,

$$\sum_{n=4}^{\infty}(n-9)^2 b_{n-3}(x-3)^{n} + \sum_{n=3}^{\infty}3(n-8)^2 b_{n-2}(x-3)^{n}$$

In this case we can't just start the first series at $n=3$ because there is not an $n-3$ sitting in that series to make the $n=3$ term zero. So, we won't be able to do this part as we did in the first part of this example.

What we'll need to do in this part is strip out the $n=3$ from the second series so they will both start at $n=4$. We will then be able to add the two series together. Stripping out the $n=3$ term from the second series gives,

$$\sum_{n=4}^{\infty}(n-9)^2 b_{n-3}(x-3)^{n} + 3(-5)^2 b_1(x-3)^3 + \sum_{n=4}^{\infty}3(n-8)^2 b_{n-2}(x-3)^{n}$$

We can now add the two series together.

$$75b_1(x-3)^3 + \sum_{n=4}^{\infty}\left[(n-9)^2 b_{n-3} + 3(n-8)^2 b_{n-2}\right](x-3)^{n}$$

This is what we're looking for. We won't worry about the extra term sitting in front of the series. When we finally get around to finding series solutions to differential equations we will see how to deal with that term there.

[Return to Problems]

There is one final fact that we need take care of before moving on. Before giving this fact for power series let's notice that the only way for

$$a + bx + cx^2 = 0$$

to be zero for all x is to have $a=b=c=0$.

We've got a similar fact for power series.

Fact

If,

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = 0$$

for all x then,

$$a_n = 0, \quad n = 0, 1, 2, \ldots$$

This fact will be key to our work with differential equations so don't forget it.

Review : Taylor Series

We are not going to be doing a whole lot with Taylor series once we get out of the review, but they are a nice way to get us back into the swing of dealing with power series. By time most students reach this stage in their mathematical career they've not had to deal with power series for at least a semester or two. Remembering how Taylor series work will be a very convenient way to get comfortable with power series before we start looking at differential equations.

Taylor Series
If *f(x)* is an infinitely differentiable function then the Taylor Series of *f(x)* about $x=x_0$ is,

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(x_0\right)}{n!}\left(x-x_0\right)^n$$

Recall that

$$f^{(0)}\left(x\right)=f\left(x\right) \qquad\qquad f^{(n)}\left(x\right)=\text{ n}^{\text{th}}\text{ derivative of } f\left(x\right)$$

Let's take a look at an example.

Example 1 Determine the Taylor series for $f\left(x\right)=\mathbf{e}^x$ about *x=0*.

Solution
This is probably one of the easiest functions to find the Taylor series for. We just need to recall that,

$$f^{(n)}\left(x\right)=\mathbf{e}^x \qquad n=0,1,2,\ldots$$

and so we get,

$$f^{(n)}\left(0\right)=1 \qquad n=0,1,2,\ldots$$

The Taylor series for this example is then,

$$\mathbf{e}^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Of course, it's often easier to find the Taylor series about *x=0* but we don't always do that.

Example 2 Determine the Taylor series for $f\left(x\right)=\mathbf{e}^x$ about *x=-4*.

Solution
This problem is virtually identical to the previous problem. In this case we just need to notice that,

$$f^{(n)}\left(-4\right)=\mathbf{e}^{-4} \quad n=0,1,2,\ldots$$

The Taylor series for this example is then,

$$\mathbf{e}^x=\sum_{n=0}^{\infty}\frac{\mathbf{e}^{-4}}{n!}\left(x+4\right)^n$$

Let's now do a Taylor series that requires a little more work.

Example 3 Determine the Taylor series for $f\left(x\right)=\cos\left(x\right)$ about *x=0*.

Solution
This time there is no formula that will give us the derivative for each *n* so let's start taking derivatives and plugging in *x=0*.

$$\begin{array}{ll}
f^{(0)}\left(x\right)=\cos\left(x\right) & f^{(0)}\left(0\right)=1 \\
f^{(1)}\left(x\right)=-\sin\left(x\right) & f^{(1)}\left(0\right)=0 \\
f^{(2)}\left(x\right)=-\cos\left(x\right) & f^{(2)}\left(0\right)=-1 \\
f^{(3)}\left(x\right)=\sin\left(x\right) & f^{(3)}\left(0\right)=0 \\
f^{(4)}\left(x\right)=\cos\left(x\right) & f^{(4)}\left(0\right)=1 \\
\vdots & \vdots
\end{array}$$

Once we reach this point it's fairly clear that there is a pattern emerging here. Just what this pattern is has yet to be determined, but it does seem fairly clear that a pattern does exist.

Let's plug what we've got into the formula for the Taylor series and see what we get.

$$\begin{aligned}
\cos\left(x\right) &= \sum_{n=0}^{\infty}\frac{f^{(n)}\left(0\right)}{n!}x^n \\
&= \frac{f^{(0)}\left(0\right)}{0!}+\frac{f^{(1)}\left(0\right)}{1!}x+\frac{f^{(2)}\left(0\right)}{2!}x^2+\frac{f^{(3)}\left(0\right)}{3!}x^3+\cdots \\
&= \frac{1}{0!}+0-\frac{x^2}{2!}+0+\frac{x^4}{4!}+0-\frac{x^6}{6!}+0+\frac{x^8}{8!}+\cdots
\end{aligned}$$

So, every other term is zero.

We would like to write this in terms of a series, however finding a formula that is zero every other term and gives the correct answer for those that aren't zero would be unnecessarily complicated. So, let's rewrite what we've got above and while were at it renumber the terms as follows,

$$\cos\left(x\right)=\underbrace{\frac{1}{0!}}_{n=0}-\underbrace{\frac{x^2}{2!}}_{n=1}+\underbrace{\frac{x^4}{4!}}_{n=2}-\underbrace{\frac{x^6}{6!}}_{n=3}+\underbrace{\frac{x^8}{8!}}_{n=4}+\cdots$$

With this "renumbering" we can fairly easily get a formula for the Taylor series of the cosine function about *x=0*.

$$\cos\left(x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n x^{2n}}{\left(2n\right)!}$$

For practice you might want to see if you can verify that the Taylor series for the sine function about *x=0* is,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We need to look at one more example of a Taylor series. This example is both tricky and very easy.

Example 4 Determine the Taylor series for $f(x) = 3x^2 - 8x + 2$ about *x=2*.

Solution

There's not much to do here except to take some derivatives and evaluate at the point.

$$\begin{aligned} f(x) &= 3x^2 - 8x + 2 & f(2) &= -2 \\ f'(x) &= 6x - 8 & f'(2) &= 4 \\ f''(x) &= 6 & f''(2) &= 6 \\ f^{(n)}(x) &= 0,\ n \ge 3 & f^{(n)}(2) &= 0,\ n \ge 3 \end{aligned}$$

So, in this case the derivatives will all be zero after a certain order. That happens occasionally and will make our work easier. Setting up the Taylor series then gives,

$$\begin{aligned} 3x^2 - 8x + 2 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \frac{f^{(0)}(2)}{0!} + \frac{f^{(1)}(2)}{1!}(x-2) + \frac{f^{(2)}(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \cdots \\ &= -2 + 4(x-2) + \frac{6}{2}(x-2)^2 + 0 \\ &= -2 + 4(x-2) + 3(x-2)^2 \end{aligned}$$

In this case the Taylor series terminates and only had three terms. Note that since we are after the Taylor series we do not multiply the 4 through on the second term or square out the third term. All the terms with the exception of the constant should contain an *x-2*.

Note in this last example that if we were to multiply the Taylor series we would get our original polynomial. This should not be too surprising as both are polynomials and they should be equal.

We now need a quick definition that will make more sense to give here rather than in the next section where we actually need it since it deals with Taylor series.

Definition

A function, *f(x)*, is called **analytic** at *x=a* if the Taylor series for *f(x)* about *x=a* has a positive radius of convergence and converges to *f(x)*.

We need to give one final note before proceeding into the next section. We started this section out by saying that we weren't going to be doing much with Taylor series after this section. While that is correct it is only correct because we are going to be keeping the problems fairly simple. For more complicated problems we would also be using quite a few Taylor series.

Series Solutions to Differential Equations

Before we get into finding series solutions to differential equations we need to determine when we can find series solutions to differential equations. So, let's start with the differential equation,

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0 \qquad \textbf{(1)}$$

This time we really do mean nonconstant coefficients. To this point we've only dealt with constant coefficients. However, with series solutions we can now have nonconstant coefficient differential equations. Also, in order to make the problems a little nicer we will be dealing only with polynomial coefficients.

Now, we say that $x=x_0$ is an **ordinary point** if provided both

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)}$$

are analytic at $x=x_0$. That is to say that these two quantities have Taylor series around $x=x_0$. We are going to be only dealing with coefficients that are polynomials so this will be equivalent to saying that

$$p(x_0) \neq 0$$

for most of the problems.

If a point is not an ordinary point we call it a **singular point**.

The basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \textbf{(2)}$$

and then try to determine what the a_n's need to be. We will only be able to do this if the point $x=x_0$, is an ordinary point. We will usually say that (2) is a series solution around $x=x_0$.

Let's start with a very basic example of this. In fact it will be so basic that we will have constant coefficients. This will allow us to check that we get the correct solution.

Example 1 Determine a series solution for the following differential equation about $x_0 = 0$.

$$y'' + y = 0$$

Solution

Notice that in this case $p(x)=1$ and so every point is an ordinary point. We will be looking for a solution in the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We will need to plug this into our differential equation so we'll need to find a couple of derivatives.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad\qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Recall from the power series review section on power series that we can start these at *n=0* if we need to, however it's almost always best to start them where we have here. If it turns out that it would have been easier to start them at *n=0* we can easily fix that up when the time comes around.

So, plug these into our differential equation. Doing this gives,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

The next step is to combine everything into a single series. To do this requires that we get both series starting at the same point and that the exponent on the x be the same in both series.

We will always start this by getting the exponent on the x to be the same. It is usually best to get the exponent to be an n. The second series already has the proper exponent and the first series will need to be shifted down by 2 in order to get the exponent up to an n. If you don't recall how to do this take a quick look at the first review section where we did several of these types of problems.

Shifting the first power series gives us,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Notice that in the process of the shift we also got both series starting at the same place. This won't always happen, but when it does we'll take it. We can now add up the two series. This gives,

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = 0$$

Now recalling the fact from the power series review section we know that if we have a power series that is zero for all x (as this is) then all the coefficients must have been zero to start with. This gives us the following,

$$(n+2)(n+1)a_{n+2} + a_n = 0, \qquad n = 0,1,2,\ldots$$

This is called the **recurrence relation** and notice that we included the values of n for which it must be true. We will always want to include the values of n for which the recurrence relation is true since they won't always start at $n = 0$ as it did in this case.

Now let's recall what we were after in the first place. We wanted to find a series solution to the differential equation. In order to do this we needed to determine the values of the a_n's. We are almost to the point where we can do that. The recurrence relation has two different a_n's in it so we can't just solve this for a_n and get a formula that will work for all n. We can however, use this to determine what all but two of the a_n's are.

To do this we first solve the recurrence relation for the a_n that has the largest subscript. Doing this gives,

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \qquad n = 0,1,2,\ldots$$

Now, at this point we just need to start plugging in some value of n and see what happens,

$$n=0 \quad a_2 = \frac{-a_0}{(2)(1)} \qquad\qquad n=1 \quad a_3 = \frac{-a_1}{(3)(2)}$$

$$n=2 \quad \begin{aligned} a_4 &= -\frac{a_2}{(4)(3)} \\ &= \frac{a_0}{(4)(3)(2)(1)} \end{aligned} \qquad\qquad n=3 \quad \begin{aligned} a_5 &= -\frac{a_3}{(5)(4)} \\ &= \frac{a_1}{(5)(4)(3)(2)} \end{aligned}$$

$$n=4 \quad \begin{aligned} a_6 &= -\frac{a_4}{(6)(5)} \\ &= \frac{-a_0}{(6)(5)(4)(3)(2)(1)} \end{aligned} \qquad\qquad n=5 \quad \begin{aligned} a_7 &= -\frac{a_5}{(7)(6)} \\ &= \frac{-a_1}{(7)(6)(5)(4)(3)(2)} \end{aligned}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}, \ k = 1,2,\ldots \qquad\qquad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}, \ k = 1,2,\ldots$$

Notice that at each step we always plugged back in the previous answer so that when the subscript was even we could always write the a_n in terms of a_0 and when the coefficient was odd we could always write the a_n in terms of a_1. Also notice that, in this case, we were able to find a general formula for a_n's with even coefficients and a_n's with odd coefficients. This won't always be possible to do.

There's one more thing to notice here. The formulas that we developed were only for $k=1,2,\ldots$ however, in this case again, they will also work for $k=0$. Again, this is something that won't always work, but does here.

Do not get excited about the fact that we don't know what a_0 and a_1 are. As you will see, we actually need these to be in the problem to get the correct solution.

Now that we've got formulas for the a_n's let's get a solution. The first thing that we'll do is write out the solution with a couple of the a_n's plugged in.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \cdots \\ &= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \cdots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^{k+1} a_1}{(2k+1)!} x^{2k+1} + \cdots \end{aligned}$$

The next step is to collect all the terms with the same coefficient in them and then factor out that coefficient.

$$y(x) = a_0\left\{1 - \frac{x^2}{2!}\cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots\right\} + a_1\left\{x - \frac{x^3}{3!} + \cdots + \frac{(-1)^{k+1}}{(2k+1)!}x^{2k+1} + \cdots\right\}$$

$$= a_0\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} + a_1\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

In the last step we also used the fact that we knew what the general formula was to write both portions as a power series. This is also our solution. We are done.

Before working another problem let's take a look at the solution to the previous example. First, we started out by saying that we wanted a series solution of the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and we didn't get that. We got a solution that contained two different power series. Also, each of the solutions had an unknown constant in them. This is not a problem. In fact, it's what we want to have happen. From our work with second order constant coefficient differential equations we know that the solution to the differential equation in the last example is,

$$y(x) = c_1\cos(x) + c_2\sin(x)$$

Solutions to second order differential equations consist of two separate functions each with an unknown constant in front of them that are found by applying any initial conditions. So, the form of our solution in the last example is exactly what we want to get. Also recall that the following Taylor series,

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad\qquad \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Recalling these we very quickly see that what we got from the series solution method was exactly the solution we got from first principles, with the exception that the functions were the Taylor series for the actual functions instead of the actual functions themselves.

Now let's work an example with nonconstant coefficients since that is where series solutions are most useful.

Example 2 Find a series solution around $x_0 = 0$ for the following differential equation.

$$y'' - xy = 0$$

Solution

As with the first example *p(x)=1* and so again for this differential equation every point is an ordinary point. Now we'll start this one out just as we did the first example. Let's write down the form of the solution and get its derivatives.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Plugging into the differential equation gives,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

Unlike the first example we first need to get all the coefficients moved into the series.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we will need to shift the first series down by 2 and the second series up by 1 to get both of the series in terms of x^n.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Next we need to get the two series starting at the same value of n. The only way to do that for this problem is to strip out the $n=0$ term.

$$(2)(1) a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1}\right] x^n = 0$$

We now need to set all the coefficients equal to zero. We will need to be careful with this however. The $n=0$ coefficient is in front of the series and the $n=1,2,3\ldots$ are all in the series. So, setting coefficient equal to zero gives,

$$\begin{aligned} n &= 0: & 2a_2 &= 0 \\ n &= 1,2,3,\ldots & (n+2)(n+1) a_{n+2} - a_{n-1} &= 0 \end{aligned}$$

Solving the first as well as the recurrence relation gives,

$$\begin{aligned} n &= 0: & a_2 &= 0 \\ n &= 1,2,3,\ldots & a_{n+2} &= \frac{a_{n-1}}{(n+2)(n+1)} \end{aligned}$$

Now we need to start plugging in values of n.

$$a_3 = \frac{a_0}{(3)(2)} \qquad a_4 = \frac{a_1}{(4)(3)} \qquad a_5 = \frac{a_2}{(5)(4)} = 0$$

$$a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)} \qquad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)} \qquad a_8 = \frac{a_5}{(8)(7)} = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$a_{3k} = \frac{a_0}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \qquad a_{3k+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \qquad a_{3k+2} = 0$$

$$k = 1,2,3,\cdots \qquad\qquad k = 1,2,3,\cdots \qquad\qquad k = 0,1,2,\cdots$$

There are a couple of things to note about these coefficients. First, every third coefficient is zero. Next, the formulas here are somewhat unpleasant and not all that easy to see the first time around. Finally, these formulas will not work for *k=0* unlike the first example.

Now, get the solution,

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{3k} x^{3k} + a_{3k+1} x^{3k+1} + \cdots \\ &= a_0 + a_1 x + \frac{a_0}{6} x^3 + \frac{a_1}{12} x^4 \cdots + \frac{a_0 x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} + \\ &\qquad \frac{a_1 x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} + \cdots \end{aligned}$$

Again, collect up the terms that contain the same coefficient, factor the coefficient out and write the results as a new series,

$$y(x) = a_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right\} + a_1 \left\{ x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right\}$$

We couldn't start our series at *k=0* this time since the general term doesn't hold for *k=0*.

Now, we need to work an example in which we use a point other that *x=0*. In fact, let's just take the previous example and rework it for a different value of x_0. We're also going to need to change up the instructions a little for this example.

Example 3 Find the first four terms in each portion of the series solution around $x_0 = -2$ for the following differential equation.

$$y'' - xy = 0$$

Solution

Unfortunately for us there is nothing from the first example that can be reused here. Changing to $x_0 = -2$ completely changes the problem. In this case our solution will be,

$$y(x) = \sum_{n=0}^{\infty} a_n (x+2)^n$$

The derivatives of the solution are,

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x+2)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2}$$

Plug these into the differential equation.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2} - x\sum_{n=0}^{\infty} a_n (x+2)^n = 0$$

We now run into our first real difference between this example and the previous example. In this case we can't just multiply the *x* into the second series since in order to combine with the series it must be $x+2$. Therefore we will first need to modify the coefficient of the second series before multiplying it into the series.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2} - (x+2-2)\sum_{n=0}^{\infty} a_n (x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2} - (x+2)\sum_{n=0}^{\infty} a_n (x+2)^n + 2\sum_{n=0}^{\infty} a_n (x+2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x+2)^{n-2} - \sum_{n=0}^{\infty} a_n (x+2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x+2)^n = 0$$

We now have three series to work with. This will often occur in these kinds of problems. Now we will need to shift the first series down by 2 and the second series up by 1 the get common exponents in all the series.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x+2)^n - \sum_{n=1}^{\infty} a_{n-1} (x+2)^n + \sum_{n=0}^{\infty} 2a_n (x+2)^n = 0$$

In order to combine the series we will need to strip out the $n=0$ terms from both the first and third series.

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x+2)^n - \sum_{n=1}^{\infty} a_{n-1} (x+2)^n + \sum_{n=1}^{\infty} 2a_n (x+2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} + 2a_n \right] (x+2)^n = 0$$

Setting coefficients equal to zero gives,

$$\begin{aligned} &n = 0 && 2a_2 + 2a_0 = 0 \\ &n = 1, 2, 3, \ldots && (n+2)(n+1) a_{n+2} - a_{n-1} + 2a_n = 0 \end{aligned}$$

We now need to solve both of these. In the first case there are two options, we can solve for a_2 or we can solve for a_0. Out of habit I'll solve for a_0. In the recurrence relation we'll solve for the term with the largest subscript as in previous examples.

$$\begin{aligned} &n = 0 && a_2 = -a_0 \\ &n = 1, 2, 3, \ldots && a_{n+2} = \frac{a_{n-1} - 2a_n}{(n+2)(n+1)} \end{aligned}$$

Notice that in this example we won't be having every third term drop out as we did in the previous example.

At this point we'll also acknowledge that the instructions for this problem are different as well. We aren't going to get a general formula for the a_n's this time so we'll have to be satisfied with just getting the first couple of terms for each portion of the solution. This is often the case for series solutions. Getting general formulas for the a_n's is the exception rather than the rule in these kinds of problems.

To get the first four terms we'll just start plugging in terms until we've got the required number of terms. Note that we will already be starting with an a_0 and an a_1 from the first two terms of the solution so all we will need are three more terms with an a_0 in them and three more terms with an a_1 in them.

$$n=0 \qquad a_2=-a_0$$

We've got two a_0's and one a_1.

$$n=1 \qquad a_3=\frac{a_0-2a_1}{(3)(2)}=\frac{a_0}{6}-\frac{a_1}{3}$$

We've got three a_0's and two a_1's.

$$n=2 \qquad a_4=\frac{a_1-2a_2}{(4)(3)}=\frac{a_1-2(-a_0)}{(4)(3)}=\frac{a_0}{6}+\frac{a_1}{12}$$

We've got four a_0's and three a_1's. We've got all the a_0's that we need, but we still need one more a_1'. So, we'll need to do one more term it looks like.

$$n=3 \qquad a_5=\frac{a_2-2a_3}{(5)(4)}=-\frac{a_0}{20}-\frac{1}{10}\left(\frac{a_0}{6}-\frac{a_1}{3}\right)=-\frac{a_0}{15}+\frac{a_1}{30}$$

We've got five a_0's and four a_1's. We've got all the terms that we need.

Now, all that we need to do is plug into our solution.

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n (x+2)^n \\ &= a_0+a_1(x+2)+a_2(x+2)^2+a_3(x+2)^3+a_4(x+2)^4+a_5(x+2)^5+\cdots \\ &= a_0+a_1(x+2)-a_0(x+2)^2+\left(\frac{a_0}{6}-\frac{a_1}{3}\right)(x+2)^3+ \\ &\qquad \left(\frac{a_0}{6}+\frac{a_1}{12}\right)(x+2)^4+\left(-\frac{a_0}{15}+\frac{a_1}{30}\right)(x+2)^5+\cdots \end{aligned}$$

Finally collect all the terms up with the same coefficient and factor out the coefficient to get,

$$\begin{aligned} y(x) = a_0 &\left\{1-(x+2)^2+\frac{1}{6}(x+2)^3+\frac{1}{6}(x+2)^4-\frac{1}{15}(x+2)^5+\cdots\right\}+ \\ a_1 &\left\{(x+2)-\frac{1}{3}(x+2)^3+\frac{1}{12}(x+2)^4+\frac{1}{30}(x+2)^5+\cdots\right\} \end{aligned}$$

That's the solution for this problem as far as we're concerned. Notice that this solution looks nothing like the solution to the previous example. It's the same differential equation, but changing x_0 completely changed the solution.

Let's work one final problem.

Example 4 Find the first four terms in each portion of the series solution around $x_0 = 0$ for the following differential equation.

$$\left(x^2+1\right)y'' - 4xy' + 6y = 0$$

Solution

We finally have a differential equation that doesn't have a constant coefficient for the second derivative.

$$p\left(x\right) = x^2 + 1 \qquad\qquad p\left(0\right) = 1 \neq 0$$

So $x_0 = 0$ is an ordinary point for this differential equation. We first need the solution and its derivatives,

$$y\left(x\right) = \sum_{n=0}^{\infty} a_n x^n \qquad y'\left(x\right) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''\left(x\right) = \sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2}$$

Plug these into the differential equation.

$$\left(x^2+1\right)\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Now, break up the first term into two so we can multiply the coefficient into the series and multiply the coefficients of the second and third series in as well.

$$\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^n + \sum_{n=2}^{\infty} n\left(n-1\right) a_n x^{n-2} - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

We will only need to shift the second series down by two to get all the exponents the same in all the series.

$$\sum_{n=2}^{\infty} n\left(n-1\right) a_n x^n + \sum_{n=0}^{\infty} \left(n+2\right)\left(n+1\right) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

At this point we could strip out some terms to get all the series starting at *n=2*, but that's actually more work than is needed. Let's instead note that we could start the third series at *n=0* if we wanted to because that term is just zero. Likewise the terms in the first series are zero for both *n=1* and *n=0* and so we could start that series at *n=0*. If we do this all the series will now start at *n=0* and we can add them up without stripping terms out of any series.

$$\sum_{n=0}^{\infty} \left[n\left(n-1\right)a_n + \left(n+2\right)\left(n+1\right)a_{n+2} - 4n a_n + 6a_n\right]x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2 - 5n + 6\right)a_n + \left(n+2\right)\left(n+1\right)a_{n+2}\right]x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n-2\right)\left(n-3\right)a_n + \left(n+2\right)\left(n+1\right)a_{n+2}\right]x^n = 0$$

Now set coefficients equal to zero.

$$\left(n-2\right)\left(n-3\right)a_n + \left(n+2\right)\left(n+1\right)a_{n+2} = 0, \qquad n = 0,1,2,\ldots$$

Solving this gives,

$$a_{n+2} = -\frac{(n-2)(n-3)a_n}{(n+2)(n+1)}, \qquad n = 0,1,2,\ldots$$

Now, we plug in values of *n*.

$$\begin{aligned} n = 0: &\qquad a_2 = -3a_0 \\ n = 1: &\qquad a_3 = -\frac{1}{3}a_1 \\ n = 2: &\qquad a_4 = -\frac{0}{12}a_2 = 0 \\ n = 3: &\qquad a_5 = -\frac{0}{20}a_3 = 0 \end{aligned}$$

Now, from this point on all the coefficients are zero. In this case both of the series in the solution will terminate. This won't always happen, and often only one of them will terminate.

The solution in this case is,

$$y(x) = a_0\left\{1 - 3x^2\right\} + a_1\left\{x - \frac{1}{3}x^3\right\}$$

Euler Equations

In this section we want to look for solutions to

$$ax^2y'' + bxy' + cy = 0 \qquad \textbf{(1)}$$

around $x_0 = 0$. These type of differential equations are called **Euler Equations**.

Recall from the previous section that a point is an ordinary point if the quotients,

$$\frac{bx}{ax^2} = \frac{b}{ax} \qquad \text{and} \qquad \frac{c}{ax^2}$$

have Taylor series around $x_0 = 0$. However, because of the *x* in the denominator neither of these will have a Taylor series around $x_0 = 0$ and so $x_0 = 0$ is a singular point. So, the method from the previous section won't work since it required an ordinary point.

However, it is possible to get solutions to this differential equation that aren't series solutions. Let's start off by assuming that *x>0* (the reason for this will be apparent after we work the first example) and that all solutions are of the form,

$$y(x) = x^r \qquad \textbf{(2)}$$

Now plug this into the differential equation to get,

$$\begin{aligned} ax^2(r)(r-1)x^{r-2} + bx(r)x^{r-1} + cx^r &= 0 \\ ar(r-1)x^r + b(r)x^r + cx^r &= 0 \\ \left(ar(r-1) + b(r) + c\right)x^r &= 0 \end{aligned}$$

Now, we assumed that *x>0* and so this will only be zero if,

$$ar(r-1) + b(r) + c = 0 \qquad \textbf{(3)}$$

So solutions will be of the form (2) provided *r* is a solution to (3). This equation is a quadratic in *r* and so we will have three cases to look at : Real, Distinct Roots, Double Roots, and Complex Roots.

Real, Distinct Roots

There really isn't a whole lot to do in this case. We'll get two solutions that will form a fundamental set of solutions (we'll leave it to you to check this) and so our general solution will be,

$$y(x) = c_1x^{r_1} + c_2x^{r_2}$$

Example 1 Solve the following IVP

$$2x^2y'' + 3xy' - 15y = 0, \qquad y(1) = 0 \quad y'(1) = 1$$

Solution

We first need to find the roots to (3).

$$2r(r-1) + 3r - 15 = 0$$

$$2r^2 + r - 15 = (2r-5)(r+3) = 0 \qquad \Rightarrow \qquad r_1 = \frac{5}{2}, \; r_2 = -3$$

The general solution is then,

$$y(x) = c_1 x^{\frac{5}{2}} + c_2 x^{-3}$$

To find the constants we differentiate and plug in the initial conditions as we did back in the second order differential equations chapter.

$$y'(x) = \frac{5}{2} c_1 x^{\frac{3}{2}} - 3c_2 x^{-4}$$

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2 \\ 1 &= y'(1) = \frac{5}{2} c_1 - 3c_2 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = \frac{2}{11},\ c_2 = -\frac{2}{11}$$

The actual solution is then,

$$y(x) = \frac{2}{11} x^{\frac{5}{2}} - \frac{2}{11} x^{-3}$$

With the solution to this example we can now see why we required $x>0$. The second term would have division by zero if we allowed $x=0$ and the first term would give us square roots of negative numbers if we allowed $x<0$.

Double Roots
This case will lead to the same problem that we've had every other time we've run into double roots (or double eigenvalues). We only get a single solution and will need a second solution. In this case it can be shown that the second solution will be,

$$y_2(x) = x^r \ln x$$

and so the general solution in this case is,

$$y(x) = c_1 x^r + c_2 x^r \ln x = x^r \left(c_1 + c_2 \ln x\right)$$

We can again see a reason for requiring $x>0$. If we didn't we'd have all sorts of problems with that logarithm.

Example 2 Find the general solution to the following differential equation.

$$x^2 y'' - 7xy' + 16y = 0$$

Solution
First the roots of (3).

$$\begin{aligned} r(r-1) - 7r + 16 &= 0 \\ r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \qquad \Rightarrow \qquad r = 4 \end{aligned}$$

So the general solution is then,

$$y(x) = c_1 x^4 + c_2 x^4 \ln x$$

Complex Roots
In this case we'll be assuming that our roots are of the form,

$$r_{1,2} = \lambda \pm \mu\, i$$

If we take the first root we'll get the following solution.

$$x^{\lambda + \mu i}$$

This is a problem since we don't want complex solutions, we only want real solutions. We can eliminate this by recalling that,

$$x^r = \mathbf{e}^{\ln x^r} = \mathbf{e}^{r \ln x}$$

Plugging the root into this gives,

$$\begin{aligned} x^{\lambda+\mu i} &= \mathbf{e}^{(\lambda+\mu i)\ln x} \\ &= \mathbf{e}^{\lambda \ln x}\mathbf{e}^{\mu i \ln x} \\ &= \mathbf{e}^{\ln x^{\lambda}}\left(\cos(\mu \ln x) + i \sin(\mu \ln x)\right) \\ &= x^{\lambda}\cos(\mu \ln x) + i x^{\lambda} \sin(\mu \ln x) \end{aligned}$$

Note that we had to use Euler formula as well to get to the final step. Now, as we've done every other time we've seen solutions like this we can take the real part and the imaginary part and use those for our two solutions.

So, in the case of complex roots the general solution will be,

$$y(x) = c_1 x^{\lambda}\cos(\mu \ln x) + c_2 x^{\lambda}\sin(\mu \ln x) = x^{\lambda}\left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)\right)$$

Once again we can see why we needed to require $x>0$.

Example 3 Find the solution to the following differential equation.

$$x^2 y'' + 3xy' + 4y = 0$$

Solution
Get the roots to (3) first as always.

$$\begin{aligned} r(r-1) + 3r + 4 &= 0 \\ r^2 + 2r + 4 &= 0 \qquad \Rightarrow \qquad r_{1,2} = -1 \pm \sqrt{3}\, i \end{aligned}$$

The general solution is then,

$$y(x) = c_1 x^{-1}\cos\left(\sqrt{3}\ln x\right) + c_2 x^{-1}\sin\left(\sqrt{3}\ln x\right)$$

We should now talk about how to deal with $x<0$ since that is a possibility on occasion. To deal with this we need to use the variable transformation,

$$\eta = -x$$

In this case since $x<0$ we will get $\eta>0$. Now, define,

$$u(\eta) = y(x) = y(-\eta)$$

Then using the chain rule we can see that,

$$u'(\eta) = -y'(x) \qquad \text{and} \qquad u''(\eta) = y''(x)$$

With this transformation the differential equation becomes,

$$\begin{aligned} a(-\eta)^2 u'' + b(-\eta)(-u') + cu &= 0 \\ a\eta^2 u'' + b\eta u' + cu &= 0 \end{aligned}$$

In other words, since $\eta > 0$ we can use the work above to get solutions to this differential equation. We'll also go back to x's by using the variable transformation in reverse.

$$\eta = -x$$

Let's just take the real, distinct case first to see what happens.

$$\begin{aligned} u(\eta) &= c_1\eta^{r_1} + c_2\eta^{r_2} \\ y(x) &= c_1(-x)^{r_1} + c_2(-x)^{r_2} \end{aligned}$$

Now, we could do this for the rest of the cases if we wanted to, but before doing that let's notice that if we recall the definition of absolute value,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

we can combine both of our solutions to this case into one and write the solution as,

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}, \qquad x \neq 0$$

Note that we still need to avoid $x=0$ since we could still get division by zero. However this is now a solution for any interval that doesn't contain $x=0$.

We can do likewise for the other two cases and the following solutions for any interval not containing $x=0$,.

$$\begin{aligned} y(x) &= c_1|x|^r + c_2|x|^r \ln|x| \\ y(x) &= c_1|x|^\lambda \cos(\mu\ln|x|) + c_2|x|^\lambda \sin(\mu\ln|x|) \end{aligned}$$

We can make one more generalization before working one more example. A more general form of an Euler Equation is,

$$a(x-x_0)^2 y'' + b(x-x_0)y' + cy = 0$$

and we can ask for solutions in any interval not containing $x = x_0$. The work for generating the solutions in this case is identical to all the above work and so isn't shown here.

The solutions in this general case for any interval not containing $x=a$ are,

$$\begin{aligned} y(x) &= c_1|x-a|^{r_1} + c_2|x-a|^{r_2} \\ y(x) &= |x-a|^r\left(c_1 + c_2\ln|x-a|\right) \\ y(x) &= |x-a|^\lambda\left(c_1\cos(\mu\ln|x-a|) + c_2\sin(\mu\ln|x-a|)\right) \end{aligned}$$

Where the roots are solutions to

$$ar(r-1)+b(r)+c=0$$

Example 4 Find the solution to the following differential equation on any interval not containing *x=-6*.

$$3(x+6)^2\, y''+25(x+6)\,y'-16y=0$$

Solution

So we get the roots from the identical quadratic in this case.

$$3r(r-1)+25r-16=0$$

$$3r^2+22r-16=0$$

$$(3r-2)(r+8)=0 \qquad \Rightarrow \qquad r_1=\frac{2}{3},\ r_2=-8$$

The general solution is then,

$$y(x)=c_1\left|x-a\right|^{\frac{2}{3}}+c_2\left|x-a\right|^{-8}$$

Higher Order Differential Equations

Introduction

In this chapter we're going to take a look at higher order differential equations. This chapter will actually contain more than most text books tend to have when they discuss higher order differential equations.

We will definitely cover the same material that most text books do here. However, in all the previous chapters all of our examples were 2nd order differential equations or 2 x 2 systems of differential equations. So, in this chapter we're also going to do a couple of examples here dealing with 3rd order or higher differential equations with Laplace transforms and series as well as a discussion of some larger systems of differential equations.

Here is a brief listing of the topics in this chapter.

Basic Concepts for *n*th Order Linear Equations – We'll start the chapter off with a quick look at some of the basic ideas behind solving higher order linear differential equations.

Linear Homogeneous Differential Equations – In this section we'll take a look at extending the ideas behind solving 2nd order differential equations to higher order.

Undetermined Coefficients – Here we'll look at undetermined coefficients for higher order differential equations.

Variation of Parameters – We'll look at variation of parameters for higher order differential equations in this section.

Laplace Transforms – In this section we're just going to work an example of using Laplace transforms to solve a differential equation on a 3rd order differential equation just so say that we looked at one with order higher than 2nd.

Systems of Differential Equations – Here we'll take a quick look at extending the ideas we discussed when solving 2 x 2 systems of differential equations to systems of size 3x3.

Series Solutions – This section serves the same purpose as the Laplace Transform section. It is just here so we can say we've worked an example using series solutions for a differential equations of order higher than 2nd.

Basic Concepts for n*th* *Order Linear Equations*

We'll start this chapter off with the material that most text books will cover in this chapter. We will take the material from the Second Order chapter and expand it out to n^{th} order linear differential equations. As we'll see almost all of the 2^{nd} order material will very naturally extend out to n^{th} order with only a little bit of new material.

So, let's start things off here with some basic concepts for n^{th} order linear differential equations. The most general n^{th} order linear differential equation is,

$$P_n(t)\,y^{(n)} + P_{n-1}(t)\,y^{(n-1)} + \cdots + P_1(t)\,y' + P_0(t)\,y = G(t) \qquad \textbf{(4)}$$

where you'll hopefully recall that,

$$y^{(m)} = \frac{d^m y}{dx^m}$$

Many of the theorems and ideas for this material require that $y^{(n)}$ has a coefficient of 1 and so if we divide out by $P_n(t)$ we get,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = g(t) \qquad \textbf{(5)}$$

As we might suspect an IVP for an n^{th} order differential equation will require the following n initial conditions.

$$y(t_0) = \overline{y}_0, \qquad y'(t_0) = \overline{y}_1, \qquad \cdots, \qquad y^{(n-1)}(t_0) = \overline{y}_{n-1} \qquad \textbf{(6)}$$

The following theorem tells us when we can expect there to be a unique solution to the IVP given by (2) and (3).

Theorem 1

Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ and $g(t)$ are all continuous in some open interval I containing t_0 then there is a unique solution to the IVP given by (2) and (3) and the solution will exist for all t in I.

This theorem is a very natural extension of a similar theorem we saw in the 1^{st} order material.

Next we need to move into a discussion of the n^{th} order linear homogeneous differential equation,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = 0 \qquad \textbf{(7)}$$

Let's suppose that we know $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions to (4) then by the an extension of the Principle of Superposition we know that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

will also be a solution to (4). The real question here is whether or not this will form a general solution to (4).

In order for this to be a general solution then we will have to be able to find constants $c_1, c_2, \ldots, c_n$ for any choice of t_0 (in the interval I from Theorem 1) and any choice of $\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n$. Or, in other words we need to be able to find $c_1, c_2, \ldots, c_n$ that will solve,

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) &= \overline{y}_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) &= \overline{y}_1 \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= \overline{y}_{n-1} \end{aligned}$$

Just as we did for 2nd order differential equations, we can use Cramer's Rule to solve this and the denominator of each the answers will be the following determinant of an n x n matrix.

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

As we did back with the 2nd order material we'll define this to be the **Wronskian** and denote it by,

$$W(y_1, y_2, \ldots y_n)(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Now that we have the definition of the Wronskian out of the way we need to get back to the question at hand. Because the Wronskian is the denominator in the solution to each of the c_i we can see that we'll have a solution provided it is not zero for any value of $t = t_0$ that we chose to evaluate the Wronskian at. The following theorem summarizes all this up.

Theorem 2

Suppose the functions $p_0, p_1, \ldots, p_{n-1}$ are all continuous on the open interval I and further suppose that $y_1(t), y_2(t), \ldots y_n(t)$ are all solutions to (4). If $W(y_1, y_2, \ldots y_n)(t) \neq 0$ for every t in I then $y_1(t), y_2(t), \ldots y_n(t)$ form a **Fundamental Set of Solutions** and the general solution to (4) is,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Recall as well that if a set of solutions form a fundamental set of solutions then they will also be a set of linearly independent functions.

We'll close this section off with a quick reminder of how we find solutions to the nonhomogeneous differential equation, (2). We first need the n^{th} order version of a theorem we saw back in the 2nd order material.

Theorem 3

Suppose that $Y_1(t)$ and $Y_2(t)$ are two solutions to (2) and that $y_1(t), y_2(t), \ldots y_n(t)$ are a fundamental set of solutions to the homogeneous differential equation (4) then,

$$Y_1(t) - Y_2(t)$$

is a solution to (4) and it can be written as

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Now, just as we did with the 2nd order material if we let $Y(t)$ be the general solution to (2) and if we let $Y_P(t)$ be any solution to (2) then using the result of this theorem we see that we must have,

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y_P(t) = y_c(t) + Y_P(t)$$

where, $y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ is called the **complementary solution** and $Y_P(t)$ is called a **particular solution**.

Over the course of the next couple of sections we'll discuss the differences in finding the complementary and particular solutions for n^{th} order differential equations in relation to what we know about 2nd order differential equations. We'll see that, for the most part, the methods are the same. The amount of work involved however will often be significantly more.

Linear Homogeneous Differential Equations

As with 2nd order differential equations we can't solve a nonhomogeneous differential equation unless we can first solve the homogeneous differential equation. We'll also need to restrict ourselves down to constant coefficient differential equations as solving non-constant coefficient differential equations is quite difficult and so we won't be discussing them here. Likewise, we'll only be looking at linear differential equations.

So, let's start off with the following differential equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Now, assume that solutions to this differential equation will be in the form $y(t) = \mathbf{e}^{rt}$ and plug this into the differential equation and with a little simplification we get,

$$\mathbf{e}^{rt}\left(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0\right) = 0$$

and so in order for this to be zero we'll need to require that

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

This is called the **characteristic polynomial/equation** and its roots/solutions will give us the solutions to the differential equation. We know that, including repeated roots, an n^{th} degree polynomial (which we have here) will have *n* roots. So, we need to go through all the possibilities that we've got for roots here.

This is where we start to see differences in how we deal with n^{th} order differential equations versus 2nd order differential equations. There are still the three main cases: real distinct roots, repeated roots and complex roots (although these can now also be repeated as we'll see). In 2nd order differential equations each differential equation could only involve one of these cases. Now, however, that will not necessarily be the case. We could very easily have differential equations that contain each of these cases.

For instance suppose that we have an 9th order differential equation. The complete list of roots could have 3 roots which only occur once in the list (*i.e.* real distinct roots), a root with multiplicity 4 (*i.e.* occurs 4 times in the list) and a set of complex conjugate roots (recall that because the coefficients are all real complex roots will always occur in conjugate pairs).

So, for each n^{th} order differential equation we'll need to form a set of *n* linearly independent functions (*i.e.* a fundamental set of solutions) in order to get a general solution. In the work that follows we'll discuss the solutions that we get from each case but we will leave it to you to verify that when we put everything together to form a general solution that we do indeed get a fundamental set of solutions. Recall that in order to this we need to verify that the Wronskian is not zero.

So, let's get started with the work here. Let's start off by assuming that in the list of roots of the characteristic equation we have $r_1, r_2, \ldots, r_k$ and they only occur once in the list. The solution from each of these will then be,

$$\mathbf{e}^{r_1 t}, \quad \mathbf{e}^{r_2 t}, \quad \cdots, \quad \mathbf{e}^{r_k t}$$

There's nothing really new here for real distinct roots.

Now let's take a look at repeated roots. The result here is a natural extension of the work we saw in the 2nd order case. Let's suppose that r is a root of multiplicity k (*i.e.* r occurs k times in the list of roots). We will then get the following k solutions to the differential equation,

$$\mathbf{e}^{rt}, \qquad t\,\mathbf{e}^{rt}, \qquad \cdots, \qquad t^{k-1}\mathbf{e}^{rt}$$

So, for repeated roots we just add in a t for each of the solutions past the first one until we have a total of k solutions. Again, we will leave it to you to compute the Wronskian to verify that these are in fact a set of linearly independent solutions.

Finally we need to deal with complex roots. The biggest issue here is that we can now have repeated complex roots for 4th order or higher differential equations. We'll start off by assuming that $r = \lambda \pm \mu\, i$ occurs only once in the list of roots. In this case we'll get the standard two solutions,

$$\mathbf{e}^{\lambda t}\cos\left(\mu\, t\right) \qquad \mathbf{e}^{\lambda t}\sin\left(\mu\, t\right)$$

Now let's suppose that $r = \lambda \pm \mu\, i$ has a multiplicity of k (*i.e.* they occur k times in the list of roots). In this case we can use the work from the repeated roots above to get the following set of $2k$ complex-valued solutions,

$$\begin{array}{lll} \mathbf{e}^{(\lambda+\mu i)t}, t\,\mathbf{e}^{(\lambda+\mu i)t}, & \cdots, & t^{k-1}\mathbf{e}^{(\lambda+\mu i)t} \\ \mathbf{e}^{(\lambda-\mu i)t}, t\,\mathbf{e}^{(\lambda-\mu i)t}, & \cdots, & t^{k-1}\mathbf{e}^{(\lambda-\mu i)t} \end{array}$$

The problem here of course is that we really want real-valued solutions. So, recall that in the case where they occurred once all we had to do was use Euler's formula on the first one and then take the real and imaginary part to get two real valued solutions. We'll do the same thing here and use Euler's formula on the first set of complex-valued solutions above, split each one into its real and imaginary parts to arrive at the following set of $2k$ real-valued solutions.

$$\begin{array}{l} \mathbf{e}^{\lambda t}\cos\left(\mu\, t\right), \quad \mathbf{e}^{\lambda t}\sin\left(\mu\, t\right), \quad t\mathbf{e}^{\lambda t}\cos\left(\mu\, t\right), \quad t\mathbf{e}^{\lambda t}\sin\left(\mu\, t\right), \quad \cdots, \\ \qquad\qquad\qquad\qquad t^{k-1}\mathbf{e}^{\lambda t}\cos\left(\mu\, t\right), \quad t^{k-1}\mathbf{e}^{\lambda t}\sin\left(\mu\, t\right) \end{array}$$

Once again we'll leave it to you to verify that these do in fact form a fundamental set of solutions.

Before we work a couple of quick examples here we should point out that the characteristic polynomial is now going to be at least a 3rd degree polynomial and finding the roots of these by hand is often a very difficult and time consuming process and in many cases if the roots are not rational (*i.e.* in the form $\frac{p}{q}$) it can be almost impossible to find them all by hand. To see a process for determining all the rational roots of a polynomial check out the Finding Zeroes of Polynomials page in my Algebra notes. In practice however, we usually use some form of computation aid such as Maple or Mathematica to find all the roots.

So, let's work a couple of example here to illustrate at least some of the ideas discussed here.

Example 1 Solve the following IVP.

$$y^{(3)} - 5y'' - 22y' + 56y = 0 \qquad y(0) = 1 \quad y'(0) = -2 \quad y''(0) = -4$$

Solution

The characteristic equation is,

$$r^3 - 5r^2 - 22r + 56 = (r+4)(r-2)(r-7) = 0 \qquad \Rightarrow \qquad r_1 = -4,\ r_2 = 2,\ r_3 = 7$$

So we have three real distinct roots here and so the general solution is,

$$y(t) = c_1 \mathbf{e}^{-4t} + c_2 \mathbf{e}^{2t} + c_3 \mathbf{e}^{7t}$$

Differentiating a couple of times and applying the initial conditions gives the following system of equations that we'll need to solve in order to find the coefficients.

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 + c_3 \\ -2 &= y'(0) = -4c_1 + 2c_2 + 7c_3 \\ -4 &= y''(0) = 16c_1 + 4c_2 + 49c_3 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} c_1 &= \tfrac{14}{33} \\ c_2 &= \tfrac{13}{15} \\ c_3 &= -\tfrac{16}{55} \end{aligned}$$

The actual solution is then,

$$y(t) = \tfrac{14}{33}\mathbf{e}^{-4t} + \tfrac{13}{15}\mathbf{e}^{2t} - \tfrac{16}{55}\mathbf{e}^{7t}$$

So, outside of needing to solve a cubic polynomial (which we left the details to you to verify) and needing to solve a system of 3 equations to find the coefficients (which we also left to you to fill in the details) the work here is pretty much identical to the work we did in solving a 2nd order IVP.

Because the initial condition work is identical to work that we should be very familiar with to this point with the exception that it involved solving larger systems we're going to not bother with solving IVP's for the rest of the examples. The main point of this section is the new ideas involved in finding the general solution to the differential equation anyway and so we'll concentrate on that for the remaining examples.

Also note that we'll not be showing very much work in solving the characteristic polynomial. We are using computational aids here and would encourage you to do the same here. Solving these higher degree polynomials is just too much work and would obscure the point of these examples.

So, let's move into a couple of examples where we have more than one case involved in the solution.

Example 2 Solve the following differential equation.

$$2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$$

Solution

The characteristic equation is,

$$2r^4 + 11r^3 + 18r^2 + 4r - 8 = (2r-1)(r+2)^3 = 0$$

So, we have two roots here, $r_1 = \frac{1}{2}$ and $r_2 = -2$ which is multiplicity of 3. Remember that we'll get three solutions for the second root and after the first one we add t's only the solution until we reach three solutions.

The general solution is then,

$$y(t) = c_1\mathbf{e}^{\frac{1}{2}t} + c_2\mathbf{e}^{-2t} + c_3 t\mathbf{e}^{-2t} + c_4 t^2\mathbf{e}^{-2t}$$

Example 3 Solve the following differential equation.

$$y^{(5)} + 12y^{(4)} + 104y^{(3)} + 408y'' + 1156y' = 0$$

Solution

The characteristic equation is,

$$r^5 + 12r^4 + 104r^3 + 408r^2 + 1156r = r\left(r^2 + 6r + 34\right)^2 = 0$$

So, we have one real root $r = 0$ and a pair of complex roots $r = -3 \pm 5i$ each with multiplicity 2. So, the solution for the real root is easy and for the complex roots we'll get a total of 4 solutions, 2 will be the *normal* solutions and two will be the normal solution each multiplied by t.

The general solution is,

$$y(t) = c_1 + c_2\mathbf{e}^{-3t}\cos(5t) + c_3\mathbf{e}^{-3t}\sin(5t) + c_4\,t\,\mathbf{e}^{-3t}\cos(5t) + c_5\,t\,\mathbf{e}^{-3t}\sin(5t)$$

Let's now work an example that contains all three of the basic cases just to say that we that we've got one work here.

Example 4 Solve the following differential equation.

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

Solution

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2\left(r^2 - 4r + 5\right) = 0$$

In this case we've got one real distinct root, $r = 1$, and double root, $r = 5$, and a pair of complex roots, $r = 2 \pm i$ that only occur once.

The general solution is then,

$$y(t) = c_1\mathbf{e}^{t} + c_2\mathbf{e}^{5t} + c_3\,t\,\mathbf{e}^{5t} + c_4\mathbf{e}^{2t}\cos(t) + c_5\mathbf{e}^{2t}\sin(t)$$

We've got one final example to work here that on the surface at least seems almost too easy. The problem here will be finding the roots as well see.

Example 5 Solve the following differential equation.

$$y^{(4)} + 16y = 0$$

Solution

The characteristic equation is

$$r^4 + 16 = 0$$

So, a really simple characteristic equation. However, in order to find the roots we need to compute the fourth root of -16 and that is something that most people haven't done at this point in their mathematical career. We'll just give the formula here for finding them, but if you're interested in seeing a little more about this you might want to check out the Powers and Roots section of my Complex Numbers Primer.

The 4 (and yes there are 4!) 4th roots of -16 can be found by evaluating the following,

$$\sqrt[4]{-16} = \left(-16\right)^{\frac{1}{4}} = \sqrt[4]{16}\mathbf{e}^{\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)i} = 2\left(\cos\left(\tfrac{\pi}{4}+\tfrac{\pi k}{2}\right)+i\sin\left(\tfrac{\pi}{4}+\tfrac{\pi k}{2}\right)\right) \qquad k = 0,1,2,3$$

Note that each value of k will give a distinct 4th root of -16. Also, note that for the 4th root (and ONLY the 4th root) of any negative number all we need to do is replace the 16 in the above formula with the absolute value of the number in question and this formula will work for those as well.

Here are the 4th roots of -16.

$$k=0:\quad 2\left(\cos\left(\tfrac{\pi}{4}\right)+i\sin\left(\tfrac{\pi}{4}\right)\right) = 2\left(\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right) = \sqrt{2}+i\sqrt{2}$$

$$k=1:\quad 2\left(\cos\left(\tfrac{3\pi}{4}\right)+i\sin\left(\tfrac{3\pi}{4}+\tfrac{\pi k}{2}\right)\right) = 2\left(-\tfrac{1}{\sqrt{2}}+\tfrac{1}{\sqrt{2}}i\right) = -\sqrt{2}+i\sqrt{2}$$

$$k=2:\quad 2\left(\cos\left(\tfrac{5\pi}{4}\right)+i\sin\left(\tfrac{5\pi}{4}\right)\right) = 2\left(-\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = -\sqrt{2}-i\sqrt{2}$$

$$k=3:\quad 2\left(\cos\left(\tfrac{7\pi}{4}\right)+i\sin\left(\tfrac{7\pi}{4}\right)\right) = 2\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right) = \sqrt{2}-i\sqrt{2}$$

So, we have two sets of complex roots : $r = \sqrt{2} \pm i\sqrt{2}$ and $r = -\sqrt{2} \pm i\sqrt{2}$. The general solution is,

$$y(t) = c_1\mathbf{e}^{\sqrt{2}t}\cos\left(\sqrt{2}\,t\right) + c_2\mathbf{e}^{\sqrt{2}t}\sin\left(\sqrt{2}\,t\right) + c_3\mathbf{e}^{-\sqrt{2}t}\cos\left(\sqrt{2}\,t\right) + c_4\mathbf{e}^{-\sqrt{2}t}\sin\left(\sqrt{2}\,t\right)$$

So, we've worked a handful of examples here of higher order differential equations that should give you a feel for how these work in most cases.

There are of course a great many different kinds of combinations of the basic cases than what we did here and of course we didn't work any case involving 6th order or higher, but once you've got an idea on how these work it's pretty easy to see that they all work pretty in pretty much the same manner. The biggest problem with the higher order differential equations is that the work in solving the characteristic polynomial and the system for the coefficients on the solution can be quite involved.

Undetermined Coefficients

We now need to start looking into determining a particular solution for n^{th} order differential equations. The two methods that we'll be looking at are the same as those that we looked at in the 2nd order chapter.

In this section we'll look at the method of Undetermined Coefficients and this will be a fairly short section. With one small extension, which we'll see in the lone example in this section, the method is identical to what we saw back when we were looking at undetermined coefficients in the 2nd order differential equations chapter.

Given the differential equation,

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \cdots + p_1(t)\, y' + p_0(t)\, y = g(t)$$

if $g(t)$ is an exponential function, polynomial, sine, cosine, sum/difference of one of these and/or a product of one of these then we guess the form of a particular solution using the same guidelines that we used in the 2nd order material. We then plug the guess into the differential equation, simplify and set the coefficients equal to solve for the constants.

The one thing that we need to recall is that we first need the complementary solution prior to making our guess for a particular solution. If any term in our guess is in the complementary solution then we need to multiply the portion of our guess that contains that term by a *t*. This is where the one extension to the method comes into play. With a 2nd order differential equation the most we'd ever need to multiply by is t^2. With higher order differential equations this may need to be more than t^2.

The work involved here is almost identical to the work we've already done and in fact it isn't even that much more difficult unless the guess is particularly messy and that makes for more mess when we take the derivatives and solve for the coefficients. Because there isn't much difference in the work here we're only going to do a single example in this section illustrating the extension. So, let's take a look at the lone example we're going to do here.

Example 1 Solve the following differential equation.

$$y^{(3)} - 12y'' + 48y' - 64y = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$$

Solution

We first need the complementary solution so the characteristic equation is,

$$r^3 - 12r^2 + 48r - 64 = (r-4)^3 = 0 \qquad \Rightarrow \qquad r = 4 \ (\text{multiplicity } 3)$$

We've got a single root of multiplicity 3 so the complementary solution is,

$$y_c(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t}$$

Now, our first guess for a particular solution is,

$$Y_P = A + B\mathbf{e}^{-8t} + C\mathbf{e}^{4t}$$

Notice that the last term in our guess is in the complementary solution so we'll need to add one at least one *t* to the third term in our guess. Also notice that multiplying the third term by either *t* or

t^2 will result in a new term that is still in the complementary solution and so we'll need to multiply the third term by t^3 in order to get a term that is not contained in the complementary solution.

Our final guess is then,

$$Y_P = A + B\mathbf{e}^{-8t} + Ct^3\mathbf{e}^{4t}$$

Now all we need to do is take three derivatives of this, plug this into the differential equation and simplify to get (we'll leave it to you to verify the work here),

$$-64A - 1728B\mathbf{e}^{-8t} + 6C\mathbf{e}^{4t} = 12 - 32\mathbf{e}^{-8t} + 2\mathbf{e}^{4t}$$

Setting coefficients equal and solving gives,

$$\begin{array}{rrcl} t^0: & -64A = 12 & & A = -\frac{3}{16} \\ \mathbf{e}^{-8t}: & -1728B = -32 & \Rightarrow & B = \frac{1}{54} \\ \mathbf{e}^{4t}: & 6C = 2 & & C = \frac{1}{3} \end{array}$$

A particular solution is then,

$$Y_P = -\frac{3}{16} + \frac{1}{54}\mathbf{e}^{-8t} + \frac{1}{3}t^3\mathbf{e}^{4t}$$

The general solution to this differential equation is then,

$$y(t) = c_1\mathbf{e}^{4t} + c_2 t\mathbf{e}^{4t} + c_3 t^2\mathbf{e}^{4t} - \frac{3}{16} + \frac{1}{54}\mathbf{e}^{-8t} + \frac{1}{3}t^3\mathbf{e}^{4t}$$

Okay, we've only worked one example here, but remember that we mentioned earlier that with the exception of the extension to the method that we used in this example the work here is identical to work we did the 2nd order material.

Variation of Parameters

We now need to take a look at the second method of determining a particular solution to a differential equation. As we did when we first saw Variation of Parameters we'll go through the whole process and derive up a set of formulas that can be used to generate a particular solution.

However, as we saw previously when looking at 2nd order differential equations this method can lead to integrals that are not easy to evaluate. So, while this method can always be used, unlike Undetermined Coefficients, to at least write down a formula for a particular solution it is not always going to be possible to actually get a solution.

So let's get started on the process. We'll start with the differential equation,

$$y^{(n)} + p_{n-1}(t)\,y^{(n-1)} + \cdots + p_1(t)\,y' + p_0(t)\,y = g(t) \qquad \textbf{(1)}$$

and assume that we've found a fundamental set of solutions, $y_1(t), y_2(t), \ldots, y_n(t)$, for the associated homogeneous differential equation.

Because we have a fundamental set of solutions to the homogeneous differential equation we now know that the complementary solution is,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

The method of variation of parameters involves trying to find a set of new functions, $u_1(t), u_2(t), \ldots, u_n(t)$ so that,

$$Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t) + \cdots + u_n(t)\,y_n(t) \qquad \textbf{(2)}$$

will be a solution to the nonhomogeneous differential equation. In order to determine if this is possible, and to find the $u_i(t)$ if it is possible, we'll need a total of *n* equations involving the unknown functions that we can (hopefully) solve.

One of the equations is easy. The guess, (2), will need to satisfy the original differential equation, (1). So, let's start taking some derivatives and as we did when we first looked at variation of parameters we'll make some assumptions along the way that will simplify our work and in the process generate the remaining equations we'll need.

The first derivative of (2) is,

$$Y'(t) = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n' + u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n$$

Note that we rearranged the results of the differentiation process a little here and we dropped the (t) part on the *u* and *y* to make this a little easier to read. Now, if we keep differentiating this it will quickly become unwieldy and so let's make as assumption to simplify things here. Because we are after the $u_i(t)$ we should probably try to avoid letting the derivatives on these become too large. So, let's make the assumption that,

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0$$

The natural question at this point is does this even make sense to do? The answer is, if we end up with a system of *n* equations that we can solve for the $u_i(t)$ then yes it does make sense to do.

Of course, the other answer is, we wouldn't be making this assumption if we didn't know that it was going to work. However to accept this answer requires that you trust us to make the correct assumptions so maybe the first answer is the best at this point.

At this point the first derivative of (2) is,

$$Y'(t) = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'$$

and so we can now take the second derivative to get,

$$Y''(t) = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n'' + u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'$$

This looks an awful lot like the original first derivative prior to us simplifying it so let's again make a simplification. We'll again want to keep the derivatives on the $u_i(t)$ to a minimum so this time let's assume that,

$$u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0$$

and with this assumption the second derivative becomes,

$$Y''(t) = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n''$$

Hopefully you're starting to see a pattern develop here. If we continue this process for the first $n-1$ derivatives we will arrive at the following formula for these derivatives.

$$Y^{(k)}(t) = u_1 y_1^{(k)} + u_2 y_2^{(k)} + \cdots + u_n y_n^{(k)} \qquad k = 1, 2, \ldots, n-1 \qquad \textbf{(3)}$$

To get to each of these formulas we also had to assume that,

$$u_1' y_1^{(k)} + u_2' y_2^{(k)} + \cdots + u_n' y_n^{(k)} = 0 \qquad k = 0, 1, \ldots n-2 \qquad \textbf{(4)}$$

and recall that the 0th derivative of a function is just the function itself. So, for example, $y_2^{(0)}(t) = y_2(t)$.

Notice as well that the set of assumptions in (4) actually give us $n-1$ equations in terms of the derivatives of the unknown functions : $u_1(t), u_2(t), \ldots, u_n(t)$.

All we need to do then is finish generating the first equation we started this process to find (*i.e.* plugging (2) into (1)). To do this we'll need one more derivative of the guess. Differentiating the $(n-1)^{\text{st}}$ derivative, which we can get from (3), to get the n^{th} derivative gives,

$$Y^{(n)}(t) = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}$$

This time we'll also not be making any assumptions to simplify this but instead just plug this along with the derivatives given in (3) into the differential equation, (1)

$$\begin{aligned}
u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} + \\
p_{n-1}(t)\left[u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right] + \\
\vdots \qquad\qquad \\
p_1(t)\left[u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'\right] + \\
p_0(t)\left[u_1 y_1 + u_2 y_2 + \cdots + u_n y_n\right] = g(t)
\end{aligned}$$

Next, rearrange this a little to get,

$$\begin{aligned}
u_1\left[y_1^{(n)} + p_{n-1}(t)\, y_1^{(n-1)} + \cdots + p_1(t)\, y_1' + p_0(t)\, y_1\right] + \\
u_2\left[y_2^{(n)} + p_{n-1}(t)\, y_2^{(n-1)} + \cdots + p_1(t)\, y_2' + p_0(t)\, y_2\right] + \\
\vdots \qquad\qquad \\
u_n\left[y_n^{(n)} + p_{n-1}(t)\, y_n^{(n-1)} + \cdots + p_1(t)\, y_n' + p_0(t)\, y_n\right] + \\
u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g(t)
\end{aligned}$$

Recall that $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions to the homogeneous differential equation and so all the quantities in the [ ] are zero and this reduces down to,

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g(t)$$

So this equation, along with those given in (4), give us the *n* equations that we needed. Let's list them all out here for the sake of completeness.

$$\begin{aligned}
u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n &= 0 \\
u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' &= 0 \\
u_1' y_1'' + u_2' y_2'' + \cdots + u_n' y_n'' &= 0 \\
&\vdots \\
u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} &= 0 \\
u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= g(t)
\end{aligned}$$

So, we've got *n* equations, but notice that just like we got when we did this for 2nd order differential equations the unknowns in the system are not $u_1(t), u_2(t), \ldots, u_n(t)$ but instead they are the derivatives, $u_1'(t), u_2'(t), \ldots, u_n'(t)$. This isn't a major problem however. Provided we can solve this system we can then just integrate the solutions to get the functions that we're after.

Also, recall that the $y_1(t), y_2(t), \ldots, y_n(t)$ are assumed to be known functions and so they along with their derivatives (which appear in the system) are all known quantities in the system.

Now, we need to think about how to solve this system. If there aren't too many equations we can just solve it directly if we want to. However, for large *n* (and it won't take much to get large here) that could be quite tedious and prone to error and it won't work at all for general *n* as we have here.

The best solution method to use at this point is then Cramer's Rule. We've used Cramer's Rule several times in this course, but the best reference for our purposes here is when we used it when we first defined Fundamental Sets of Solutions back in the 2nd order material.

Upon using Cramer's Rule to solve the system the resulting solution for each u_i' will be a quotient of two determinants of *n* x *n* matrices. The denominator of each solution will be the determinant of the matrix of the known coefficients,

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W\left(y_1, y_2, \ldots y_n\right)(t)$$

This however, is just the Wronskian of $y_1(t), y_2(t), \ldots, y_n(t)$ as noted above and because we have assumed that these form a fundamental set of solutions we also know that the Wronskian will not be zero. This in turn tells us that the system above is in fact solvable and all of the assumptions we apparently made out of the blue above did in fact work.

The numerators of the solution for u_i' will be the determinant of the matrix of coefficients with the i^{th} column replaced with the column $\left(0,0,0,\ldots,0,g(t)\right)$. For example, the numerator for the first one, u_1' is,

$$\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ g(t) & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Now, by a nice property of determinants if we factor something out of one of the columns of a matrix then the determinant of the resulting matrix will be the factor times the determinant of new matrix. In other words, if we factor $g(t)$ out of this matrix we arrive at,

$$\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ g(t) & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = g(t)\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

We did this only for the first one, but we could just as easily done this with any of the *n* solutions. So, let W_i represent the determinant we get by replacing the i^{th} column of the Wronskian with the column (0,0,0,…,0,1) and the solution to the system can then be written as,

$$u_1' = \frac{g(t)W_1(t)}{W(t)}, \quad u_2' = \frac{g(t)W_2(t)}{W(t)}, \quad \cdots, \quad u_n' = \frac{g(t)W_n(t)}{W(t)}$$

Wow! That was a lot of effort to generate and solve the system but we're almost there. With the solution to the system in hand we can now integrate each of these terms to determine just what the unknown functions, $u_1(t), u_2(t), \ldots, u_n(t)$ we've after all along are.

$$u_1 = \int \frac{g(t)W_1(t)}{W(t)}\,dt, \quad u_2 = \int \frac{g(t)W_2(t)}{W(t)}\,dt, \quad \cdots, \quad u_n = \int \frac{g(t)W_n(t)}{W(t)}\,dt$$

Finally, a particular solution to (1) is then given by,

$$Y(t) = y_1(t)\int \frac{g(t)W_1(t)}{W(t)}\,dt + y_2(t)\int \frac{g(t)W_2(t)}{W(t)}\,dt + \cdots + y_n(t)\int \frac{g(t)W_n(t)}{W(t)}\,dt$$

We should also note that in the derivation process here we assumed that the coefficient of the $y^{(n)}$ term was a one and that has been factored into the formula above. If the coefficient of this term is not one then we'll need to make sure and divide it out before trying to use this formula.

Before we work an example here we really should note that while we can write this formula down actually computing these integrals may be all but impossible to do.

Okay let's take a look at a quick example.

Example 1 Solve the following differential equation.

$$y^{(3)} - 2y'' - 21y' - 18y = 3 + 4\mathbf{e}^{-t}$$

Solution

The characteristic equation is,

$$r^3 - 2r^2 - 21r - 18 = (r+3)(r+1)(r-6) = 0 \qquad \Rightarrow \qquad r_1 = -3,\ r_2 = -1,\ r_3 = 6$$

So we have three real distinct roots here and so the general solution is,

$$y_c(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t}$$

Okay, we've now got several determinants to compute. We'll leave it to you to verify the following determinant computations.

$$W = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 126\mathbf{e}^{2t} \qquad W_1 = \begin{vmatrix} 0 & \mathbf{e}^{-t} & \mathbf{e}^{6t} \\ 0 & -\mathbf{e}^{-t} & 6\mathbf{e}^{6t} \\ 1 & \mathbf{e}^{-t} & 36\mathbf{e}^{6t} \end{vmatrix} = 7\mathbf{e}^{5t}$$

$$W_2 = \begin{vmatrix} \mathbf{e}^{-3t} & 0 & \mathbf{e}^{6t} \\ -3\mathbf{e}^{-3t} & 0 & 6\mathbf{e}^{6t} \\ 9\mathbf{e}^{-3t} & 1 & 36\mathbf{e}^{6t} \end{vmatrix} = -9\mathbf{e}^{3t} \qquad W_3 = \begin{vmatrix} \mathbf{e}^{-3t} & \mathbf{e}^{-t} & 0 \\ -3\mathbf{e}^{-3t} & -\mathbf{e}^{-t} & 0 \\ 9\mathbf{e}^{-3t} & \mathbf{e}^{-t} & 1 \end{vmatrix} = 2\mathbf{e}^{-4t}$$

Now, given that $g(t) = 3 + 4\mathbf{e}^{-t}$ we can compute each of the u_i. Here are those integrals.

$$u_1 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(7\mathbf{e}^{5t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{18}\int 3\mathbf{e}^{3t} + 4\mathbf{e}^{2t}\,dt = \frac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)$$

$$u_2 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(-9\mathbf{e}^{3t}\right)}{126\mathbf{e}^{2t}}\,dt = -\frac{1}{14}\int 3\mathbf{e}^{t} + 4\,dt = -\frac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)$$

$$u_3 = \int \frac{\left(3+4\mathbf{e}^{-t}\right)\left(2\mathbf{e}^{-4t}\right)}{126\mathbf{e}^{2t}}\,dt = \frac{1}{63}\int 3\mathbf{e}^{-6t} + 4\mathbf{e}^{-7t}\,dt = \frac{1}{63}\left(-\frac{1}{2}\mathbf{e}^{-6t} - \frac{4}{7}\mathbf{e}^{-7t}\right)$$

Note that we didn't include the constants of integration in each of these because including them would just have introduced a term that would get absorbed into the complementary solution just as we saw when we were dealing with 2nd order differential equations.

Finally, a particular solution for this differential equation is then,

$$\begin{aligned} Y_P &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= \tfrac{1}{18}\left(\mathbf{e}^{3t} + 2\mathbf{e}^{2t}\right)\mathbf{e}^{-3t} - \tfrac{1}{14}\left(3\mathbf{e}^{t} + 4t\right)\mathbf{e}^{-t} + \tfrac{1}{63}\left(-\tfrac{1}{2}\mathbf{e}^{-6t} - \tfrac{4}{7}\mathbf{e}^{-7t}\right)\mathbf{e}^{6t} \\ &= -\tfrac{1}{6} + \tfrac{5}{49}\mathbf{e}^{-t} - \tfrac{2}{7}t\mathbf{e}^{-t} \end{aligned}$$

The general solution is then,

$$y(t) = c_1\mathbf{e}^{-3t} + c_2\mathbf{e}^{-t} + c_3\mathbf{e}^{6t} - \tfrac{1}{6} + \tfrac{5}{49}\mathbf{e}^{-t} - \tfrac{2}{7}t\mathbf{e}^{-t}$$

We're only going to do a single example in this section to illustrate the process more than anything so with that we'll close out this section.

Laplace Transforms

There really isn't all that much to this section. All we're going to do here is work a quick example using Laplace transforms for a 3rd order differential equation so we can say that we worked at least one problem for a differential equation whose order was larger than 2.

Everything that we know from the Laplace Transforms chapter is still valid. The only new bit that we'll need here is the Laplace transform of the third derivative. We can get this from the general formula that we gave when we first started looking at solving IVP's with Laplace transforms. Here is that formula,

$$\mathfrak{L}\{y'''\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Here's the example for this section.

Example 1 Solve the following IVP.

$$y''' - 4y'' = 4t + 3u_6(t)\mathbf{e}^{30-5t}, \quad y(0) = -3 \quad y'(0) = 1 \quad y''(0) = 4$$

Solution

As always we first need to make sure the function multiplied by the Heaviside function has been properly shifted.

$$y''' - 4y'' = 4t + 3u_6(t)\mathbf{e}^{-5(t-6)}$$

It has been properly shifted and we can see that we're shifting $\mathbf{e}^{-5t}$. All we need to do now is take the Laplace transform of everything, plug in the initial conditions and solve for $Y(s)$. Doing all of this gives,

$$\begin{aligned}
s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 4\left(s^2Y(s) - sy(0) - y'(0)\right) &= \frac{4}{s^2} + \frac{3\mathbf{e}^{-6s}}{s+5} \\
\left(s^3 - 4s^2\right)Y(s) + 3s^2 - 13s &= \frac{4}{s^2} + \frac{3\mathbf{e}^{-6s}}{s+5} \\
\left(s^3 - 4s^2\right)Y(s) &= \frac{4}{s^2} - 3s^2 + 13s + \frac{3\mathbf{e}^{-6s}}{s+5} \\
\left(s^3 - 4s^2\right)Y(s) &= \frac{4 - 3s^4 + 13s^3}{s^2} + \frac{3\mathbf{e}^{-6s}}{s+5} \\
Y(s) &= \frac{4 - 3s^4 + 13s^3}{s^4(s-4)} + \frac{3\mathbf{e}^{-6s}}{s^2(s-4)(s+5)} \\
Y(s) &= F(s) + 3\mathbf{e}^{-6s}G(s)
\end{aligned}$$

Now we need to partial fraction and inverse transform *F(s)* and *G(s)*. We'll leave it to you to verify the details.

$$F(s) = \frac{4 - 3s^4 + 13s^3}{s^4(s-4)} = \frac{\frac{17}{64}}{s-4} - \frac{\frac{209}{64}}{s} - \frac{\frac{1}{16}}{s^2} - \frac{\frac{1}{4}\left(\frac{2!}{2!}\right)}{s^3} - \frac{1\left(\frac{3!}{3!}\right)}{s^4}$$

$$f(t) = \tfrac{17}{64}\mathbf{e}^{4t} - \tfrac{209}{64} - \tfrac{1}{16}t - \tfrac{1}{8}t^2 - \tfrac{1}{6}t^3$$

$$G(s) = \frac{1}{s^2(s-4)(s+5)} = \frac{\frac{1}{144}}{s-4} - \frac{\frac{1}{225}}{s+5} - \frac{\frac{1}{400}}{s} - \frac{\frac{1}{20}}{s^2}$$

$$g(t) = \tfrac{1}{144}\mathbf{e}^{4t} - \tfrac{1}{225}\mathbf{e}^{-5t} - \tfrac{1}{400} - \tfrac{1}{20}t$$

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

$$Y(s) = F(s) + 3\mathbf{e}^{-6s}G(s) \qquad \Rightarrow \qquad y(t) = f(t) + 3u_6(t)\,g(t-6)$$

where *f(t)* and *g(t)* are the functions shown above.

Okay, there is the one Laplace transform example with a differential equation with order greater than 2. As you can see the work in identical except for the fact that the partial fraction work (which we didn't show here) is liable to be messier and more complicated.

Systems of Differential Equations

In this section we want to take a brief look at systems of differential equations that are larger than 2 x 2. The problem here is that unlike the first few sections where we looked at n^{th} order differential equations we can't really come up with a set of formulas that will always work for every system. So, with that in mind we're going to look at all possible cases for a 3 x 3 system (leaving some details for you to verify at times) and then a couple of quick comments about 4 x 4 systems to illustrate how to extend things out to even larger systems and then we'll leave it to you to actually extend things out if you'd like to.

We will also not be doing any actual examples in this section. The point of this section is just to show how to extend out what we know about 2 x 2 systems to larger systems.

Initially the process is identical regardless of the size of the system. So, for a system of 3 differential equations with 3 unknown functions we first put the system into matrix form,

$$\vec{x}' = A\vec{x}$$

where the coefficient matrix, A, is a 3 x 3 matrix. We next need to determine the eigenvalues and eigenvectors for A and because A is a 3 x 3 matrix we know that there will be 3 eigenvalues (including repeated eigenvalues if there are any).

This is where the process from the 2 x 2 systems starts to vary. We will need a total of 3 linearly independent solutions to form the general solution. Some of what we know from the 2 x 2 systems can be brought forward to this point. For instance, we know that solutions corresponding to simple eigenvalues (*i.e.* they only occur once in the list of eigenvalues) will be linearly independent. We know that solutions from a set of complex conjugate eigenvalues will be linearly independent. We also know how to get a set of linearly independent solutions from a double eigenvalue with a single eigenvector.

There are also a couple of facts about eigenvalues/eigenvectors that we need to review here as well. First, provided A has only real entries (which it always will here) all complex eigenvalues will occur in conjugate pairs (*i.e.* $\lambda = \alpha \pm \beta i$) and their associated eigenvectors will also be complex conjugates of each other. Next, if an eigenvalue has multiplicity $k \geq 2$ (*i.e.* occurs at least twice in the list of eigenvalues) then there will be anywhere from 1 to k linearly independent eigenvectors for the eigenvalue.

With all these ideas in mind let's start going through all the possible combinations of eigenvalues that we can possibly have for a 3 x 3 case. Let's also note that for a 3 x 3 system it is impossible to have only 2 real distinct eigenvalues. The only possibilities are to have 1 or 3 real distinct eigenvalues.

Here are all the possible cases.

3 Real Distinct Eigenvalues

In this case we'll have the real, distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and their associated eigenvectors, $\vec{\eta}_1$, $\vec{\eta}_2$ and $\vec{\eta}_3$ are guaranteed to be linearly independent and so the three linearly independent solutions we get from this case are,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1 \qquad \mathbf{e}^{\lambda_2 t}\vec{\eta}_2 \qquad \mathbf{e}^{\lambda_3 t}\vec{\eta}_3$$

1 Real and 2 Complex Eigenvalues
From the real eigenvalue/eigenvector pair, λ_1 and $\vec{\eta}_1$, we get one solution,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1$$

We get the other two solutions in the same manner that we did with the 2 x 2 case. If the eigenvalues are $\lambda_{2,3} = \alpha \pm \beta\, i$ with eigenvectors $\vec{\eta}_2$ and $\vec{\eta}_3 = \overline{\left(\vec{\eta}_2\right)}$ we can get two real-valued solution by using Euler's formula to expand,

$$\mathbf{e}^{\lambda_2 t}\vec{\eta}_2 = \mathbf{e}^{(\alpha+\beta i)t}\vec{\eta}_2 = \mathbf{e}^{\alpha t}\left(\cos\left(\beta t\right) + i\sin\left(\beta t\right)\right)\vec{\eta}_2$$

into its real and imaginary parts, $\vec{u} + i\,\vec{v}$. The final two real valued solutions we need are then,

$$\vec{u} \qquad\qquad\qquad \vec{v}$$

1 Real Distinct and 1 Double Eigenvalue with 1 Eigenvector
From the real eigenvalue/eigenvector pair, λ_1 and $\vec{\eta}_1$, we get one solution,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1$$

From our work in the 2 x 2 systems we know that from the double eigenvalue λ_2 with single eigenvector, $\vec{\eta}_2$, we get the following two solutions,

$$\mathbf{e}^{\lambda_2 t}\vec{\eta}_2 \qquad\qquad\qquad t\,\mathbf{e}^{\lambda_2 t}\vec{\xi} + \mathbf{e}^{\lambda_2 t}\vec{\rho}$$

where $\vec{\xi}$ and $\vec{\rho}$ must satisfy the following equations,

$$\left(A - \lambda_2 I\right)\vec{\xi} = \vec{0} \qquad\qquad\qquad \left(A - \lambda_2 I\right)\vec{\rho} = \vec{\xi}$$

Note that the first equation simply tells us that $\vec{\xi}$ must be the single eigenvector for this eigenvalue, $\vec{\eta}_2$, and we usually just say that the second solution we get from the double root case is,

$$t\,\mathbf{e}^{\lambda_2 t}\vec{\eta}_2 + \mathbf{e}^{\lambda_2 t}\vec{\rho} \qquad\qquad \text{where } \vec{\rho} \text{ satisfies } \left(A - \lambda_2 I\right)\vec{\rho} = \vec{\eta}_2$$

1 Real Distinct and 1 Double Eigenvalue with 2 Linearly Independent Eigenvectors
We didn't look at this case back when we were examining the 2 x 2 systems but it is easy enough to deal with. In this case we'll have a single real distinct eigenvalue/eigenvector pair, λ_1 and $\vec{\eta}_1$, as well as a double eigenvalue λ_2 and the double eigenvalue has two linearly independent eigenvectors, $\vec{\eta}_2$ and $\vec{\eta}_3$.

In this case all three eigenvectors are linearly independent and so we get the following three linearly independent solutions,

$$\mathbf{e}^{\lambda_1 t}\vec{\eta}_1 \qquad\qquad \mathbf{e}^{\lambda_2 t}\vec{\eta}_2 \qquad\qquad \mathbf{e}^{\lambda_2 t}\vec{\eta}_3$$

We are now out of the cases that compare to those that we did with 2 x 2 systems and we now need to move into the brand new case that we pick up for 3 x 3 systems. This new case involves eigenvalues with multiplicity of 3. As we noted above we can have 1, 2, or 3 linearly independent eigenvectors and so we actually have 3 sub cases to deal with here. So let's go through these final 3 cases for a 3 x 3 system.

1 Triple Eigenvalue with 1 Eigenvector
The eigenvalue/eigenvector pair in this case are λ and $\vec{\eta}$. Because the eigenvalue is real we know that the first solution we need is,

$$\mathbf{e}^{\lambda t}\vec{\eta}$$

We can use the work from the double eigenvalue with one eigenvector to get that a second solution is,

$$t\,\mathbf{e}^{\lambda t}\vec{\eta}+\mathbf{e}^{\lambda t}\vec{\rho} \qquad\qquad \text{where } \vec{\rho} \text{ satisfies } \left(A-\lambda I\right)\vec{\rho}=\vec{\eta}$$

For a third solution we can take a clue from how we dealt with n^{th} order differential equations with roots multiplicity 3. In those cases we multiplied the original solution by a t^2. However, just as with the double eigenvalue case that won't be enough to get us a solution. In this case the third solution will be,

$$\tfrac{1}{2}t^2\mathbf{e}^{\lambda t}\vec{\xi}+t\mathbf{e}^{\lambda t}\vec{\rho}+\mathbf{e}^{\lambda t}\vec{\mu}$$

where $\vec{\xi}$, $\vec{\rho}$, and $\vec{\mu}$ must satisfy,

$$\left(A-\lambda I\right)\vec{\xi}=\vec{0} \qquad \left(A-\lambda I\right)\vec{\rho}=\vec{\xi} \qquad \left(A-\lambda I\right)\vec{\mu}=\vec{\rho}$$

You can verify that this is a solution and the conditions by taking a derivative and plugging into the system.

Now, the first condition simply tells us that $\vec{\xi}=\vec{\eta}$ because we only have a single eigenvector here and so we can reduce this third solution to,

$$\tfrac{1}{2}t^2\mathbf{e}^{\lambda t}\vec{\eta}+t\mathbf{e}^{\lambda t}\vec{\rho}+\mathbf{e}^{\lambda t}\vec{\mu}$$

where $\vec{\rho}$, and $\vec{\mu}$ must satisfy,

$$\left(A-\lambda I\right)\vec{\rho}=\vec{\eta} \qquad \left(A-\lambda I\right)\vec{\mu}=\vec{\rho}$$

and finally notice that we would have solved the new first condition in determining the second solution above and so all we really need to do here is solve the final condition.

As a final note in this case, the $\tfrac{1}{2}$ is in the solution solely to keep any extra constants from appearing in the conditions which in turn allows us to reuse previous results.

1 Triple Eigenvalue with 2 Linearly Independent Eigenvectors
In this case we'll have the eigenvalue λ with the two linearly independent eigenvectors $\vec{\eta}_1$ and $\vec{\eta}_2$ so we get the following two linearly independent solutions,

$$\mathbf{e}^{\lambda t}\vec{\eta}_1 \qquad\qquad \mathbf{e}^{\lambda t}\vec{\eta}_2$$

We now need a third solution. The third solution will be in the form,

$$t\,\mathbf{e}^{\lambda_2 t}\vec{\xi}+\mathbf{e}^{\lambda_2 t}\vec{\rho}$$

where $\vec{\xi}$ and $\vec{\rho}$ must satisfy the following equations,

$$\left(A-\lambda_2 I\right)\vec{\xi}=\vec{0} \qquad\qquad \left(A-\lambda_2 I\right)\vec{\rho}=\vec{\xi}$$

We've already verified that this will be a solution with these conditions in the double eigenvalue case (that work only required a repeated eigenvalue, not necessarily a double one).

However, unlike the previous times we've seen this we can't just say that $\vec{\xi}$ is an eigenvalue. In all the previous cases in which we've seen this condition we had a single eigenvalue and this time we have two linearly independent eigenvalues. This means that the most general possible solution to the first condition is,

$$\vec{\xi}=c_1\vec{\eta}_1+c_2\vec{\eta}_2$$

This creates problems in solving the second condition. The second condition will not have solutions for every choice of c_1 and c_2 and the choice that we use will be dependent upon the eigenvectors. So upon solving the first condition we would need to plug the general solution into the second condition and then proceed to determine conditions on c_1 and c_2 that would allow us to solve the second condition.

1 Triple Eigenvalue with 3 Linearly Independent Eigenvectors

In this case we'll have the eigenvalue λ with the three linearly independent eigenvectors $\vec{\eta}_1$, $\vec{\eta}_2$, and $\vec{\eta}_3$ so we get the following three linearly independent solutions,

$$\mathbf{e}^{\lambda t}\vec{\eta}_1 \qquad \mathbf{e}^{\lambda t}\vec{\eta}_2 \qquad \mathbf{e}^{\lambda t}\vec{\eta}_3$$

4 x 4 Systems

We'll close this section out with a couple of comments about 4 x 4 systems. In these cases we will have 4 eigenvalues and will need 4 linearly independent solutions in order to get a general solution. The vast majority of the cases here are natural extensions of what 3 x 3 systems cases and in fact will use a vast majority of that work.

Here are a couple of new cases that we should comment briefly on however. With 4 x 4 systems it will now be possible to have two different sets of double eigenvalues and two different sets of complex conjugate eigenvalues. In either of these cases we can treat each one as a separate case and use our previous knowledge about double eigenvalues and complex eigenvalues to get the solutions we need.

It is also now possible to have a "double" complex eigenvalue. In other words we can have $\lambda=\alpha\pm\beta\, i$ each occur twice in the list of eigenvalues. The solutions for this case aren't too bad. We get two solutions in the normal way of dealing with complex eigenvalues. The remaining two solutions will come from the work we did for a double eigenvalue. The work we did in that case did not require that the eigenvalue/eigenvector pair to be real. Therefore if the eigenvector associated with $\lambda=\alpha+\beta\, i$ is $\vec{\eta}$ then the second solution will be,

$$t\,\mathbf{e}^{(\alpha+\beta i)t}\vec{\eta}+\mathbf{e}^{(\alpha+\beta i)t}\vec{\rho} \qquad\qquad \text{where } \vec{\rho} \text{ satisfies } \left(A-\lambda I\right)\vec{\rho}=\vec{\eta}$$

and once we've determined $\vec{\rho}$ we can again split this up into its real and imaginary parts using Euler's formula to get two new real valued solutions.

Finally with 4 x 4 systems we can now have eigenvalues with multiplicity of 4. In these cases we can have 1, 2, 3, or 4 linearly independent eigenvectors and we can use our work with 3 x 3 systems to see how to generate solutions for these cases. The one issue that you'll need to pay attention to is the conditions for the 2 and 3 eigenvector cases will have the same complications that the 2 eigenvector case has in the 3 x 3 systems.

So, we've discussed some of the issues involved in systems larger than 2 x 2 and it is hopefully clear that when we move into larger systems the work can be become vastly more complicated.

Series Solutions

The purpose of this section is not to do anything new with a series solution problem. Instead it is here to illustrate that moving into a higher order differential equation does not really change the process outside of making it a little longer.

Back in the Series Solution chapter we only looked at 2nd order differential equations so we're going to do a quick example here involving a 3rd order differential equation so we can make sure and say that we've done at least one example with an order larger than 2.

Example 1 Find the series solution around $x_0 = 0$ for the following differential equation.

$$y''' + x^2 y' + xy = 0$$

Solution

Recall that we can only find a series solution about $x_0 = 0$ if this point is an ordinary point, or in other words, if the coefficient of the highest derivative term is not zero at $x_0 = 0$. That is clearly the case here so let's start with the form of the solutions as well as the derivatives that we'll need for this solution.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

Plugging into the differential equation gives,

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

Now, move all the coefficients into the series and do appropriate shifts so that all the series are in terms of x^n.

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Next, let's notice that we can start the second series at $n = 1$ since that term will be zero. So let's do that and then we can combine the second and third terms to get,

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=1}^{\infty} \left[(n-1)+1\right] a_{n-1} x^n = 0$$

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n + \sum_{n=1}^{\infty} n a_{n-1} x^n = 0$$

So, we got a nice simplification in the new series that will help with some further simplification. The new second series can now be started at $n = 0$ and then combined with the first series to get,

$$\sum_{n=0}^{\infty} \left[(n+3)(n+2)(n+1) a_{n+3} + n a_{n-1}\right] x^n = 0$$

We can now set the coefficients equal to get a fairly simply recurrence relation.

$$(n+3)(n+2)(n+1)a_{n+3}+na_{n-1}=0 \qquad n=0,1,2,\ldots$$

Solving the recurrence relation gives,

$$a_{n+3}=\frac{-na_{n-1}}{(n+1)(n+2)(n+3)} \qquad n=0,1,2,\ldots$$

Now we need to start plugging in values of *n* and this will be one of the main areas where we can see a somewhat significant increase in the amount of work required when moving into a higher order differential equation.

$$n=0: \quad a_3=0$$

$$n=1: \quad a_4=\frac{-a_0}{(2)(3)(4)}$$

$$n=2: \quad a_5=\frac{-2a_1}{(3)(4)(5)}$$

$$n=3: \quad a_6=\frac{-3a_2}{(4)(5)(6)}$$

$$n=4: \quad a_7=\frac{-4a_3}{(5)(6)(7)}=0$$

$$n=5: \quad a_8=\frac{-5a_4}{(6)(7)(8)}=\frac{5a_0}{(2)(3)(4)(6)(7)(8)}$$

$$n=6: \quad a_9=\frac{-6a_5}{(7)(8)(9)}=\frac{(2)(6)a_1}{(3)(4)(5)(7)(8)(9)}$$

$$n=7: \quad a_{10}=\frac{-7a_6}{(8)(9)(10)}=\frac{(3)(7)a_2}{(4)(5)(6)(8)(9)(10)}$$

$$n=8: \quad a_{11}=\frac{-8a_7}{(9)(10)(11)}=0$$

$$n=9: \quad a_{12}=\frac{-9a_8}{(10)(11)(12)}=\frac{-(5)(9)a_0}{(2)(3)(4)(6)(7)(8)(10)(11)(12)}$$

$$n=10: \quad a_{13}=\frac{-10a_9}{(11)(12)(13)}=\frac{-(2)(6)(10)a_1}{(3)(4)(5)(7)(8)(9)(11)(12)(13)}$$

$$n=11: \quad a_{14}=\frac{-11a_{10}}{(12)(13)(14)}=\frac{-(3)(7)(11)a_2}{(4)(5)(6)(8)(9)(10)(12)(13)(14)}$$

Okay, we can now break the coefficients down into 4 sub cases given by a_{4k}, a_{4k+1}, a_{4k+2} and a_{4k+3} for $k=0,1,2,3,\ldots$ We'll give a semi-detailed derivation for a_{4k} and then leave the rest to you with only couple of comments as they are nearly identical derivations.

First notice that all the a_{4k} terms have a_0 in them and they will alternate in sign. Next notice that we can turn the denominator into a factorial, $(4k)!$ to be exact, if we multiply top and bottom by the numbers that are already in the numerator and so this will turn these numbers into squares. Next notice that the product in the top will start at 1 and increase by 4 until we reach $4k-3$. So, taking all of this into account we get,

$$a_{4k} = \frac{(-1)^k (1)^2 (5)^2 \cdots (4k-3)^2 a_0}{(4k)!} \qquad k = 1, 2, 3, \ldots$$

and notice that this will only work starting with $k=1$ as we won't get a_0 out of this formula as we should by plugging in $k=0$.

Now, for a_{4k+1} the derivation is almost identical and so the formula is,

$$a_{4k+1} = \frac{(-1)^k (2)^2 (6)^2 \cdots (4k-2)^2 a_1}{(4k+1)!} \qquad k = 1, 2, 3, \ldots$$

and again notice that this won't work for $k=0$

The formula for a_{4k+2} is again nearly identical except for this one note that we also need to multiply top and bottom by a 2 in order to get the factorial to appear in the denominator and so the formula here is,

$$a_{4k+2} = \frac{2(-1)^k (3)^2 (7)^2 \cdots (4k-1)^2 a_2}{(4k+2)!} \qquad k = 1, 2, 3, \ldots$$

noticing yet one more time that this won't work for $k=0$.

Finally, we have $a_{4k+3} = 0$ for $k = 0, 1, 2, 3, \ldots$

Now that we have all the coefficients let's get the solution,

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{4k} x^{4k} + a_{4k+1} x^{4k+1} + a_{4k+3} x^{4k+3} + a_{4k+3} x^{4k+3} + \cdots \\ &= a_0 + a_1 x + a_2 x^2 \cdots + \frac{(-1)^k (1)^2 (5)^2 \cdots (4k-3)^2 a_0}{(4k)!} x^{4k} + \\ &\qquad \frac{(-1)^k (2)^2 (6)^2 \cdots (4k-2)^2 a_1}{(4k+1)!} x^{4k+1} + \\ &\qquad \frac{2(-1)^k (3)^2 (7)^2 \cdots (4k-1)^2 a_2}{(4k+2)!} x^{4k+2} + \cdots \end{aligned}$$

Collecting up the terms that contain the same coefficient (except for the first one in each case since the formula won't work for those) and writing everything as a set of series gives us our solution,

$$y(x) = a_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (1)^2 (5)^2 \cdots (4k-3)^2 \, x^{4k}}{(4k)!} \right\} + a_1 \left\{ x + \sum_{k=1}^{\infty} \frac{(-1)^k (2)^2 (6)^2 \cdots (4k-2)^2 \, x^{4k+1}}{(4k+1)!} \right\} + a_2 \left\{ x^2 + \sum_{k=1}^{\infty} \frac{2(-1)^k (3)^2 (7)^2 \cdots (4k-1)^2 \, x^{4k+2}}{(4k+2)!} \right\}$$

So, there we have it. As we can see the work in getting formulas for the coefficients was a little messy because we had three formulas to get, but individually they were not as bad as even some of them could be when dealing with 2nd order differential equations. Also note that while we got lucky with this problem and we were able to get general formulas for the terms the higher the order the less likely this will become.

Boundary Value Problems & Fourier Series

Introduction

In this chapter we'll be taking a quick and very brief look at a couple of topics. The two main topics in this chapter are Boundary Value Problems and Fourier Series. We'll also take a look at a couple of other topics in this chapter. The main point of this chapter is to get some of the basics out of the way that we'll need in the next chapter where we'll take a look at one of the more common solution methods for partial differential equations.

It should be pointed out that both of these topics are far more in depth than what we'll be covering here. In fact you can do whole courses on each of these topics. What we'll be covering here are simply the basics of these topics that well need in order to do the work in the next chapter. There are whole areas of both of these topics that we'll not be even touching on.

Here is a brief listing of the topics in this chapter.

Boundary Value Problems

Before we start off this section we need to make it very clear that we are only going to scratch the surface of the topic of boundary value problems. There is enough material in the topic of boundary value problems that we could devote a whole class to it. The intent of this section is to give a brief (and we mean very brief) look at the idea of boundary value problems and to give enough information to allow us to do some basic partial differential equations in the next chapter.

Now, with that out of the way, the first thing that we need to do is to define just what we mean by a boundary value problem (BVP for short). With initial value problems we had a differential equation and we specified the value of the solution and an appropriate number of derivatives at the same point (collectively called initial conditions). For instance for a second order differential equation the initial conditions are,

$$y\left(t_0\right) = y_0 \qquad\qquad y'\left(t_0\right) = y'_0$$

With boundary value problems we will have a differential equation and we will specify the function and/or derivatives at *different* points, which we'll call boundary values. For second order differential equations, which will be looking at pretty much exclusively here, any of the following can, and will, be used for boundary conditions.

$$y\left(x_0\right) = y_0 \qquad\qquad y\left(x_1\right) = y_1 \tag{1}$$

$$y'\left(x_0\right) = y_0 \qquad\qquad y'\left(x_1\right) = y_1 \tag{2}$$

$$y'\left(x_0\right) = y_0 \qquad\qquad y\left(x_1\right) = y_1 \tag{3}$$

$$y\left(x_0\right) = y_0 \qquad\qquad y'\left(x_1\right) = y_1 \tag{4}$$

As mentioned above we'll be looking pretty much exclusively at second order differential equations. We will also be restricting ourselves down to linear differential equations. So, for the purposes of our discussion here we'll be looking almost exclusively at differential equations in the form,

$$y'' + p\left(x\right) y' + q\left(x\right) y = g\left(x\right) \tag{5}$$

along with one of the sets of boundary conditions given in (1) – (4). We will, on occasion, look at some different boundary conditions but the differential equation will always be on that can be written in this form.

As we'll soon see much of what we know about initial value problems will not hold here. We can, of course, solve (5) provided the coefficients are constant and for a few cases in which they aren't. None of that will change. The changes (and perhaps the problems) arise when we move from initial conditions to boundary conditions.

One of the first changes is a definition that we saw all the time in the earlier chapters. In the earlier chapters we said that a differential equation was homogeneous if $g\left(x\right) = 0$ for all *x*. Here we will say that a boundary value problem is **homogeneous** if in addition to $g\left(x\right) = 0$ we also have $y_0 = 0$ and $y_1 = 0$ (regardless of the boundary conditions we use). If any of these are not zero we will call the BVP **nonhomogeneous**.

It is important to now remember that when we say homogeneous (or nonhomogeneous) we are saying something not only about the differential equation itself but also about the boundary conditions as well.

The biggest change that we're going to see here comes when we go to solve the boundary value problem. When solving linear initial value problems a unique solution will be guaranteed under very mild conditions. We only looked at this idea for first order IVP's but the idea does extend to higher order IVP's. In that section we saw that all we needed to guarantee a unique solution was some basic continuity conditions. With boundary value problems we will often have no solution or infinitely many solutions even for very nice differential equations that would yield a unique solution if we had initial conditions instead of boundary conditions.

Before we get into solving some of these let's next address the question of why we're even talking about these in the first place. As we'll see in the next chapter in the process of solving some partial differential equations we will run into boundary value problems that will need to be solved as well. In fact, a large part of the solution process there will be in dealing with the solution to the BVP. In these cases the boundary conditions will represent things like the temperature at either end of a bar, or the heat flow into/out of either end of a bar. Or maybe they will represent the location of ends of a vibrating string. So, the boundary conditions there will really be conditions on the boundary of some process.

So, with some of basic stuff out of the way let's find some solutions to a few boundary value problems. Note as well that there really isn't anything new here yet. We know how to solve the differential equation and we know how to find the constants by applying the conditions. The only difference is that here we'll be applying boundary conditions instead of initial conditions.

Example 1 Solve the following BVP.

$$y'' + 4y = 0 \qquad\qquad y(0) = -2 \qquad y\left(\frac{\pi}{4}\right) = 10$$

Solution

Okay, this is a simple differential equation so solve and so we'll leave it to you to verify that the general solution to this is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Now all that we need to do is apply the boundary conditions.

$$-2 = y(0) = c_1$$
$$10 = y\left(\tfrac{\pi}{4}\right) = c_2$$

The solution is then,

$$y(x) = -2\cos(2x) + 10\sin(2x)$$

We mentioned above that some boundary value problems can have no solutions or infinite solutions we had better do a couple of examples of those as well here. This next set of examples will also show just how small of a change to the BVP it takes to move into these other possibilities.

Example 2 Solve the following BVP.
$$y'' + 4y = 0 \qquad y(0) = -2 \qquad y(2\pi) = -2$$

Solution
We're working with the same differential equation as the first example so we still have,
$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Upon applying the boundary conditions we get,
$$-2 = y(0) = c_1$$
$$-2 = y(2\pi) = c_1$$

So in this case, unlike previous example, both boundary conditions tell us that we have to have $c_1 = -2$ and neither one of them tell us anything about c_2. Remember however that all we're asking for is a solution to the differential equation that satisfies the two given boundary conditions and the following function will do that,

$$y(x) = -2\cos(2x) + c_2 \sin(2x)$$

In other words, regardless of the value of c_2 we get a solution and so, in this case we get infinitely many solutions to the boundary value problem.

Example 3 Solve the following BVP.
$$y'' + 4y = 0 \qquad y(0) = -2 \qquad y(2\pi) = 3$$

Solution
Again, we have the following general solution,
$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

This time the boundary conditions give us,
$$-2 = y(0) = c_1$$
$$3 = y(2\pi) = c_1$$

In this case we have a set of boundary conditions each of which requires a different value of c_1 in order to be satisfied. This, however, is not possible and so in this case have **no solution**.

So, with Examples 2 and 3 we can see that only a small change to the boundary conditions, in relation to each other and to Example 1, can completely change the nature of the solution. All three of these examples used the same differential equation and yet a different set of initial conditions yielded, no solutions, one solution, or infinitely many solutions.

Note that this kind of behavior is not always unpredictable however. If we use the conditions $y(0)$ and $y(2\pi)$ the only way we'll ever get a solution to the boundary value problem is if we have,
$$y(0) = a \qquad y(2\pi) = a$$

for any value of a. Also, note that if we do have these boundary conditions we'll in fact get infinitely many solutions.

All the examples we've worked to this point involved the same differential equation and the same type of boundary conditions so let's work a couple more just to make sure that we've got some more examples here. Also, note that with each of these we could tweak the boundary conditions a little to get any of the possible solution behaviors to show up (*i.e.* zero, one or infinitely many solutions).

Example 4 Solve the following BVP.

$$y'' + 3y = 0 \qquad\qquad y(0) = 7 \qquad y(2\pi) = 0$$

Solution

The general solution for this differential equation is,

$$y(x) = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Applying the boundary conditions gives,

$$7 = y(0) = c_1$$

$$0 = y(2\pi) = c_1 \cos\left(2\sqrt{3}\,\pi\right) + c_2 \sin\left(2\sqrt{3}\,\pi\right) \quad \Rightarrow \qquad c_2 = -7\cot\left(2\sqrt{3}\,\pi\right)$$

In this case we get a single solution,

$$y(x) = 7\cos\left(\sqrt{3}\,x\right) - 7\cot\left(2\sqrt{3}\,\pi\right)\sin\left(\sqrt{3}\,x\right)$$

Example 5 Solve the following BVP.

$$y'' + 25y = 0 \qquad\qquad y'(0) = 6 \qquad y'(\pi) = -9$$

Solution

Here the general solution is,

$$y(x) = c_1 \cos(5x) + c_2 \sin(5x)$$

and we'll need the derivative to apply the boundary conditions,

$$y'(x) = -5c_1 \sin(5x) + 5c_2 \cos(5x)$$

Applying the boundary conditions gives,

$$6 = y'(0) = 5c_2 \qquad \Rightarrow \qquad c_2 = \tfrac{6}{5}$$

$$-9 = y'(\pi) = -5c_2 \qquad \Rightarrow \qquad c_2 = \tfrac{9}{5}$$

This is not possible and so in this case have **no solution**.

All of the examples worked to this point have been nonhomogeneous because at least one of the boundary conditions have been non-zero. Let's work one nonhomogeneous example where the differential equation is also nonhomogeneous before we work a couple of homogeneous examples.

Example 6 Solve the following BVP.

$$y'' + 9y = \cos x \qquad y'(0) = 5 \qquad y\left(\frac{\pi}{2}\right) = -\frac{5}{3}$$

Solution

The complementary solution for this differential equation is,

$$y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

Using Undetermined Coefficients or Variation of Parameters it is easy to show (we'll leave the details to you to verify) that a particular solution is,

$$Y_P(x) = \tfrac{1}{8}\cos x$$

The general solution and its derivative (since we'll need that for the boundary conditions) are,

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) + \tfrac{1}{8}\cos x$$
$$y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \tfrac{1}{8}\sin x$$

Applying the boundary conditions gives,

$$5 = y'(0) = 3c_2 \qquad \Rightarrow \qquad c_2 = \tfrac{5}{3}$$
$$-\tfrac{5}{3} = y\left(\tfrac{\pi}{2}\right) = -c_2 \qquad \Rightarrow \qquad c_2 = \tfrac{5}{3}$$

The boundary conditions then tell us that we must have $c_2 = \frac{5}{3}$ and they don't tell us anything about c_1 and so it is can be arbitrarily chosen. The solution is then,

$$y(x) = c_1 \cos(3x) + \tfrac{5}{3}\sin(3x) + \tfrac{1}{8}\cos x$$

and there will be infinitely many solutions to the BVP.

Let's now work a couple of homogeneous examples that will also be helpful to have worked once we get to the next section.

Example 7 Solve the following BVP.

$$y'' + 4y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0$$

Solution

Here the general solution is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Applying the boundary conditions gives,

$$0 = y(0) = c_1$$
$$0 = y(2\pi) = c_1$$

So c_2 is arbitrary and the solution is,

$$y(x) = c_2 \sin(2x)$$

and in this case we'll get infinitely many solutions.

Example 8 Solve the following BVP.

$$y'' + 3y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0$$

Solution

The general solution in this case is,

$$y(x) = c_1 \cos\left(\sqrt{3}\,x\right) + c_2 \sin\left(\sqrt{3}\,x\right)$$

Applying the boundary conditions gives,

$$0 = y(0) = c_1$$

$$0 = y(2\pi) = c_2 \sin\left(2\sqrt{3}\,\pi\right) \qquad \Rightarrow \qquad c_2 = 0$$

In this case we found both constants to be zero and so the solution is,

$$y(x) = 0$$

In the previous example the solution was $y(x) = 0$. Notice however, that this will always be a solution to any homogenous system given by (5) and any of the (homogeneous) boundary conditions given by (1) – (4). Because of this we usually call this solution the **trivial solution**. Sometimes, as in the case of the last example the trivial solution is the only solution however we generally prefer solutions to be non-trivial. This will be a major idea in the next section.

Before we leave this section an important point needs to be made. In each of the examples, with one exception, the differential equation that we solved was in the form,

$$y'' + \lambda y = 0$$

The one exception to this still solved this differential equation except it was not a homogeneous differential equation and so we were still solving this basic differential equation in some manner.

So, there are probably several natural questions that can arise at this point. Do all BVP's involve this differential equation and if not why did we spend so much time solving this one to the exclusion of all the other possible differential equations?

The answers to these questions are fairly simple. First, this differential equation is most definitely not the only one used in boundary value problems. It does however exhibit all of the behavior that we wanted to talk about here and has the added bonus of being very easy to solve. So, by using this differential equation almost exclusively we can see and discuss the important behavior that we need to discuss and frees us up from lots of potentially messy solution details and or messy solutions. We will, on occasion, look at other differential equations in the rest of this chapter, but we will still be working almost exclusively with this one.

There is another important reason for looking at this differential equation. When we get to the next chapter and take a brief look at solving partial differential equations we will see that almost every one of the examples that we'll work there come down to exactly this differential equation. Also, in those problems we will be working some "real" problems that are actually solved in places and so are not just "made up" problems for the purposes of examples. Admittedly they will have some simplifications in them, but they do come close to realistic problem in some cases.

Eigenvalues and Eigenfunctions

As we did in the previous section we need to again note that we are only going to give a brief look at the topic of eigenvalues and eigenfunctions for boundary value problems. There are quite a few ideas that we'll not be looking at here. The intent of this section is simply to give you an idea of the subject and to do enough work to allow us to solve some basic partial differential equations in the next chapter.

Now, before we start talking about the actual subject of this section let's recall a topic from Linear Algebra that we briefly discussed previously in these notes. For a given square matrix, A, if we could find values of λ for which we could find nonzero solutions, *i.e.* $\vec{x} \neq \vec{0}$, to,

$$A\vec{x} = \lambda\vec{x}$$

then we called λ an eigenvalue of A and $\vec{x}$ was its corresponding eigenvector.

It's important to recall here that in order for λ to be an eigenvalue then we had to be able to find nonzero solutions to the equation.

So, just what does this have to do with boundary value problems? Well go back to the previous section and take a look at Example 7 and Example 8. In those two examples we solved homogeneous (and that's important!) BVP's in the form,

$$y'' + \lambda y = 0 \qquad\qquad y(0) = 0 \qquad y(2\pi) = 0 \tag{1}$$

In Example 7 we had $\lambda = 4$ and we found nontrivial (*i.e.* nonzero) solutions to the BVP. In Example 8 we used $\lambda = 3$ and the only solution was the trivial solution (*i.e.* $y(t) = 0$). So, this homogeneous BVP (recall this also means the boundary conditions are zero) seems to exhibit similar behavior to the behavior in the matrix equation above. There are values of λ that will give nontrivial solutions to this BVP and values of λ that will only admit the trivial solution.

So, for those values of λ that give nontrivial solutions we'll call λ an **eigenvalue** for the BVP and the nontrivial solutions will be called **eigenfunctions** for the BVP corresponding to the given eigenvalue.

We now know that for the homogeneous BVP given in (1) $\lambda = 4$ is an eigenvalue (with eigenfunctions $y(x) = c_2 \sin(2x)$) and that $\lambda = 3$ is not an eigenvalue.

Eventually we'll try to determine if there are any other eigenvalues for (1), however before we do that let's comment briefly on why it is so important for the BVP to be homogeneous in this discussion. In Example 2 and Example 3 of the previous section we solved the homogeneous differential equation

$$y'' + 4y = 0$$

with two different nonhomogeneous boundary conditions in the form,

$$y(0) = a \qquad\qquad y(2\pi) = b$$

In these two examples we saw that by simply changing the value of a and/or b we were able to get either nontrivial solutions or to force no solution at all. In the discussion of eigenvalues/eigenfunctions we need solutions to exist and the only way to assure this behavior is to require that the boundary conditions also be homogeneous. In other words, we need for the BVP to be homogeneous.

There is one final topic that we need to discuss before we move into the topic of eigenvalues and eigenfunctions and this is more of a notational issue that will help us with some of the work that we'll need to do.

Let's suppose that we have a second order differential equations and its characteristic polynomial has two real, distinct roots and that they are in the form

$$r_1 = \alpha \qquad\qquad r_2 = -\alpha$$

Then we know that the solution is,

$$y(x) = c_1 \mathbf{e}^{r_1 x} + c_2 \mathbf{e}^{r_2 x} = c_1 \mathbf{e}^{\alpha x} + c_2 \mathbf{e}^{-\alpha x}$$

While there is nothing wrong with this solution let's do a little rewriting of this. We'll start by splitting up the terms as follows,

$$\begin{aligned} y(x) &= c_1 \mathbf{e}^{\alpha x} + c_2 \mathbf{e}^{-\alpha x} \\ &= \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} \end{aligned}$$

Now we'll add/subtract the following terms (note we're "mixing" the c_i and $\pm\alpha$ up in the new terms) to get,

$$y(x) = \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_1}{2}\mathbf{e}^{\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} + \frac{c_2}{2}\mathbf{e}^{-\alpha x} + \left(\frac{c_1}{2}\mathbf{e}^{-\alpha x} - \frac{c_1}{2}\mathbf{e}^{-\alpha x}\right) + \left(\frac{c_2}{2}\mathbf{e}^{\alpha x} - \frac{c_2}{2}\mathbf{e}^{\alpha x}\right)$$

Next, rearrange terms around a little,

$$y(x) = \frac{1}{2}\left(c_1\mathbf{e}^{\alpha x} + c_1\mathbf{e}^{-\alpha x} + c_2\mathbf{e}^{\alpha x} + c_2\mathbf{e}^{-\alpha x}\right) + \frac{1}{2}\left(c_1\mathbf{e}^{\alpha x} - c_1\mathbf{e}^{-\alpha x} - c_2\mathbf{e}^{\alpha x} + c_2\mathbf{e}^{-\alpha x}\right)$$

Finally, the quantities in parenthesis factor and we'll move the location of the fraction as well. Doing this, as well as renaming the new constants we get,

$$\begin{aligned} y(x) &= (c_1 + c_2)\frac{\mathbf{e}^{\alpha x} + \mathbf{e}^{-\alpha x}}{2} + (c_1 - c_2)\frac{\mathbf{e}^{\alpha x} - \mathbf{e}^{-\alpha x}}{2} \\ &= \overline{c}_1\frac{\mathbf{e}^{\alpha x} + \mathbf{e}^{-\alpha x}}{2} + \overline{c}_2\frac{\mathbf{e}^{\alpha x} - \mathbf{e}^{-\alpha x}}{2} \end{aligned}$$

All this work probably seems very mysterious and unnecessary. However there really was a reason for it. In fact you may have already seen the reason, at least in part. The two "new" functions that we have in our solution are in fact two of the hyperbolic functions. In particular,

$$\cosh(x) = \frac{\mathbf{e}^{x} + \mathbf{e}^{-x}}{2} \qquad\qquad \sinh(x) = \frac{\mathbf{e}^{x} - \mathbf{e}^{-x}}{2}$$

So, another way to write the solution to a second order differential equation whose characteristic polynomial has two real, distinct roots in the form $r_1 = \alpha,\ r_2 = -\alpha$ is,

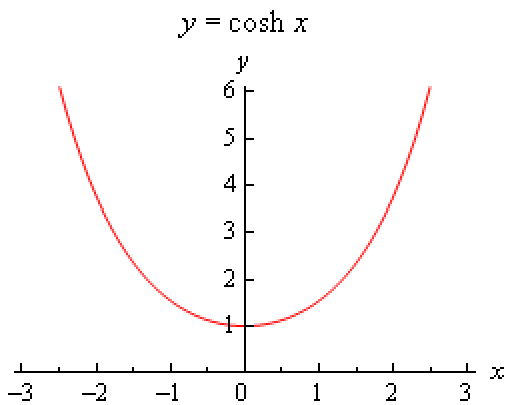
$$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

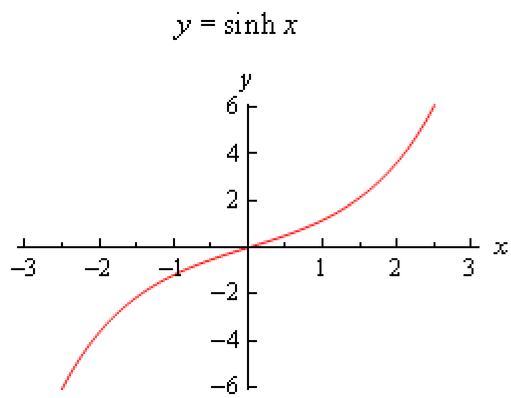
Having the solution in this form for some (actually most) of the problems we'll be looking will make our life a lot easier. The hyperbolic functions have some very nice properties that we can (and will) take advantage of.

First, since we'll be needing them later on, the derivatives are,

$$\frac{d}{dx}\left(\cosh(x)\right) = \sinh(x) \qquad\qquad \frac{d}{dx}\left(\sinh(x)\right) = \cosh(x)$$

Next let's take a quick look at the graphs of these functions.





Note that $\cosh(0) = 1$ and $\sinh(0) = 0$. Because we'll often be working with boundary conditions at $x = 0$ these will be useful evaluations.

Next, and possibly more importantly, let's notice that $\cosh(x) > 0$ for all x and so the hyperbolic cosine will never be zero. Likewise, we can see that $\sinh(x) = 0$ only if $x = 0$. We will be using both of these facts in some of our work so we shouldn't forget them.

Okay, now that we've got all that out of the way let's work an example to see how we go about finding eigenvalues/eigenfunctions for a BVP.

Example 1 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad\qquad y(0) = 0 \qquad y(2\pi) = 0$$

Solution

We started off this section looking at this BVP and we already know one eigenvalue ($\lambda = 4$) and we know one value of λ that is not an eigenvalue ($\lambda = 3$). As we go through the work here we need to remember that we will get an eigenvalue for a particular value of λ if we get non-trivial solutions of the BVP for that particular value of λ.

In order to know that we've found all the eigenvalues we can't just start randomly trying values of λ to see if we get non-trivial solutions or not. Luckily there is a way to do this that's not too bad and will give us all the eigenvalues/eigenfunctions. We are going to have to do some cases however. The three cases that we will need to look at are : $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Each of these cases gives a specific form of the solution to the BVP to which we can then apply the boundary conditions to see if we'll get non-trivial solutions or not. So, let's get started on the cases.

$\underline{\lambda > 0}$

In this case the characteristic polynomial we get from the differential equation is,

$$r^2 + \lambda = 0 \qquad \Rightarrow \qquad r_{1,2} = \pm\sqrt{-\lambda}$$

In this case since we know that $\lambda > 0$ these roots are complex and we can write them instead as,

$$r_{1,2} = \pm\sqrt{\lambda}\, i$$

The general solution to the differential equation is then,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives us,

$$0 = y(0) = c_1$$

So, taking this into account and applying the second boundary condition we get,

$$0 = y(2\pi) = c_2 \sin\left(2\pi\sqrt{\lambda}\right)$$

This means that we have to have one of the following,

$$c_2 = 0 \qquad \text{or} \qquad \sin\left(2\pi\sqrt{\lambda}\right) = 0$$

However, recall that we want non-trivial solutions and if we have the first possibility we will get the trivial solution for all values of $\lambda > 0$. Therefore let's assume that $c_2 \neq 0$. This means that we have,

$$\sin\left(2\pi\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad 2\pi\sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

In other words, taking advantage of the fact that we know where sine is zero we can arrive at the second equation. Also note that because we are assuming that $\lambda > 0$ we know that $2\pi\sqrt{\lambda} > 0$ and so *n* can only be a positive integer for this case.

Now all we have to do is solve this for λ and we'll have all the positive eigenvalues for this BVP.

The positive eigenvalues are then,

$$\lambda_n = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4} \qquad n = 1, 2, 3, \ldots$$

and the eigenfunctions that correspond to these eigenvalues are,

$$y_n(x) = \sin\left(\frac{n\,x}{2}\right) \qquad\qquad n = 1, 2, 3, \ldots$$

Note that we subscripted an *n* on the eigenvalues and eigenfunctions to denote the fact that there is one for each of the given values of *n*. Also note that we dropped the c_2 on the eigenfunctions. For eigenfunctions we are only interested in the function itself and not the constant in front of it and so we generally drop that.

Let's now move into the second case.

$\underline{\lambda = 0}$

In this case the BVP becomes,

$$y'' = 0 \qquad\qquad y(0) = 0 \qquad\qquad y(2\pi) = 0$$

and integrating the differential equation a couple of times gives us the general solution,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = y(0) = c_1$$

Applying the second boundary condition as well as the results of the first boundary condition gives,

$$0 = y(2\pi) = 2c_2\pi$$

Here, unlike the first case, we don't have a choice on how to make this zero. This will only be zero if $c_2 = 0$.

Therefore, for this BVP (and that's important), if we have $\lambda = 0$ the only solution is the trivial solution and so $\lambda = 0$ cannot be an eigenvalue for this BVP.

Now let's look at the final case.

$\underline{\lambda < 0}$

In this case the characteristic equation and its roots are the same as in the first case. So, we know that,

$$r_{1,2} = \pm\sqrt{-\lambda}$$

However, because we are assuming $\lambda < 0$ here these are now two real distinct roots and so using our work above for these kinds of real, distinct roots we know that the general solution will be,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Note that we could have used the exponential form of the solution here, but our work will be significantly easier if we use the hyperbolic form of the solution here.

Now, applying the first boundary condition gives,

$$0 = y(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1(1) + c_2(0) = c_1 \qquad \Rightarrow \qquad c_1 = 0$$

Applying the second boundary condition gives,

$$0 = y(2\pi) = c_2 \sinh\left(2\pi\sqrt{-\lambda}\right)$$

Because we are assuming $\lambda < 0$ we know that $2\pi\sqrt{-\lambda} \neq 0$ and so we also know that $\sinh\left(2\pi\sqrt{-\lambda}\right) \neq 0$. Therefore, much like the second case, we must have $c_2 = 0$.

So, for this BVP (again that's important), if we have $\lambda < 0$ we only get the trivial solution and so there are no negative eigenvalues.

In summary then we will have the following eigenvalues/eigenfunctions for this BVP.

$$\lambda_n = \frac{n^2}{4} \qquad y_n(x) = \sin\left(\frac{n\,x}{2}\right) \qquad n = 1, 2, 3, \ldots$$

Let's take a look at another example with slightly different boundary conditions.

Example 2 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad y'(0) = 0 \qquad y'(2\pi) = 0$$

Solution

Here we are going to work with derivative boundary conditions. The work is pretty much identical to the previous example however so we won't put in quite as much detail here. We'll need to go through all three cases just as the previous example so let's get started on that.

$\underline{\lambda > 0}$

The general solution to the differential equation is identical to the previous example and so we have,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\,x\right) + c_2 \sin\left(\sqrt{\lambda}\,x\right)$$

Applying the first boundary condition gives us,

$$0 = y'(0) = \sqrt{\lambda}\,c_2 \qquad \Rightarrow \qquad c_2 = 0$$

Recall that we are assuming that $\lambda > 0$ here and so this will only be zero if $c_2 = 0$. Now, the second boundary condition gives us,

$$0 = y'(2\pi) = -\sqrt{\lambda}\,c_1 \sin\left(2\pi\sqrt{\lambda}\right)$$

Recall that we don't want trivial solutions and that $\lambda > 0$ so we will only get non-trivial solution if we require that,

$$\sin\left(2\pi\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad 2\pi\sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

Solving for λ and we see that we get exactly the same positive eigenvalues for this BVP that we got in the previous example.

$$\lambda_n = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4} \qquad\qquad n = 1, 2, 3, \ldots$$

The eigenfunctions that correspond to these eigenvalues however are,

$$y_n(x) = \cos\left(\frac{n\,x}{2}\right) \qquad\qquad n = 1, 2, 3, \ldots$$

So, for this BVP we get cosines for eigenfunctions corresponding to positive eigenvalues.

Now the second case.

$\underline{\lambda = 0}$
The general solution is,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = y'(0) = c_2$$

Using this the general solution is then,

$$y(x) = c_1$$

and note that this will trivially satisfy the second boundary condition,

$$0 = y'(2\pi) = 0$$

Therefore, unlike the first example, $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$y(x) = 1$$

Again, note that we dropped the arbitrary constant for the eigenfunctions.

Finally let's take care of the third case.

$\underline{\lambda < 0}$
The general solution here is,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\,x\right) + c_2 \sinh\left(\sqrt{-\lambda}\,x\right)$$

Applying the first boundary condition gives,

$$0 = y'(0) = \sqrt{-\lambda}\,c_1 \sinh(0) + \sqrt{-\lambda}\,c_2 \cosh(0) = \sqrt{-\lambda}\,c_2 \qquad \Rightarrow \qquad c_2 = 0$$

Applying the second boundary condition gives,

$$0 = y'(2\pi) = \sqrt{-\lambda}\,c_1 \sinh\left(2\pi\sqrt{-\lambda}\right)$$

As with the previous example we again know that $2\pi\sqrt{-\lambda} \neq 0$ and so $\sinh\left(2\pi\sqrt{-\lambda}\right) \neq 0$. Therefore we must have $c_1 = 0$.

So, for this BVP we again have no negative eigenvalues.

In summary then we will have the following eigenvalues/eigenfunctions for this BVP.

$$\lambda_n = \frac{n^2}{4} \qquad y_n(x) = \cos\left(\frac{n\,x}{2}\right) \qquad n = 1,2,3,\ldots$$
$$\lambda_0 = 0 \qquad y_0(x) = 1$$

Notice as well that we can actually combine these if we allow the list of *n*'s for the first one to start at zero instead of one. This will often not happen, but when it does we'll take advantage of it. So the "official" list of eigenvalues/eigenfunctions for this BVP is,

$$\lambda_n = \frac{n^2}{4} \qquad y_n(x) = \cos\left(\frac{n\,x}{2}\right) \qquad n = 0,1,2,3,\ldots$$

So, in the previous two examples we saw that we generally need to consider different cases for λ as different values will often lead to different general solutions. Do not get too locked into the cases we did here. We will mostly be solving this particular differential equation and so it will be tempting to assume that these are always the cases that we'll be looking at, but there are BVP's that will require other/different cases.

Also, as we saw in the two examples sometimes one or more of the cases will not yield any eigenvalues. This will often happen, but again we shouldn't read anything into the fact that we didn't have negative eigenvalues for either of these two BVP's. There are BVP's that will have negative eigenvalues.

Let's take a look at another example with a very different set of boundary conditions. These are not the traditional boundary conditions that we've been looking at to this point, but we'll see in the next chapter how these can arise from certain physical problems.

Example 3 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad y(-\pi) = y(\pi) \qquad y'(-\pi) = y'(\pi)$$

Solution

So, in this example we aren't actually going to specify the solution or its derivative at the boundaries. Instead we'll simply specify that the solution must be the same at the two boundaries and the derivative of the solution must also be the same at the two boundaries. Also, this type of boundary condition will typically be on an interval of the form [-L,L] instead of [0,L] as we've been working on to this point.

As mentioned above these kind of boundary conditions arise very naturally in certain physical problems and we'll see that in the next chapter.

As with the previous two examples we still have the standard three cases to look at.

$\underline{\lambda > 0}$
The general solution for this case is,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition and using the fact that cosine is an even function (*i.e.* $\cos(-x) = \cos(x)$) and that sine is an odd function (*i.e.* $\sin(-x) = -\sin(x)$). gives us,

$$\begin{aligned} c_1 \cos\left(-\pi\sqrt{\lambda}\right) + c_2 \sin\left(-\pi\sqrt{\lambda}\right) &= c_1 \cos\left(\pi\sqrt{\lambda}\right) + c_2 \sin\left(\pi\sqrt{\lambda}\right) \\ c_1 \cos\left(\pi\sqrt{\lambda}\right) - c_2 \sin\left(\pi\sqrt{\lambda}\right) &= c_1 \cos\left(\pi\sqrt{\lambda}\right) + c_2 \sin\left(\pi\sqrt{\lambda}\right) \\ -c_2 \sin\left(\pi\sqrt{\lambda}\right) &= c_2 \sin\left(\pi\sqrt{\lambda}\right) \\ 0 &= 2c_2 \sin\left(\pi\sqrt{\lambda}\right) \end{aligned}$$

This time, unlike the previous two examples this doesn't really tell us anything. We could have $\sin\left(\pi\sqrt{\lambda}\right) = 0$ but it is also completely possible, at this point in the problem anyway, for us to have $c_2 = 0$ as well.

So, let's go ahead and apply the second boundary condition and see if we get anything out of that.

$$\begin{aligned} -\sqrt{\lambda}\, c_1 \sin\left(-\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(-\pi\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\pi\sqrt{\lambda}\right) \\ \sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\pi\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\pi\sqrt{\lambda}\right) \\ \sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) \\ 2\sqrt{\lambda}\, c_1 \sin\left(\pi\sqrt{\lambda}\right) &= 0 \end{aligned}$$

So, we get something very similar to what we got after applying the first boundary condition. Since we are assuming that $\lambda > 0$ this tells us that either $\sin\left(\pi\sqrt{\lambda}\right) = 0$ or $c_1 = 0$.

Note however that if $\sin\left(\pi\sqrt{\lambda}\right) \neq 0$ then we will have to have $c_1 = c_2 = 0$ and we'll get the trivial solution. We therefore need to require that $\sin\left(\pi\sqrt{\lambda}\right) = 0$ and so just as we've done for the previous two examples we can now get the eigenvalues,

$$\pi\sqrt{\lambda} = n\pi \qquad \Rightarrow \qquad \lambda = n^2 \quad n = 1, 2, 3, \ldots$$

Recalling that $\lambda > 0$ and we can see that we do need to start the list of possible *n*'s at one instead of zero.

So, we now know the eigenvalues for this case, but what about the eigenfunctions. The solution for a given eigenvalue is,

$$y(x) = c_1 \cos(n\,x) + c_2 \sin(n\,x)$$

and we've got no reason to believe that either of the two constants are zero or non-zero for that matter. In cases like these we get two sets of eigenfunctions, one corresponding to each constant. The two sets of eigenfunctions for this case are,

$$y_n(x) = \cos(n\,x) \qquad\qquad y_n(x) = \sin(n\,x) \qquad n = 1, 2, 3, \ldots$$

Now the second case.

$\underline{\lambda = 0}$

The general solution is,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$\begin{aligned} c_1 + c_2(-\pi) &= c_1 + c_2(\pi) \\ 2\pi c_2 &= 0 \qquad\qquad \Rightarrow \qquad\qquad c_2 = 0 \end{aligned}$$

Using this the general solution is then,

$$y(x) = c_1$$

and note that this will trivially satisfy the second boundary condition just as we saw in the second example above. Therefore we again have $\lambda = 0$ as an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$y(x) = 1$$

Finally let's take care of the third case.

$\underline{\lambda < 0}$

The general solution here is,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\,x\right) + c_2 \sinh\left(\sqrt{-\lambda}\,x\right)$$

Applying the first boundary condition and using the fact that hyperbolic cosine is even and hyperbolic sine is odd gives,

$$\begin{aligned} c_1 \cosh\left(-\pi\sqrt{-\lambda}\right) + c_2 \sinh\left(-\pi\sqrt{-\lambda}\right) &= c_1 \cosh\left(\pi\sqrt{-\lambda}\right) + c_2 \sinh\left(\pi\sqrt{-\lambda}\right) \\ -c_2 \sinh\left(-\pi\sqrt{-\lambda}\right) &= c_2 \sinh\left(\pi\sqrt{-\lambda}\right) \\ 2c_2 \sinh\left(\pi\sqrt{-\lambda}\right) &= 0 \end{aligned}$$

Now, in this case we are assuming that $\lambda < 0$ and so we know that $\pi\sqrt{-\lambda} \neq 0$ which in turn tells us that $\sinh\left(\pi\sqrt{-\lambda}\right) \neq 0$. We therefore must have $c_2 = 0$.

Let's now apply the second boundary condition to get,

$$\sqrt{-\lambda}\, c_1 \sinh\left(-\pi\sqrt{-\lambda}\right) = \sqrt{-\lambda}\, c_1 \sinh\left(\pi\sqrt{-\lambda}\right)$$

$$2\sqrt{-\lambda}\, c_1 \sinh\left(\pi\sqrt{-\lambda}\right) = 0 \qquad \Rightarrow \qquad c_1 = 0$$

By our assumption on λ we again have no choice here but to have $c_1 = 0$.

Therefore, in this case the only solution is the trivial solution and so, for this BVP we again have no negative eigenvalues.

In summary then we will have the following eigenvalues/eigenfunctions for this BVP.

$$\lambda_n = n^2 \qquad y_n(x) = \sin(n\,x) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = n^2 \qquad y_n(x) = \cos(n\,x) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_0 = 0 \qquad y_0(x) = 1$$

Note that we've acknowledged that for $\lambda > 0$ we had two sets of eigenfunctions by listing them each separately. Also, we can again combine the last two into one set of eigenvalues and eigenfunctions. Doing so gives the following set of eigenvalues and eigenfunctions.

$$\lambda_n = n^2 \qquad y_n(x) = \sin(n\,x) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = n^2 \qquad y_n(x) = \cos(n\,x) \qquad n = 0, 1, 2, 3, \ldots$$

Once again we've got an example with no negative eigenvalues. We can't stress enough that this is more a function of the differential equation we're working with than anything and there will be examples in which we may get negative eigenvalues.

Now, to this point we've only worked with one differential equation so let's work an example with a different differential equation just to make sure that we don't get too locked into this one differential equation.

Before working this example let's note that we will still be working the vast majority of our examples with the one differential equation we've been using to this point. We're working with this other differential equation just to make sure that we don't get too locked into using one single differential equation.

Example 4 Find all the eigenvalues and eigenfunctions for the following BVP.

$$x^2 y'' + 3xy' + \lambda y = 0 \qquad y(1) = 0 \qquad y(2) = 0$$

Solution

This is an Euler differential equation and so we know that we'll need to find the roots of the following quadratic.

$$r(r-1) + 3r + \lambda = r^2 + 2r + \lambda = 0$$

The roots to this quadratic are,

$$r_{1,2} = \frac{-2 \pm \sqrt{4-4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}$$

Now, we are going to again have some cases to work with here, however they won't be the same as the previous examples. The solution will depend on whether or not the roots are real distinct, double or complex and these cases will depend upon the sign/value of $1-\lambda$. So, let's go through the cases.

$1-\lambda < 0,\ \lambda > 1$

In this case the roots will be complex and we'll need to write them as follows in order to write down the solution.

$$r_{1,2} = -1 \pm \sqrt{1-\lambda} = -1 \pm \sqrt{-(\lambda-1)} = -1 \pm i\sqrt{\lambda-1}$$

By writing the roots in this fashion we know that $\lambda - 1 > 0$ and so $\sqrt{\lambda-1}$ is now a real number, which we need in order to write the following solution,

$$y(x) = c_1 x^{-1} \cos\left(\ln(x)\sqrt{\lambda-1}\right) + c_2 x^{-1} \sin\left(\ln(x)\sqrt{\lambda-1}\right)$$

Applying the first boundary condition gives us,

$$0 = y(1) = c_1\cos(0) + c_2\sin(0) = c_1 \qquad \Rightarrow \qquad c_1 = 0$$

The second boundary condition gives us,

$$0 = y(2) = \tfrac{1}{2} c_2 \sin\left(\ln(2)\sqrt{\lambda-1}\right)$$

In order to avoid the trivial solution for this case we'll require,

$$\sin\left(\ln(2)\sqrt{\lambda-1}\right) = 0 \qquad \Rightarrow \qquad \ln(2)\sqrt{\lambda-1} = n\pi \qquad n = 1,2,3,\ldots$$

This is much more complicated of a condition than we've seen to this point, but other than that we do the same thing. So, solving for λ gives us the following set of eigenvalues for this case.

$$\lambda_n = 1 + \left(\frac{n\pi}{\ln 2}\right)^2 \qquad n = 1,2,3,\ldots$$

Note that we need to start the list of n's off at one and not zero to make sure that we have $\lambda > 1$ as we're assuming for this case.

The eigenfunctions that correspond to these eigenvalues are,

$$y_n(x) = \sin\left(\frac{n\pi}{\ln 2}\ln(x)\right) \qquad n = 1,2,3,\ldots$$

Now the second case.

$1-\lambda=0,\ \ \lambda=1$

In this case we get a double root of $r_{1,2}=-1$ and so the solution is,

$$y(x)=c_1x^{-1}+c_2x^{-1}\ln(x)$$

Applying the first boundary condition gives,

$$0=y(1)=c_1$$

The second boundary condition gives,

$$0=y(2)=\tfrac{1}{2}c_2\ln(2) \qquad\Rightarrow\qquad c_2=0$$

We therefore have only the trivial solution for this case and so $\lambda=1$ is not an eigenvalue.

Let's now take care of the third (and final) case.

$1-\lambda>0,\ \lambda<1$

This case will have two real distinct roots and the solution is,

$$y(x)=c_1x^{-1+\sqrt{1-\lambda}}+c_2x^{-1-\sqrt{1-\lambda}}$$

Applying the first boundary condition gives,

$$0=y(1)=c_1+c_2 \qquad\Rightarrow\qquad c_2=-c_1$$

Using this our solution becomes,

$$y(x)=c_1x^{-1+\sqrt{1-\lambda}}-c_1x^{-1-\sqrt{1-\lambda}}$$

Applying the second boundary condition gives,

$$0=y(2)=c_1 2^{-1+\sqrt{1-\lambda}}-c_1 2^{-1-\sqrt{1-\lambda}}=c_1\left(2^{-1+\sqrt{1-\lambda}}-2^{-1-\sqrt{1-\lambda}}\right)$$

Now, because we know that $\lambda\neq1$ for this case the exponents on the two terms in the parenthesis are not the same and so the term in the parenthesis is not the zero. This means that we can only have,

$$c_1=c_2=0$$

and so in this case we only have the trivial solution and there are no eigenvalues for which $\lambda<1$.

The only eigenvalues for this BVP then come from the first case.

So, we've now worked an example using a differential equation other than the "standard" one we've been using to this point. As we saw in the work however, the basic process was pretty much the same. We determined that there were a number of cases (three here, but it won't always be three) that gave different solutions. We examined each case to determine if non-trivial solutions were possible and if so found the eigenvalues and eigenfunctions corresponding to that case.

We need to work one last example in this section before we leave this section for some new topics. The four examples that we've worked to this point were all fairly simple (with simple being relative of course…), however we don't want to leave without acknowledging that many eigenvalue/eigenfunctions problems are so easy.

In many examples it is not even possible to get a complete list of all possible eigenvalues for a BVP. Often the equations that we need to solve to get the eigenvalues are difficult if not impossible to solve exactly. So, let's take a look at one example like this to see what kinds of things can be done to at least get an idea of what the eigenvalues look like in these kinds of cases.

Example 5 Find all the eigenvalues and eigenfunctions for the following BVP.

$$y'' + \lambda y = 0 \qquad\qquad y(0) = 0 \qquad y'(1) + y(1) = 0$$

Solution

The boundary conditions for this BVP are fairly different from those that we've worked with to this point. However, the basic process is the same. So let's start off with the first case.

$\underline{\lambda > 0}$

The general solution to the differential equation is identical to the first few examples and so we have,

$$y(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives us,

$$0 = y(0) = c_1 \qquad\qquad \Rightarrow \qquad\qquad c_1 = 0$$

The second boundary condition gives us,

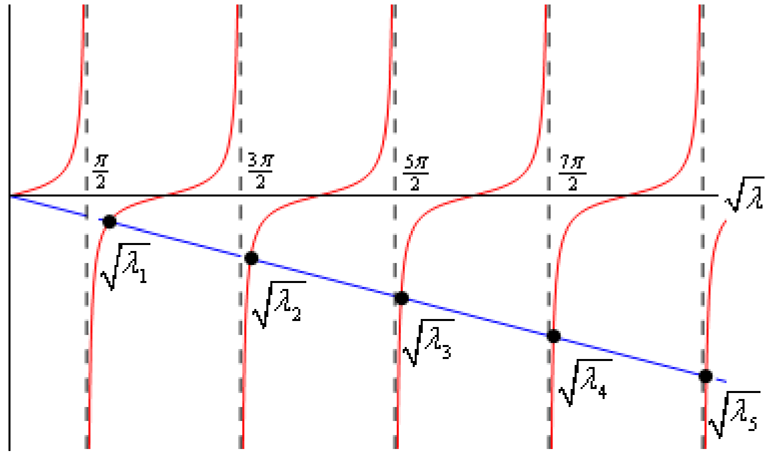
$$\begin{aligned} 0 = y(1) + y'(1) &= c_2 \sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(\sqrt{\lambda}\right) \\ &= c_2\left(\sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}\right)\right) \end{aligned}$$

So, if we let $c_2 = 0$ we'll get the trivial solution and so in order to satisfy this boundary condition we'll need to require instead that,

$$\begin{aligned} 0 &= \sin\left(\sqrt{\lambda}\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) \\ \sin\left(\sqrt{\lambda}\right) &= -\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) \\ \tan\left(\sqrt{\lambda}\right) &= -\sqrt{\lambda} \end{aligned}$$

Now, this equation has solutions but we'll need to use some numerical techniques in order to get them. In order to see what's going on here let's graph $\tan\left(\sqrt{\lambda}\right)$ and $-\sqrt{\lambda}$ on the same graph.

Here is that graph and note that the horizontal axis really is values of $\sqrt{\lambda}$ as that will make things a little easier to see and relate to values that we're familiar with.



So, eigenvalues for this case will occur where the two curves intersect. We've shown the first five on the graph and again what is showing on the graph is really the square root of the actual eigenvalue as we've noted.

The interesting thing to note here is that the farther out on the graph the closer the eigenvalues come to the asymptotes of tangent and so we'll take advantage of that and say that for large enough *n* we can approximate the eigenvalues with the (very well known) locations of the asymptotes of tangent.

How large the value of *n* is before we start using the approximation will depend on how much accuracy we want, but since we know the location of the asymptotes and as *n* increases the accuracy of the approximation will increase so it will be easy enough to check for a given accuracy.

For the purposes of this example we found the first five numerically and then we'll use the approximation of the remaining eigenvalues. Here are those values/approximations.

$$\begin{array}{lll}
\sqrt{\lambda_1} = 2.0288 & \lambda_1 = 4.1160 & (2.4674) \\
\sqrt{\lambda_2} = 4.9132 & \lambda_2 = 24.1395 & (22.2066) \\
\sqrt{\lambda_3} = 7.9787 & \lambda_3 = 63.6597 & (61.6850) \\
\sqrt{\lambda_4} = 11.0855 & \lambda_4 = 122.8883 & (120.9027) \\
\sqrt{\lambda_5} = 14.2074 & \lambda_5 = 201.8502 & (199.8595) \\
\sqrt{\lambda_n} \approx \frac{2n-1}{2}\pi & \lambda_n \approx \frac{(2n-1)^2}{4}\pi^2 & n = 6,7,8,\ldots
\end{array}$$

The number in parenthesis after the first five is the approximate value of the asymptote. As we can see they are a little off, but by the time we get to $n = 5$ the error in the approximation is 0.9862%. So less than 1% error by the time we get to $n = 5$ and it will only get better for larger value of *n*.

The eigenfunctions for this case are,

$$y_n(x) = \sin\left(\sqrt{\lambda_n}\, x\right) \qquad\qquad n = 1, 2, 3, \ldots$$

where the values of λ_n are given above.

So, now that all that work is out of the way let's take a look at the second case.

$\underline{\lambda = 0}$

The general solution is,

$$y(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = y(0) = c_1$$

Using this the general solution is then,

$$y(x) = c_2 x$$

Applying the second boundary condition to this gives,

$$0 = y'(1) + y(1) = c_2 + c_2 = 2c_2 \qquad\qquad \Rightarrow \qquad\qquad c_2 = 0$$

Therefore for this case we get only the trivial solution and so $\lambda = 0$ is not an eigenvalue. Note however that had the second boundary condition been $y'(1) - y(1) = 0$ then $\lambda = 0$ would have been an eigenvalue (with eigenfunctions $y(x) = x$) and so again we need to be careful about reading too much into our work here.

Finally let's take care of the third case.

$\underline{\lambda < 0}$

The general solution here is,

$$y(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = y(0) = c_1 \cosh(0) + c_2 \sinh(0) = c_1 \qquad\qquad \Rightarrow \qquad c_1 = 0$$

Using this the general solution becomes,

$$y(x) = c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the second boundary condition to this gives,

$$\begin{aligned} 0 = y'(1) + y(1) &= \sqrt{-\lambda}\, c_2 \cosh\left(\sqrt{-\lambda}\right) + c_2 \sinh\left(\sqrt{-\lambda}\right) \\ &= c_2\left(\sqrt{-\lambda} \cosh\left(\sqrt{-\lambda}\right) + \sinh\left(\sqrt{-\lambda}\right)\right) \end{aligned}$$

Now, by assumption we know that $\lambda < 0$ and so $\sqrt{-\lambda} > 0$. This in turn tells us that $\sinh\left(\sqrt{-\lambda}\right) > 0$ and we know that $\cosh\left(x\right) > 0$ for all *x*. Therefore,

$$\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}\right) + \sinh\left(\sqrt{-\lambda}\right) \neq 0$$

and so we must have $c_2 = 0$ and once again in this third case we get the trivial solution and so this BVP will have no negative eigenvalues.

In summary the only eigenvalues for this BVP come from assuming that $\lambda > 0$ and they are given above.

So, we've worked several eigenvalue/eigenfunctions examples in this section. Before leaving this section we do need to note once again that there are a vast variety of different problems that we can work here and we've really only shown a bare handful of examples and so please do not walk away from this section believing that we've shown you everything.

The whole purpose of this section is to prepare us for the types of problems that we'll be seeing in the next chapter. Also, in the next chapter we will again be restricting ourselves down to some pretty basic and simple problems in order to illustrate one of the more common methods for solving partial differential equations.

Periodic Functions, Even/Odd Functions and Orthogonal Functions

This is going to be a short section. We just need to have a brief discussion about a couple of ideas that we'll be dealing with on occasion as we move into the next topic of this chapter.

Periodic Function

The first topic we need to discuss is that of a periodic function. A function is said to be **periodic** with **period *T*** if the following is true,

$$f\left(x+T\right)=f\left(x\right) \qquad \text{for all } x$$

The following is a nice little fact about periodic functions.

Fact 1

If f and g are both periodic functions with period T then so is $f+g$ and fg.

This is easy enough to prove so let's do that.

$$\left(f+g\right)\left(x+T\right)=f\left(x+T\right)+g\left(x+T\right)=f\left(x\right)+g\left(x\right)=\left(f+g\right)\left(x\right)$$
$$\left(fg\right)\left(x+T\right)=f\left(x+T\right)g\left(x+T\right)=f\left(x\right)g\left(x\right)=\left(fg\right)\left(x\right)$$

The two periodic functions that most of us are familiar are sine and cosine and in fact we'll be using these two functions regularly in the remaining sections of this chapter. So, having said that let's close off this discussion of periodic functions with the following fact,

Fact 2

$\sin\left(\omega x\right)$ and $\cos\left(\omega x\right)$ are periodic functions with period $T=\dfrac{2\pi}{\omega}$.

Even and Odd Functions

The next quick idea that we need to discuss is that of even and odd functions.

Recall that a function is said to be **even** if,

$$f\left(-x\right)=f\left(x\right)$$

and a function is said to be **odd** if,

$$f\left(-x\right)=-f\left(x\right)$$

The standard examples of even functions are $f\left(x\right)=x^2$ and $g\left(x\right)=\cos\left(x\right)$ while the standard examples of odd functions are $f\left(x\right)=x^3$ and $g\left(x\right)=\sin\left(x\right)$. The following fact about certain integrals of even/odd functions will be useful in some of our work.

Fact 3

1. If $f(x)$ is an even function then,

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

2. If $f(x)$ is an odd function then,

$$\int_{-L}^{L} f(x)\,dx = 0$$

Note that this fact is only valid on a "symmetric" interval, *i.e.* an interval in the form $[-L, L]$. If we aren't integrating on a "symmetric" interval then the fact may or may not be true.

Orthogonal Functions

The final topic that we need to discuss here is that of orthogonal functions. This idea will be integral to what we'll be doing in the remainder of this chapter and in the next chapter as we discuss one of the basic solution methods for partial differential equations.

Let's first get the definition of orthogonal functions out of the way.

Definition

1. Two non-zero functions, $f(x)$ and $g(x)$, are said to be **orthogonal** on $a \le x \le b$ if,

$$\int_{a}^{b} f(x)\,g(x)\,dx = 0$$

2. A set of non-zero functions, $\{f_i(x)\}$, is said to be **mutually orthogonal** on $a \le x \le b$ (or just an **orthogonal set** if we're being lazy) if $f_i(x)$ and $f_j(x)$ are orthogonal for every $i \ne j$. In other words,

$$\int_{a}^{b} f_i(x)\,f_j(x)\,dx = \begin{cases} 0 & i \ne j \\ c > 0 & i = j \end{cases}$$

Note that in the case of $i = j$ for the second definition we know that we'll get a positive value from the integral because,

$$\int_{a}^{b} f_i(x)\,f_i(x)\,dx = \int_{a}^{b} \left[f_i(x) \right]^2 dx > 0$$

Recall that when we integrate a positive function we know the result will be positive as well.

Also note that the non-zero requirement is important because otherwise the integral would be trivially zero regardless of the other function we were using.

Before we work some examples there are a nice set of trig formulas that we'll need to help us with some of the integrals.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha-\beta\right)+\sin\left(\alpha+\beta\right)\right]$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)+\cos\left(\alpha+\beta\right)\right]$$

Now let's work some examples that we'll need over the course of the next couple of sections.

Example 1 Show that $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ is mutually orthogonal on $-L \le x \le L$.

Solution
This is not too difficult to do. All we really need to do is evaluate the following integral.

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

Before we start evaluating this integral let's notice that the integrand is the product of two even functions and so must also be even. This means that we can use Fact 3 above to write the integral as,

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

There are two reasons for doing this. First having a limit of zero will often make the evaluation step a little easier and that will be the case here. We'll discuss the second reason after we're done with the example.

Now, in order to do this integral we'll actually need to consider three cases.

$\underline{n = m = 0}$
In this case the integral is very easy and is,

$$\int_{-L}^{L}dx = 2\int_{0}^{L}dx = 2L$$

$\underline{n = m \neq 0}$
This integral is a little harder than the first case, but not by much (provided we recall a simple trig formula). The integral for this case is,

$$\begin{aligned}\int_{-L}^{L}\cos^{2}\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L}\cos^{2}\left(\frac{n\pi x}{L}\right)dx &= \int_{0}^{L}1+\cos\left(\frac{2n\pi x}{L}\right)dx\\ &= \left(x+\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right)\Bigg|_{0}^{L} = L+\frac{L}{2n\pi}\sin\left(2n\pi\right)\end{aligned}$$

Now, at the point we need to recall that *n* is an integer and so $\sin\left(2n\pi\right)=0$ and our final value for the is,

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx=2\int_{0}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx=L$$

The first two cases are really just showing that if $n=m$ the integral is not zero (as it shouldn't be) and depending upon the value of *n* (and hence *m*) we get different values of the integral. Now we need to do the third case and this, in some ways, is the important case since we must get zero out of this integral in order to know that the set is an orthogonal set. So, let's take care of the final case.

$\underline{n\neq m}$
This integral is the "messiest" of the three that we've had to do here. Let's just start off by writing the integral down.

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

In this case we can't combine/simplify as we did in the previous two cases. We can however, acknowledge that we've got a product of two cosines with different arguments and so we can use one of the trig formulas above to break up the product as follows,

$$\begin{aligned}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx&=2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\\&=\int_{0}^{L}\cos\left(\frac{\left(n-m\right)\pi x}{L}\right)+\cos\left(\frac{\left(n+m\right)\pi x}{L}\right)dx\\&=\left[\frac{L}{\left(n-m\right)\pi}\sin\left(\frac{\left(n-m\right)\pi x}{L}\right)+\frac{L}{\left(n+m\right)\pi}\sin\left(\frac{\left(n+m\right)\pi x}{L}\right)\right]_{0}^{L}\\&=\frac{L}{\left(n-m\right)\pi}\sin\left(\left(n-m\right)\pi\right)+\frac{L}{\left(n+m\right)\pi}\sin\left(\left(n+m\right)\pi\right)\end{aligned}$$

Now, we know that *n* and *m* are both integers and so $n-m$ and $n+m$ are also integers and so both of the sines above must be zero and all together we get,

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=2\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=0$$

So, we've shown that if $n\neq m$ the integral is zero and if $n=m$ the value of the integral is a positive constant and so the set is mutually orthogonal.

In all of the work above we kept both forms of the integral at every step. Let's discuss why we did this a little bit. By keeping both forms of the integral around we were able to show that not only is $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ mutually orthogonal on $-L \le x \le L$ but it is also mutually orthogonal on $0 \le x \le L$. The only difference is the value of the integral when $n = m$ and we can get those values from the work above.

Let's take a look at another example.

Example 2 Show that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $-L \le x \le L$ and on $0 \le x \le L$.

Solution
First we'll acknowledge from the start this time that we'll be showing orthogonality on both of the intervals. Second, we need to start this set at $n = 1$ because if we used $n = 0$ the first function would be zero and we don't want the zero function to show up on our list.

As with the first example all we really need to do is evaluate the following integral.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

In this case integrand is the product of two odd functions and so must be even. This means that we can again use Fact 3 above to write the integral as,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 2\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

We only have two cases to do for the integral here.

$\underline{n = m}$
Not much to this integral. It's pretty similar to the previous examples second case.

$$\begin{aligned}\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx &= \int_{0}^{L} 1-\cos\left(\frac{2n\pi x}{L}\right)dx \\ &= \left(x-\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right)\Bigg|_{0}^{L} = L-\frac{L}{2n\pi}\sin\left(2n\pi\right) = L\end{aligned}$$

Summarizing up we get,

$$\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = 2\int_{0}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = L$$

$\underline{n \ne m}$
As with the previous example this can be a little messier but it is also nearly identical to the third case from the previous example so we'll not show a lot of the work.

$$\begin{aligned}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx &= 2\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \\ &= \int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right)-\cos\left(\frac{(n+m)\pi x}{L}\right)dx \\ &= \left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)-\frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]_0^L \\ &= \frac{L}{(n-m)\pi}\sin\left((n-m)\pi\right)-\frac{L}{(n+m)\pi}\sin\left((n+m)\pi\right)\end{aligned}$$

As with the previous example we know that n and m are both integers a and so both of the sines above must be zero and all together we get,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 2\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$

So, we've shown that if $n \neq m$ the integral is zero and if $n = m$ the value of the integral is a positive constant and so the set is mutually orthogonal.

We've now shown that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ mutually orthogonal on $-L \le x \le L$ and on $0 \le x \le L$.

We need to work one more example in this section.

Example 3 Show that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ and $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ are mutually orthogonal on $-L \le x \le L$.

Solution

This example is a little different from the previous two examples. Here we want to show that together both sets are mutually orthogonal on $-L \le x \le L$. To show this we need to show three things. First (and second actually) we need to show that individually each set is mutually orthogonal and we've already done that in the previous two examples. The third (and only) thing we need to show here is that if we take one function from one set and another function from the other set and we integrate them we'll get zero.

Also, note that this time we really do only want to do the one interval as the two sets, taken together, are not mutually orthogonal on $0 \le x \le L$. You might want to do the integral on this interval to verify that it won't always be zero.

So, let's take care of the one integral that we need to do here and there isn't a lot to do. Here is the integral.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

The integrand in this case is the product of an odd function (the sine) and an even function (the cosine) and so the integrand is an odd function. Therefore, since the integral is on a symmetric interval, *i.e.* $-L \le x \le L$, and so by Fact 3 above we know the integral must be zero or,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

So, in previous examples we've shown that on the interval $-L \le x \le L$ the two sets are mutually orthogonal individually and here we've shown that integrating a product of a sine and a cosine gives zero. Therefore, as a combined set they are also mutually orthogonal.

We've now worked three examples here dealing with orthogonality and we should note that these were not just pulled out of the air as random examples to work. In the following sections (and following chapter) we'll need the results from these examples. So, let's summarize those results up here.

1. $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are mutually orthogonal on $-L \le x \le L$ as individual sets and as a combined set.

2. $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ is mutually orthogonal on $0 \le x \le L$.

3. $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $0 \le x \le L$.

We will also be needing the results of the integrals themselves, both on $-L \le x \le L$ and on $0 \le x \le L$ so let's also summarize those up here as well so we can refer to them when we need to.

1. $$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

2. $$\int_{0}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m = 0 \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

3. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

4. $$\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

5. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

With this summary we'll leave this section and move off into the second major topic of this chapter : Fourier Series.

Fourier Sine Series

In this section we are going to start taking a look at Fourier series. We should point out that this is a subject that can span a whole class and what we'll be doing in this section (as well as the next couple of sections) is intended to be nothing more than a very brief look at the subject. The point here is to do just enough to allow us to do some basic solutions to partial differential equations in the next chapter. There are many topics in the study of Fourier series that we'll not even touch upon here.

So, with that out of the way let's get started, although we're not going to start off with Fourier series. Let's instead think back to our Calculus class where we looked at Taylor Series. With Taylor Series we wrote a series representation of a function, $f\left(x\right)$, as a series whose terms were powers of $x-a$ for some $x=a$. With some conditions we were able to show that,

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^{n}$$

and that the series will converge to $f\left(x\right)$ on $\left|x-a\right|<R$ for some R that will be dependent upon the function itself.

There is nothing wrong with this, but it does require that derivatives of all orders exist at $x=a$. Or in other words $f^{(n)}\left(a\right)$ exists for $n=0,1,2,3,\ldots$ Also for some functions the value of R may end up being quite small.

These two issues (along with a couple of others) mean that this is not always the best way of writing a series representation for a function. In many cases it works fine and there will be no reason to need a different kind of series. There are times however where another type of series is either preferable or required.

We're going to build up an alternative series representation for a function over the course of the next couple of sections. The ultimate goal for the rest of this chapter will be to write down a series representation for a function in terms of sines and cosines.

We'll start things off by assuming that the function, $f\left(x\right)$, we want to write a series representation for is an odd function (*i.e.* $f\left(-x\right)=-f\left(x\right)$). Because $f\left(x\right)$ is odd it makes some sense that should be able to write a series representation for this in terms of sines only (since they are also odd functions).

What we'll try to do here is write $f\left(x\right)$ as the following series representation, called a **Fourier sine series**, on $-L\le x\le L$.

$$\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

There are a couple of issues to note here. First, at this point, we are going to assume that the series representation will converge to $f\left(x\right)$ on $-L\le x\le L$. We will be looking into whether or not it will actually converge in a later section. However, assuming that the series does converge to $f(x)$ it is interesting to note that, unlike Taylor Series, this representation will always converge on the same interval and that the interval does not depend upon the function.

Second, the series representation will not involve powers of sine (again contrasting this with Taylor Series) but instead will involve sines with different arguments.

Finally, the argument of the sines, $\frac{n\pi x}{L}$, may seem like an odd choice that was arbitrarily chosen and in some ways it was. For Fourier sine series the argument doesn't have to necessarily be this but there are several reasons for the choice here. First, this is the argument that will naturally arise in the next chapter when we use Fourier series (in general and not necessarily Fourier sine series) to help us solve some basic partial differential equations.

The next reason for using this argument is that fact that the set of functions that we chose to work with, $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ in this case, need to be orthogonal on the given interval, $-L \le x \le L$ in this case, and note that in the last section we showed that in fact they are. In other words, the choice of functions we're going to be working with and the interval we're working on will be tied together in some way. We can use a different argument, but will need to also choose an interval on which we can prove that the sines (with the different argument) are orthogonal.

So, let's start off by assuming that given an odd function, $f(x)$, we can in fact find a Fourier sine series, of the form given above, to represent the function on $-L \le x \le L$. This means we will have,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

As noted above we'll discuss whether or not this even can be done and if the series representation does in fact converge to the function in later section. At this point we're simply going to assume that it can be done. The question now is how to determine the coefficients, B_n, in the series.

Let's start with the series above and multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ where *m* is a fixed integer in the range $\{1, 2, 3, \ldots\}$. In other words we multiply both sides by any of the sines in the set of sines that we're working with here. Doing this gives,

$$f(x)\sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

Now, let's integrate both sides of this from $x = -L$ to $x = L$.

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_{-L}^{L} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

At this point we've got a small issue to deal with. We know from Calculus that an integral of a finite series (more commonly called a finite sum….) is nothing more than the (finite) sum of the integrals of the pieces. In other words for finite series we can interchange an integral and a series.

For infinite series however, we cannot always do this. For some integrals of infinite series we cannot interchange an integral and a series. Luckily enough for us we actually can interchange the integral and the series in this case. Doing this and factoring the constant, B_n, out of the integral gives,

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx &= \sum_{n=1}^{\infty}\int_{-L}^{L} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx \\ &= \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\end{aligned}$$

Now, recall from the last section we proved that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is orthogonal on $-L \le x \le L$ and that,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

So, what does this mean for us. As we work through the various values of n in the series and compute the value of the integrals all but one of the integrals will be zero. The only non-zero integral will come when we have $n = m$, in which case the integral has the value of L. Therefore, the only non-zero term in the series will come when we have $n = m$ and our equation becomes,

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m L$$

Finally all we need to do is divide by L and we now have an equation for each of the coefficients.

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Next, note that because we're integrating two odd functions the integrand of this integral is even and so we also know that,

$$B_m = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Summarizing all this work up the Fourier sine series of an odd function $f(x)$ on $-L \le x \le L$ is given by,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad \begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, 3, \ldots \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n = 1, 2, 3, \ldots\end{aligned}$$

Let's take a quick look at an example.

Example 1 Find the Fourier sine series for $f\left(x\right)=x$ on $-L\le x\le L$.

Solution
First note that the function we're working with is in fact an odd function and so this is something we can do. There really isn't much to do here other than to compute the coefficients for $f\left(x\right)=x$.

Here is that work and note that we're going to leave the integration by parts details to you to verify. Don't forget that *n*, *L*, and π are constants!

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right)-n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{n^2\pi^2}\left(L\sin\left(n\pi\right)-n\pi L\cos\left(n\pi\right)\right) \end{aligned}$$

These integrals can, on occasion, be somewhat messy especially when we use a general *L* for the endpoints of the interval instead of a specific number.

Now, taking advantage of the fact that *n* is an integer we know that $\sin\left(n\pi\right)=0$ and that $\cos\left(n\pi\right)=\left(-1\right)^n$. We therefore have,

$$B_n = \frac{2}{n^2\pi^2}\left(-n\pi L\left(-1\right)^n\right)=\frac{\left(-1\right)^{n+1}2L}{n\pi} \qquad\qquad n=1,2,3\ldots$$

The Fourier sine series is then,

$$x=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

At this point we should probably point out that we'll be doing most, if not all, of our work here on a general interval ($-L\le x\le L$ or $0\le x\le L$) instead of intervals with specific numbers for the endpoints. There are a couple of reasons for this. First, it gives a much more general formula that will work for any interval of that form which is always nice. Secondly, when we run into this kind of work in the next chapter it will also be on general intervals so we may as well get used to them now.

Now, finding the Fourier sine series of an odd function is fine and good but what if, for some reason, we wanted to find the Fourier sine series for a function that is not odd? To see how to do this we're going to have to make a change. The above work was done on the interval $-L\le x\le L$. In the case of a function that is not odd we'll be working on the interval $0\le x\le L$. The reason for this will be made apparent in a bit.

So, we are now going to do is to try to find a series representation for $f(x)$ on the interval $0 \le x \le L$ that is in the form,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

or in other words,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

As we did with the Fourier sine series on $-L \le x \le L$ we are going to assume that the series will in fact converge to $f(x)$ and we'll hold off discussing the convergence of the series for a later section.

There are two methods of generating formulas for the coefficients, B_n, although we'll see in a bit that they really the same way, just looked at from different perspectives.

The first method is to just ignore the fact that $f(x)$ is not odd and proceed in the same manner that we did above only this time we'll take advantage of the fact that we proved in the previous section that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ also forms an orthogonal set on $0 \le x \le L$ and that,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \dfrac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

So, if we do this then all we need to do is multiply both sides of our series by $\sin\left(\frac{m\pi x}{L}\right)$, integrate from 0 to L and interchange the integral and series to get,

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

Now, plugging in for the integral we arrive at,

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m\left(\frac{L}{2}\right)$$

Upon solving for the coefficient we arrive at,

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, 3, \ldots$$

Note that this is identical to the second form of the coefficients that we arrived at above by assuming $f(x)$ was odd and working on the interval $-L \le x \le L$. The fact that we arrived at essentially the same coefficients is not actually all the surprising as we've see once we've looked the second method of generating the coefficients.

Before we look at the second method of generating the coefficients we need to take a brief look at another concept. Given a function, $f(x)$, we define the **odd extension** of $f(x)$ to be the new function,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0 \end{cases}$$

It's pretty easy to see that this is an odd function.

$$g(-x) = -f(-(-x)) = -f(x) = -g(x) \qquad\qquad \text{for } 0 < x < L$$

and we can also know that on $0 \le x \le L$ we have that $g(x) = f(x)$. Also note that if $f(x)$ is already an odd function then we in fact get $g(x) = f(x)$ on $-L \le x \le L$.

Let's take a quick look at a couple of odd extensions before we proceed any further.

Example 2 Sketch the odd extension of each of the given functions.

(a) $f(x) = L - x$ on $0 \le x \le L$ [Solution]

(b) $f(x) = 1 + x^2$ on $0 \le x \le L$ [Solution]

(c) $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$ [Solution]

Solution

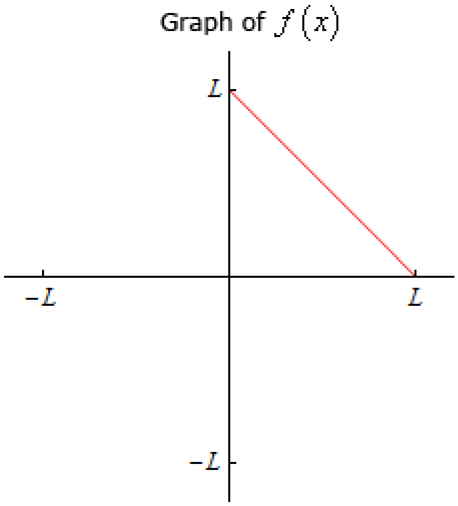
Not much to do with these other than to define the odd extension and then sketch it.

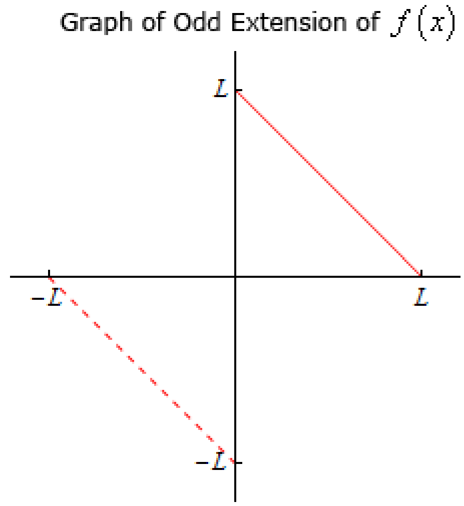
(a) $f(x) = L - x$ on $0 \le x \le L$

Here is the odd extension of this function.

$$\begin{aligned} g(x) &= \begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0 \end{cases} \\ &= \begin{cases} L - x & \text{if } 0 \le x \le L \\ -L - x & \text{if } -L \le x \le 0 \end{cases} \end{aligned}$$

Below is the graph of both the function and its odd extension. Note that we've put the "extension" in with a dashed line to make it clear the portion of the function that is being added to allow us to get the odd extension.





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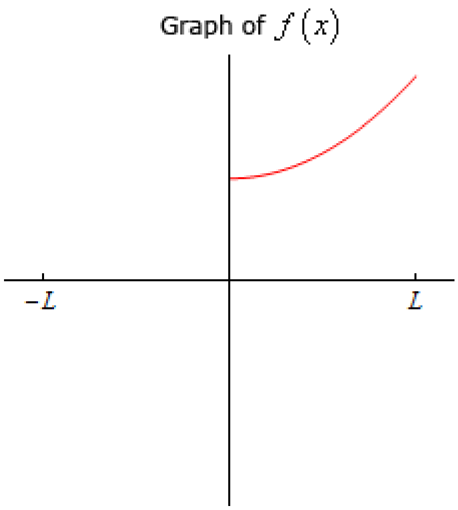
(b) $f(x) = 1 + x^2$ on $0 \le x \le L$

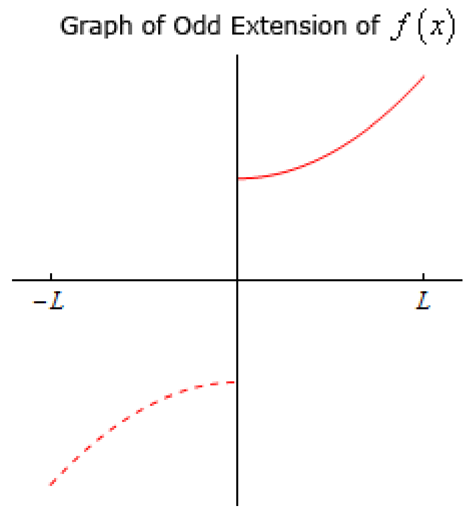
First note that this is clearly an even function. That does not however mean that we can't define the odd extension for it. The odd extension for this function is,

$$g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ -f(-x) & \text{if } -L \le x \le 0 \end{cases}$$

$$= \begin{cases} 1 + x^2 & \text{if } 0 \le x \le L \\ -1 - x^2 & \text{if } -L \le x \le 0 \end{cases}$$

The sketch of the original function and its odd extension are ,





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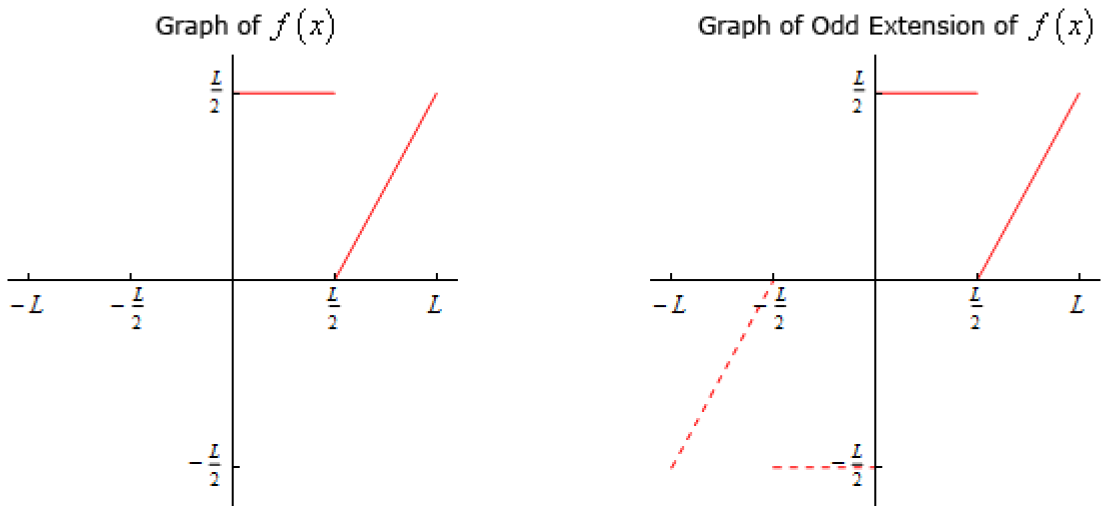
(c) $f(x)=\begin{cases} \frac{L}{2} & \text{if } 0\le x\le \frac{L}{2} \\ x-\frac{L}{2} & \text{if } \frac{L}{2}\le x\le L \end{cases}$

Let's first write down the odd extension for this function.

$$g(x)=\begin{cases} f(x) & \text{if } 0\le x\le L \\ -f(-x) & \text{if } -L\le x\le 0 \end{cases} = \begin{cases} x-\frac{L}{2} & \text{if } \frac{L}{2}\le x\le L \\ \frac{L}{2} & \text{if } 0\le x\le \frac{L}{2} \\ -\frac{L}{2} & \text{if } -\frac{L}{2}\le x\le 0 \\ x+\frac{L}{2} & \text{if } -L\le x\le -\frac{L}{2} \end{cases}$$

The sketch of the original function and its odd extension are,



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With the definition of the odd extension (and a couple of examples) out of the way we can now take a look at the second method for getting formulas for the coefficients of the Fourier sine series for a function $f(x)$ on $0\le x\le L$. First, given such a function define its odd extension as above. At this point, because $g(x)$ is an odd function, we know that on $-L\le x\le L$ the Fourier sine series for $g(x)$ (and NOT $f(x)$ yet) is,

$$g(x)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad\qquad B_n=\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n=1,2,3,\ldots$$

However, because we know that $g(x)=f(x)$ on $0\le x\le L$ we can also see that as long as we are on $0\le x\le L$ we have,

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad\qquad B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad n=1,2,3,\ldots$$

So, exactly the same formula for the coefficients regardless of how we arrived at the formula and the second method justifies why they are the same here as they were when we derived them for the Fourier sine series for an odd function.

Now, let's find the Fourier sine series for each of the functions that we looked at in Example 2.

Note that again we are working on general intervals here instead of specific numbers for the right endpoint to get a more general formula for any interval of this form and because again this is the kind of work we'll be doing in the next chapter.

Also, we'll again be leaving the actually integration details up to you to verify. In most cases it will involve some fairly simple integration by parts complicated by all the constants (*n*, *L*, π, etc.) that show up in the integral.

Example 3 Find the Fourier sine series for $f(x) = L - x$ on $0 \le x \le L$.

Solution
There really isn't much to do here other than computing the coefficients so here they are,

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L (L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L \\ &= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}\left(n\pi - \sin(n\pi)\right)\right] = \frac{2L}{n\pi} \end{aligned}$$

In the simplification process don't forget that *n* is an integer.

So, with the coefficients we get the following Fourier sine series for this function.

$$f(x) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

In the next example it is interesting to note that while we started out this section looking only at odd functions we're now going to be finding the Fourier sine series of an even function on $0 \le x \le L$. Recall however that we're really finding the Fourier sine series of the odd extension of this function and so we're okay.

Example 4 Find the Fourier sine series for $f(x) = 1 + x^2$ on $0 \le x \le L$.

Solution
In this case the coefficients are liable to be somewhat messy given the fact that the integrals will involve integration by parts twice. Here is the work for the coefficients.

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \left(1+x^2\right)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left[\left(2L^2 - n^2\pi^2\left(1+x^2\right)\right)\cos\left(\frac{n\pi x}{L}\right) + 2Ln\pi x\sin\left(\frac{n\pi x}{L}\right)\right]_0^L \\ &= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left[\left(2L^2 - n^2\pi^2\left(1+L^2\right)\right)\cos(n\pi) + 2L^2n\pi\sin(n\pi) - \left(2L^2 - n^2\pi^2\right)\right] \\ &= \frac{2}{n^3\pi^3}\left[\left(2L^2 - n^2\pi^2\left(1+L^2\right)\right)(-1)^n - 2L^2 + n^2\pi^2\right] \end{aligned}$$

As noted above the coefficients are not the most pleasant ones, but there they are. The Fourier sine series for this function is then,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi^3}\left[\left(2L^2 - n^2\pi^2\left(1+L^2\right)\right)(-1)^n - 2L^2 + n^2\pi^2\right]\sin\left(\frac{n\pi x}{L}\right)$$

In the last example of this section we'll be finding the Fourier sine series of a piecewise function and can definitely complicate the integrals a little but they do show up on occasion and so we need to be able to deal with them.

Example 5 Find the Fourier sine series for $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$ on $0 \le x \le L$.

Solution
Here is the integral for the coefficients.

$$\begin{aligned} B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{2}{L}\left[\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{L}{2}\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^L \left(x - \frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)dx\right] \end{aligned}$$

Note that we need to split the integral up because of the piecewise nature of the original function. Let's do the two integrals separately

$$\int_0^{\frac{L}{2}} \frac{L}{2}\sin\left(\frac{n\pi x}{L}\right)dx = -\left(\frac{L}{2}\right)\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} = \frac{L^2}{2n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$\int_{\frac{L}{2}}^{L} \left(x-\frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{n^2\pi^2}\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{\frac{L}{2}}^{L}$$

$$= \frac{L}{n^2\pi^2}\left[L\sin\left(n\pi\right) - \frac{n\pi L}{2}\cos\left(n\pi\right) - L\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{L^2}{n^2\pi^2}\left[\frac{n\pi\left(-1\right)^n}{2} + \sin\left(\frac{n\pi}{2}\right)\right]$$

Putting all of this together gives,

$$B_n = \frac{2}{L}\int_0^L f\left(x\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{L^2}{2n\pi}\right)\left[1+\left(-1\right)^{n+1} - \cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{L}{n\pi}\left[1+\left(-1\right)^{n+1} - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

So, the Fourier sine series for this function is,

$$f\left(x\right) = \sum_{n=1}^{\infty}\frac{L}{n\pi}\left[1+\left(-1\right)^{n+1} - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

As the previous two examples has shown the coefficients for these can be quite messy but that will often be the case and so we shouldn't let that get us too excited.

Fourier Cosine Series

In this section we're going to take a look at Fourier cosine series. We'll start off much as we did in the previous section where we looked at Fourier sine series. Let's start by assuming that the function, $f(x)$, we'll be working with initially is an even function (*i.e.* $f(-x) = f(x)$) and that we want to write a series representation for this function on $-L \le x \le L$ in terms of cosines (which are also even). In other words we are going to look for the following,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

This series is called a **Fourier cosine series** and note that in this case (unlike with Fourier sine series) we're able to start the series representation at $n = 0$ since that term will not be zero as it was with sines. Also, as with Fourier Sine series, the argument of $\frac{n\pi x}{L}$ in the cosines is being used only because it is the argument that we'll be running into in the next chapter. The only real requirement here is that the given set of functions we're using be orthogonal on the interval we're working on.

Note as well that we're assuming that the series will in fact converge to $f(x)$ on $-L \le x \le L$ at this point. In a later section we'll be looking into the convergence of this series in more detail.

So, to determine a formula for the coefficients, A_n, we'll use the fact that $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ do form an orthogonal set on the interval $-L \le x \le L$ as we showed in a previous section. In that section we also derived the following formula that we'll need in a bit.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \ne 0 \\ 0 & \text{if } n \ne m \end{cases}$$

We'll get a formula for the coefficients in almost exactly the same fashion that we did in the previous section. We'll start with the representation above and multiply both sides by $\cos\left(\frac{m\pi x}{L}\right)$ where *m* is a fixed integer in the range $\{0,1,2,3,\ldots\}$. Doing this gives,

$$f(x)\cos\left(\frac{m\pi x}{L}\right) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)$$

Next, we integrate both sides from $x = -L$ to $x = L$ and as we were able to do with the Fourier Sine series we can again interchange the integral and the series.

$$\begin{aligned} \int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx &= \int_{-L}^{L} \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx \\ &= \sum_{n=0}^{\infty} A_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx \end{aligned}$$

We now know that the all of the integrals on the right side will be zero except when $n = m$ because the set of cosines form an orthogonal set on the interval $-L \le x \le L$. However, we need to be careful about the value of *m* (or *n* depending on the letter you want to use). So, after evaluating all of the integrals we arrive at the following set of formulas for the coefficients.

$m = 0$:

$$\int_{-L}^{L} f(x)\,dx = A_0(2L) \qquad \Rightarrow \qquad A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$m \ne 0$:

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_m(L) \qquad \Rightarrow \qquad A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

Summarizing everything up then, the Fourier cosine series of an even function, $f(x)$ on $-L \le x \le L$ is given by,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \begin{cases} \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx & n = 0 \\ \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

Finally, before we work an example, let's notice that because both $f(x)$ and the cosines are even the integrand in both of the integrals above is even and so we can write the formulas for the A_n's as follows,

$$A_n = \begin{cases} \dfrac{1}{L}\displaystyle\int_{0}^{L} f(x)\,dx & n = 0 \\ \dfrac{2}{L}\displaystyle\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

Now let's take a look at an example.

Example 1 Find the Fourier cosine series for $f(x) = x^2$ on $-L \le x \le L$.

Solution

We clearly have an even function here and so all we really need to do is compute the coefficients and they are liable to be a little messy because we'll need to do integration by parts twice. We'll leave most of the actual integration details to you to verify.

$$A_0 = \frac{1}{L}\int_{0}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} x^2\,dx = \frac{1}{L}\left(\frac{L^3}{3}\right) = \frac{L^2}{3}$$

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left(2Ln\pi x\cos\left(\frac{n\pi x}{L}\right) + \left(n^2\pi^2x^2 - 2L^2\right)\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\
&= \frac{2}{n^3\pi^3}\left(2L^2n\pi\cos(n\pi) + \left(n^2\pi^2L^2 - 2L^2\right)\sin(n\pi)\right) \\
&= \frac{4L^2(-1)^n}{n^2\pi^2} \qquad\qquad n = 1,2,3,\ldots
\end{aligned}$$

The coefficients are then,

$$A_0 = \frac{L^2}{3} \qquad\qquad A_n = \frac{4L^2(-1)^n}{n^2\pi^2}, \quad n = 1,2,3,\ldots$$

The Fourier cosine series is then,

$$x^2 = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$

Note that we'll often strip out the $n = 0$ from the series as we've done here because it will almost always be different from the other coefficients and it allows us to actually plug the coefficients into the series.

Now, just as we did in the previous section let's ask what we need to do in order to find the Fourier cosine series of a function that is not even. As with Fourier sine series when we make this change we'll need to move onto the interval $0 \le x \le L$ now instead of $-L \le x \le L$ and again we'll assume that the series will converge to $f(x)$ at this point and leave the discussion of the convergence of this series to a later section.

We could go through the work to find the coefficients here twice as we did with Fourier sine series, however there's no real reason to. So, while we could redo all the work above to get formulas for the coefficients let's instead go straight to the second method of finding the coefficients.

In this case, before we actually proceed with this we'll need to define the even extension of a function, $f(x)$ on $-L \le x \le L$. So, given a function $f(x)$ we'll define the even extension of the function as,

$$g(x) = \begin{cases} f(x) & \text{if } \ 0 \le x \le L \\ f(-x) & \text{if } \ -L \le x \le 0 \end{cases}$$

Showing that this is an even function is simple enough.

$$g(-x) = f(-(-x)) = f(x) = g(x) \qquad\qquad \text{for } 0 < x < L$$

and we can see that $g(x) = f(x)$ on $0 \le x \le L$ and if $f(x)$ is already an even function we get $g(x) = f(x)$ on $-L \le x \le L$.

Let's take a look at some functions and sketch the even extensions for the functions.

Example 2 Sketch the even extension of each of the given functions.

(a) $f(x) = L - x$ on $0 \le x \le L$ [Solution]

(b) $f(x) = x^3$ on $0 \le x \le L$ [Solution]

(c) $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$ [Solution]

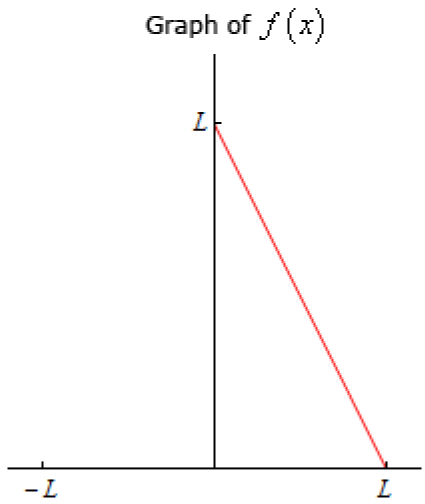
Solution

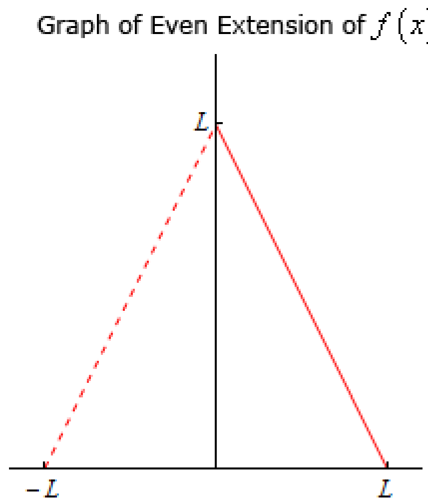
(a) $f(x) = L - x$ on $0 \le x \le L$

Here is the even extension of this function.

$$\begin{aligned} g(x) &= \begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x \le 0 \end{cases} \\ &= \begin{cases} L - x & \text{if } 0 \le x \le L \\ L + x & \text{if } -L \le x \le 0 \end{cases} \end{aligned}$$

Here is the graph of both the original function and its even extension. Note that we've put the "extension" in with a dashed line to make it clear the portion of the function that is being added to allow us to get the even extension





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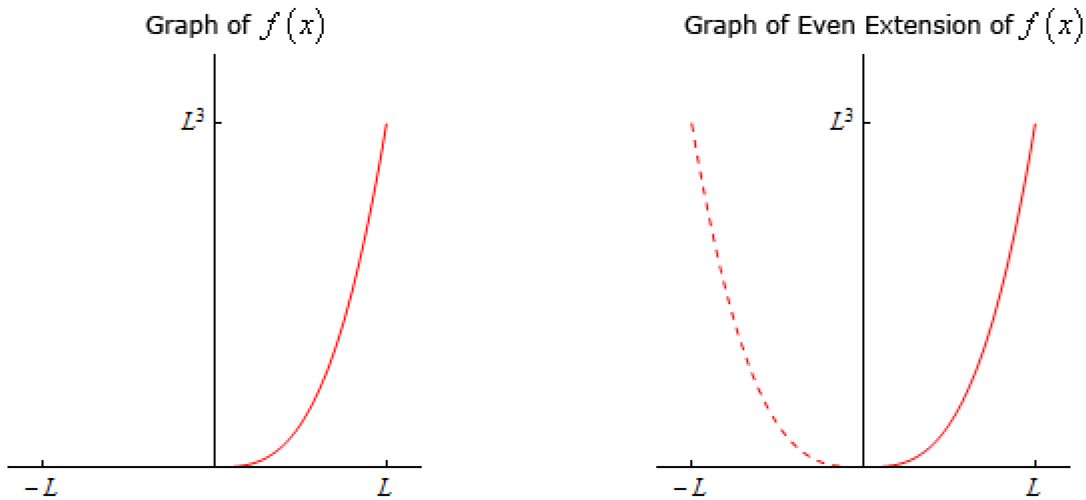
(b) $f\left(x\right)=x^3$ on $0\le x\le L$

The even extension of this function is,

$$g\left(x\right)=\begin{cases}f\left(x\right) & \text{if } \ 0\le x\le L\\ f\left(-x\right) & \text{if } \ -L\le x\le 0\end{cases}$$

$$=\begin{cases}x^3 & \text{if } \ 0\le x\le L\\ -x^3 & \text{if } \ -L\le x\le 0\end{cases}$$

The sketch of the function and the even extension is,



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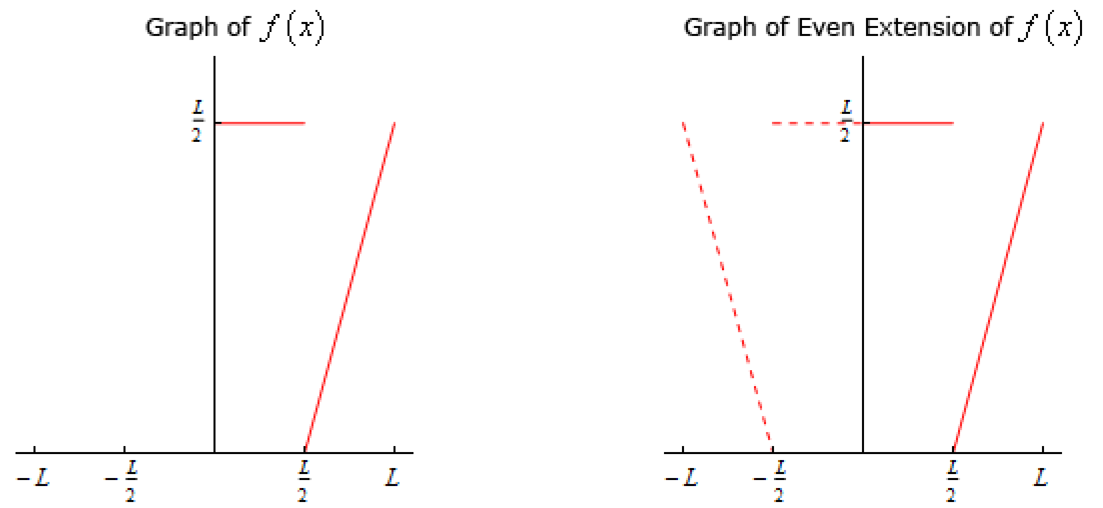
(c) $f\left(x\right)=\begin{cases}\frac{L}{2} & \text{if } \ 0\le x\le \frac{L}{2}\\ x-\frac{L}{2} & \text{if } \ \frac{L}{2}\le x\le L\end{cases}$

Here is the even extension of this function,

$$g\left(x\right)=\begin{cases}f\left(x\right) & \text{if } \ 0\le x\le L\\ f\left(-x\right) & \text{if } \ -L\le x\le 0\end{cases}$$

$$=\begin{cases}x-\frac{L}{2} & \text{if } \ \frac{L}{2}\le x\le L\\ \frac{L}{2} & \text{if } \ 0\le x\le \frac{L}{2}\\ \frac{L}{2} & \text{if } \ -\frac{L}{2}\le x\le 0\\ -x-\frac{L}{2} & \text{if } \ -L\le x\le -\frac{L}{2}\end{cases}$$

The sketch of the function and the even extension is,



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Okay, let's now think about how we can use the even extension of a function to find the Fourier cosine series of any function $f(x)$ on $0 \le x \le L$.

So, given a function $f(x)$ we'll let $g(x)$ be the even extension as defined above. Now, $g(x)$ is an even function on $-L \le x \le L$ and so we can write down its Fourier cosine series. This is,

$$g(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)\,dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

and note that we'll use the second form of the integrals to compute the constants.

Now, because we know that on $0 \le x \le L$ we have $f(x) = g(x)$ and so the Fourier cosine series of $f(x)$ on $0 \le x \le L$ is also given by,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)\,dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \ne 0 \end{cases}$$

Let's take a look at a couple of examples.

Example 3 Find the Fourier cosine series for $f(x) = L - x$ on $0 \le x \le L$.

Solution
All we need to do is compute the coefficients so here is the work for that,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L L - x\,dx = \frac{L}{2}$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(-L\cos(n\pi) + L\right) = \frac{2L}{n^2\pi^2}\left(1+(-1)^{n+1}\right) \qquad n = 1,2,3,\ldots \end{aligned}$$

The Fourier cosine series is then,

$$f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left(1+(-1)^{n+1}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Note that as we did with the first example in this section we stripped out the A_0 term before we plugged in the coefficients.

Next, let's find the Fourier cosine series of an odd function. Note that this is doable because we are really finding the Fourier cosine series of the even extension of the function.

Example 4 Find the Fourier cosine series for $f(x) = x^3$ on $0 \le x \le L$.

Solution
The integral for A_0 is simple enough but the integral for the rest will be fairly messy as it will require three integration by parts. We'll leave most of the details of the actual integration to you to verify. Here's the work,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L x^3\,dx = \frac{L^3}{4}$$

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^3\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n^4\pi^4}\right)\left(n\pi x\left(n^2\pi^2x^2 - 6L^2\right)\sin\left(\frac{n\pi x}{L}\right) + \left(3Ln^2\pi^2x^2 - 6L^3\right)\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\ &= \frac{2}{L}\left(\frac{L}{n^4\pi^4}\right)\left(n\pi L\left(n^2\pi^2L^2 - 6L^2\right)\sin(n\pi) + \left(3L^3n^2\pi^2 - 6L^3\right)\cos(n\pi) + 6L^3\right) \\ &= \frac{2}{L}\left(\frac{3L^4}{n^4\pi^4}\right)\left(2 + \left(n^2\pi^2 - 2\right)(-1)^n\right) = \frac{6L^3}{n^4\pi^4}\left(2 + \left(n^2\pi^2-2\right)(-1)^n\right) \qquad n = 1,2,3,\ldots \end{aligned}$$

The Fourier cosine series for this function is then,

$$f\left(x\right)=\frac{L^3}{4}+\sum_{n=1}^{\infty}\frac{6L^3}{n^4\pi^4}\left(2+\left(n^2\pi^2-2\right)\left(-1\right)^n\right)\cos\left(\frac{n\pi x}{L}\right)$$

Finally, let's take a quick look at a piecewise function.

Example 5 Find the Fourier cosine series for $f\left(x\right)=\begin{cases}\frac{L}{2} & \text{if } 0\le x\le\frac{L}{2}\\ x-\frac{L}{2} & \text{if } \frac{L}{2}\le x\le L\end{cases}$ on $0\le x\le L$.

Solution

We'll need to split up the integrals for each of the coefficients here. Here are the coefficients.

$$\begin{aligned}A_0&=\frac{1}{L}\int_0^L f\left(x\right)dx=\frac{1}{L}\left[\int_0^{\frac{L}{2}}f\left(x\right)dx+\int_{\frac{L}{2}}^{L}f\left(x\right)dx\right]\\&=\frac{1}{L}\left[\int_0^{\frac{L}{2}}\frac{L}{2}dx+\int_{\frac{L}{2}}^{L}x-\frac{L}{2}dx\right]=\frac{1}{L}\left[\frac{L^2}{4}+\frac{L^2}{8}\right]=\frac{1}{L}\left[\frac{3L^2}{8}\right]=\frac{3L}{8}\end{aligned}$$

For the rest of the coefficients here is the integral we'll need to do.

$$\begin{aligned}A_n=\frac{2}{L}\int_0^L f\left(x\right)\cos\left(\frac{n\pi x}{L}\right)dx&=\frac{2}{L}\left[\int_0^{\frac{L}{2}}f\left(x\right)\cos\left(\frac{n\pi x}{L}\right)dx+\int_{\frac{L}{2}}^{L}f\left(x\right)\cos\left(\frac{n\pi x}{L}\right)dx\right]\\&=\frac{2}{L}\left[\int_0^{\frac{L}{2}}\frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx+\int_{\frac{L}{2}}^{L}\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx\right]\end{aligned}$$

To make life a little easier let's do each of these separately.

$$\int_0^{\frac{L}{2}}\frac{L}{2}\cos\left(\frac{n\pi x}{L}\right)dx=\frac{L}{2}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}}=\frac{L}{2}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)=\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\begin{aligned}\int_{\frac{L}{2}}^{L}\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx&=\frac{L}{n\pi}\left(\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\left(x-\frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{\frac{L}{2}}^{L}\\&=\frac{L}{n\pi}\left(\frac{L}{n\pi}\cos\left(n\pi\right)+\frac{L}{2}\sin\left(n\pi\right)-\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)\\&=\frac{L^2}{n^2\pi^2}\left(\left(-1\right)^n-\cos\left(\frac{n\pi}{2}\right)\right)\end{aligned}$$

Putting these together gives,

$$A_n = \frac{2}{L}\left[\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\left(\left(-1\right)^n - \cos\left(\frac{n\pi}{2}\right)\right)\right]$$
$$= \frac{2L}{n^2\pi^2}\left[\left(-1\right)^n - \cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right]$$

So, after all that work the Fourier cosine series is then,

$$f\left(x\right) = \frac{3L}{8} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[\left(-1\right)^n - \cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

Note that much as we saw with the Fourier sine series many of the coefficients will quite messy to deal with.

Fourier Series

Okay, in the previous two sections we've looked at Fourier sine and Fourier cosine series. It is now time to look at a Fourier series. With a Fourier series we are going to try to write a series representation for $f(x)$ on $-L \le x \le L$ in the form,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

So, a Fourier series is, in some way a combination of the Fourier sine and Fourier cosine series. Also, like the Fourier sine/cosine series we'll not worry about whether or not the series will actually converge to $f(x)$ or not at this point. For now we'll just assume that it will converge and we'll discuss the convergence of the Fourier series in a later section.

Determining formulas for the coefficients, A_n and B_n, will be done in exactly the same manner as we did in the previous two sections. We will take advantage of the fact that $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are mutually orthogonal on $-L \le x \le L$ as we proved earlier. We'll also need the following formulas that we derived when we proved the two sets were mutually orthogonal.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \ne 0 \\ 0 & \text{if } n \ne m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \ne m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

So, let's start off by multiplying both sides of the series above by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating from $-L$ to L. Doing this gives,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \int_{-L}^{L} \sum_{n=0}^{\infty} A_n \cos\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx + \int_{-L}^{L} \sum_{n=1}^{\infty} B_n \sin\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

Now, just as we've been able to do in the last two sections we can interchange the integral and the summation. Doing this gives,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_{-L}^{L} \cos\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\tfrac{n\pi x}{L}\right)\cos\left(\tfrac{m\pi x}{L}\right)dx$$

We can now take advantage of the fact that the sines and cosines are mutually orthogonal. The integral in the second series will always be zero and in the first series the integral will be zero if $n \neq m$ and so this reduces to,

$$\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx = \begin{cases} A_m(2L) & \text{if } n = m = 0 \\ A_m(L) & \text{if } n = m \neq 0 \end{cases}$$

Solving for A_m gives,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{m\pi x}{L}\right)dx \qquad\qquad m = 1, 2, 3, \ldots$$

Now, do it all over again only this time multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$, integrate both sides from –*L* to *L* and interchange the integral and summation to get,

$$\int_{-L}^{L} f(x)\sin\left(\tfrac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \int_{-L}^{L}\cos\left(\tfrac{n\pi x}{L}\right)\sin\left(\tfrac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L}\sin\left(\tfrac{n\pi x}{L}\right)\sin\left(\tfrac{m\pi x}{L}\right)dx$$

In this case the integral in the first series will always be zero and the second will be zero if $n \neq m$ and so we get,

$$\int_{-L}^{L} f(x)\sin\left(\tfrac{m\pi x}{L}\right)dx = B_m(L)$$

Finally, solving for B_m gives,

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{m\pi x}{L}\right)dx \qquad\qquad m = 1, 2, 3, \ldots$$

In the previous two sections we also took advantage of the fact that the integrand was even to give a second form of the coefficients in terms of an integral from 0 to *L*. However, in this case we don't know anything about whether $f(x)$ will be even, odd, or more likely neither even nor odd. Therefore, this is the only form of the coefficients for the Fourier series.

Before we start examples let's remind ourselves of a couple of formulas that we'll make heavy use of here in this section, as we've done in the previous two sections as well. Provided *n* in an integer then,

$$\cos(n\pi) = (-1)^n \qquad\qquad \sin(n\pi) = 0$$

In all of the work that we'll be doing here *n* will be an integer and so we'll use these without comment in the problems so be prepared for them.

Also don't forget that sine is an odd function, *i.e.* $\sin(-x) = -\sin(x)$ and that cosine is an even function, *i.e.* $\cos(-x) = \cos(x)$. We'll also be making heavy use of these ideas without comment in many of the integral evaluations so be ready for these as well.

Now let's take a look at an example.

Example 1 Find the Fourier series for $f(x) = L - x$ on $-L \le x \le L$.

Solution
So, let's go ahead and just run through formulas for the coefficients.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} L - x\,dx = L$$

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} (L-x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(n\pi(L-x)\sin\left(\frac{n\pi x}{L}\right) - L\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} \\ &= \frac{1}{L}\left(\frac{L}{n^2\pi^2}\right)\left(-2n\pi L\sin(-n\pi)\right) = 0 \qquad\qquad n = 1, 2, 3, \ldots \end{aligned}$$

$$\begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} (L-x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{1}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L} \\ &= \frac{1}{L}\left[\frac{L^2}{n^2\pi^2}\left(2n\pi\cos(n\pi) - 2\sin(n\pi)\right)\right] = \frac{2L(-1)^n}{n\pi} \qquad\qquad n = 1, 2, 3, \ldots \end{aligned}$$

Note that in this case we had $A_0 \ne 0$ and $A_n = 0,\ n = 1, 2, 3, \ldots$ This will happen on occasion so don't get excited about this kind of thing when it happens.

The Fourier series is then,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) \\ &= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = L + \sum_{n=1}^{\infty}\frac{2L(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

As we saw in the previous example sometimes we'll get $A_0 \ne 0$ and $A_n = 0,\ n = 1, 2, 3, \ldots$ Whether or not this will happen will depend upon the function $f(x)$ and often won't happen, but when it does don't get excited about it.

Let's take a look at another problem.

Example 2 Find the Fourier series for $f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$ on $-L \le x \le L$.

Solution
Because of the piece-wise nature of the function the work for the coefficients is going to be a little unpleasant but let's get on with it.

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\left[\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\right] \\ &= \frac{1}{2L}\left[\int_{-L}^{0} L\,dx + \int_{0}^{L} 2x\,dx\right] = \frac{1}{2L}\left[L^2 + L^2\right] = L \end{aligned}$$

$$\begin{aligned} A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{1}{L}\left[\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx\right] \end{aligned}$$

At this point it will probably be easier to do each of these individually.

$$\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx = \left(\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0} = \frac{L^2}{n\pi}\sin(n\pi) = 0$$

$$\begin{aligned} \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\cos\left(\frac{n\pi x}{L}\right) + n\pi x\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L} \\ &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\cos(n\pi) + n\pi L\sin(n\pi) - L\cos(0)\right) \\ &= \left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n - 1\right) \end{aligned}$$

So, if we put all of this together we have,

$$\begin{aligned} A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx &= \frac{1}{L}\left[0 + \left(\frac{2L^2}{n^2\pi^2}\right)\left((-1)^n - 1\right)\right] \\ &= \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right), \qquad n = 1, 2, 3, \ldots \end{aligned}$$

So, we've gotten the coefficients for the cosines taken care of and now we need to take care of the coefficients for the sines.

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$
$$= \frac{1}{L}\left[\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

As with the coefficients for the cosines will probably be easier to do each of these individually.

$$\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx = \left(-\frac{L^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0} = \frac{L^2}{n\pi}\left(-1+\cos(n\pi)\right) = \frac{L^2}{n\pi}\left((-1)^n - 1\right)$$

$$\int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx = \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L}$$
$$= \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin(n\pi) - n\pi L\cos(n\pi)\right)$$
$$= \left(\frac{2L^2}{n^2\pi^2}\right)\left(-n\pi(-1)^n\right) = -\frac{2L^2}{n\pi}(-1)^n$$

So, if we put all of this together we have,

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\frac{L^2}{n\pi}\left((-1)^n - 1\right) - \frac{2L^2}{n\pi}(-1)^n\right]$$
$$= \frac{L}{n\pi}\left[-1-(-1)^n\right] = -\frac{L}{n\pi}\left(1+(-1)^n\right) \qquad n = 1, 2, 3, \ldots$$

So, after all that work the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)$$
$$= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)$$
$$= L + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty}\frac{L}{n\pi}\left(1+(-1)^n\right)\sin\left(\frac{n\pi x}{L}\right)$$

As we saw in the previous example there is often quite a bit of work involved in computing the integrals involved here.

The next couple of examples are here so we can make a nice observation about some Fourier series and their relation to Fourier sine/cosine series

Example 3 Find the Fourier series for $f(x) = x$ on $-L \le x \le L$.

Solution

Let's start with the integrals for A_n.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} x\,dx = 0$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

In both cases note that we are integrating an odd function (x is odd and cosine is even so the product is odd) over the interval $[-L, L]$ and so we know that both of these integrals will be zero.

Next here is the integral for B_n

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

In this case we're integrating an even function (x and sine are both odd so the product is even) on the interval $[-L, L]$ and so we can "simplify" the integral as shown above. The reason for doing this here is not actually to simplify the integral however. It is instead done so that we can note that we did this integral back in the Fourier sine series section and so don't need to redo it in this section. Using the previous result we get,

$$B_n = \frac{(-1)^{n+1}\,2L}{n\pi} \qquad\qquad n = 1, 2, 3, \ldots$$

In this case the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

If you go back and take a look at Example 1 in the Fourier sine series section, the same example we used to get the integral out of, you will see that in that example we were finding the Fourier sine series for $f(x) = x$ on $-L \le x \le L$. The important thing to note here is that the answer that we got in that example is identical to the answer we got here.

If you think about it however, this should not be too surprising. In both cases we were using an odd function on $-L \le x \le L$ and because we know that we had an odd function the coefficients of the cosines in the Fourier series, A_n, will involve integrating and odd function over a symmetric interval, $-L \le x \le L$, and so will be zero. So, in these cases the Fourier sine series of an odd function on $-L \le x \le L$ is really just a special case of a Fourier series.

Note however that when we moved over to doing the Fourier sine series of any function on $0 \le x \le L$ we should no longer expect to get the same results. You can see this by comparing Example 1 above with Example 3 in the Fourier sine series section. In both examples we are finding the series for $f(x) = x - L$ and yet got very different answers.

So, why did we get different answers in this case? Recall that when we find the Fourier sine series of a function on $0 \le x \le L$ we are really finding the Fourier sine series of the odd extension of the function on $-L \le x \le L$ and then just restricting the result down to $0 \le x \le L$. For a Fourier series we are actually using the whole function on $-L \le x \le L$ instead of its odd extension. We should therefore not expect to get the same results since we are really using different functions (at least on part of the interval) in each case.

So, if the Fourier sine series of an odd function is just a special case of a Fourier series it makes some sense that the Fourier cosine series of an even function should also be a special case of a Fourier series. Let's do a quick example to verify this.

Example 4 Find the Fourier series for $f(x) = x^2$ on $-L \le x \le L$.

Solution

Here are the integrals for the A_n and in this case because both the function and cosine are even we'll be integrating an even function and so can "simplify" the integral.

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\int_{0}^{L} x^2\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)dx$$

As with the previous example both of these integrals were done in Example 1 in the Fourier cosine series section and so we'll not bother redoing them here. The coefficients are,

$$A_0 = \frac{L^2}{3} \qquad\qquad A_n = \frac{4L^2(-1)^n}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

Next here is the integral for the B_n

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

In this case the function is even and sine is odd so the product is odd and we're integrating over $-L \le x \le L$ and so the integral is zero.

The Fourier series is then,

$$f(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)$$

As suggested before we started this example the result here is identical to the result from Example 1 in the Fourier cosine series section and so we can see that the Fourier cosine series of an even function is just a special case a Fourier series.

Convergence of Fourier Series

Over the last few sections we've spent a fair amount of time to computing Fourier series, but we've avoided discussing the topic of convergence of the series. In other words, will the Fourier series converge to the function on the given interval?

In this section we're going to address this issue as well as a couple of other issues about Fourier series. We'll be giving a fair number of theorems in this section but are not going to be proving any of them. We'll also not be doing a whole lot of in the way of examples in this section.

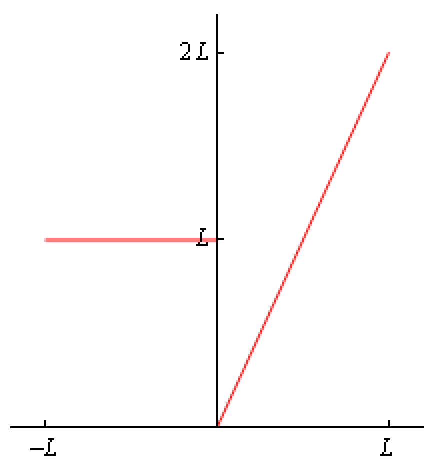
Before we get into the topic of convergence we need to first define a couple of terms that we'll run into in the rest of the section. First, we say that $f(x)$ has a **jump discontinuity** at $x = a$ if the limit of the function from the left, denoted $f(a^-)$, and the limit of the function from the right, denoted $f(a^+)$, both exist and $f(a^-) \neq f(a^+)$.

Next, we say that $f(x)$ is **piecewise smooth** if the function can be broken into distinct pieces and on each piece both the function and its derivative, $f'(x)$, are continuous. A piecewise smooth function may not be continuous everywhere however the only discontinuities that are allowed are a finite number of jump discontinuities.

Let's consider the function,

$$f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$$

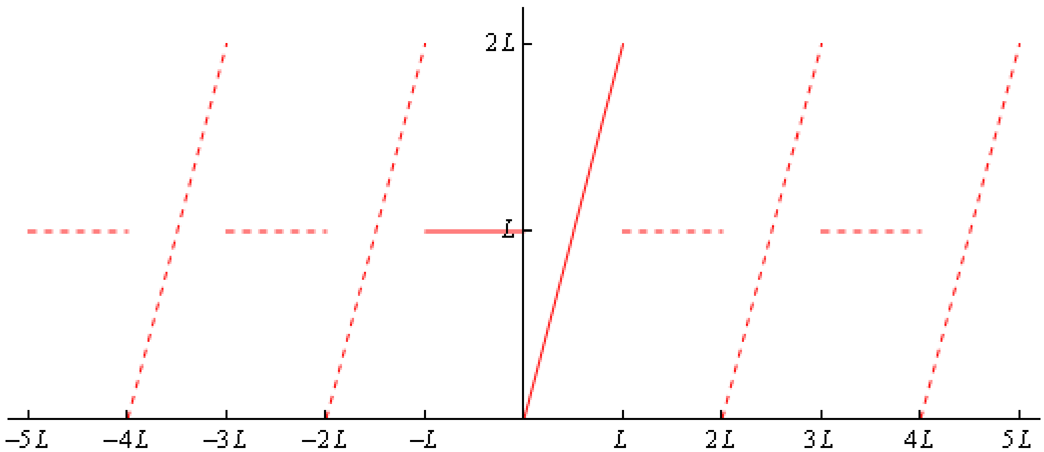
We found the Fourier series for this function in Example 2 of the previous section. Here is a sketch of this function on the interval on which it is defined, *i.e.* $-L \le x \le L$.



This function has a jump discontinuity at $x = 0$ because $f(0^-) = L \neq 0 = f(0^+)$ and note that on the intervals $-L \le x \le 0$ and $0 \le x \le L$ both the function and its derivative are continuous. This is therefore an example of a piecewise smooth function. Note that the function itself is not continuous at $x=0$ but because this point of discontinuity is a jump discontinuity the function is still piecewise smooth.

The last term we need to define is that of **periodic extension**. Given a function, $f(x)$, defined on some interval, we'll be using $-L \le x \le L$ exclusively here, the periodic extension of this function is the new function we get by taking the graph of the function on the given interval and then repeating that graph to the right and left of the graph of the original function on the given interval.

It is probably best to see an example of a periodic extension at this point to help make the words above a little clearer. Here is a sketch of the period extension of the function we looked at above,



The original function is the solid line in the range $-L \le x \le L$. We then got the periodic extension of this by picking this piece up and copying it every interval of length 2*L* to the right and left of the original graph. This is shown with the two sets of dashed lines to either side of the original graph.

Note that the resulting function that we get from defining the periodic extension is in fact a new periodic function that is equal to the original function on $-L \le x \le L$.

With these definitions out of the way we can now proceed to talk a little bit about the convergence of Fourier series. We will start off with the convergence of a Fourier series and once we have that taken care of the convergence of Fourier Sine/Cosine series will follow as a direct consequence. Here then is the theorem giving the convergence of a Fourier series.

Convergence of Fourier series

Suppose $f(x)$ is a piecewise smooth on the interval $-L \le x \le L$. The Fourier series of $f(x)$ will then converge to,

1. the periodic extension of $f(x)$ if the periodic extension is continuous.

2. the average of the two one-sided limits, $\frac{1}{2}\left[f\left(a^-\right)+f\left(a^+\right)\right]$, if the periodic extension has a jump discontinuity at $x=a$.

The first thing to note about this is that on the interval $-L \le x \le L$ both the function and the periodic extension are equal and so where the function is continuous on $-L \le x \le L$ the periodic extension will also be continuous and hence at these points the Fourier series will in fact converge to the function. The only points in the interval $-L \le x \le L$ where the Fourier series will not converge to the function is where the function has a jump discontinuity.

Let's again consider Example 2 of the previous section. In that section we found that the Fourier series of,

$$f(x) = \begin{cases} L & \text{if } -L \le x \le 0 \\ 2x & \text{if } 0 \le x \le L \end{cases}$$

on $-L \le x \le L$ to be,

$$f(x) = L + \sum_{n=1}^{\infty} \frac{2L}{n^2 \pi^2} \left((-1)^n - 1 \right) \cos\left(\frac{n\pi x}{L} \right) - \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(1 + (-1)^n \right) \sin\left(\frac{n\pi x}{L} \right)$$

We now know that in the intervals $-L < x < 0$ and $0 < x < L$ the function and hence the periodic extension are both continuous and so on these two intervals the Fourier series will converge to the periodic extension and hence will converge to the function itself.

At the point $x = 0$ the function has a jump discontinuity and so the periodic extension will also have a jump discontinuity at this point. That means that at $x = 0$ the Fourier series will converge to,

$$\frac{1}{2}\left[f\left(0^-\right) + f\left(0^+\right) \right] = \frac{1}{2}\left[L + 0 \right] = \frac{L}{2}$$

At the two endpoints of the interval, $x = -L$ and $x = L$, we can see from the sketch of the periodic extension above that the periodic extension has a jump discontinuity here and so the Fourier series will not converge to the function there but instead the averages of the limits.

So, at $x = -L$ the Fourier series will converge to,

$$\frac{1}{2}\left[f\left(-L^-\right) + f\left(-L^+\right) \right] = \frac{1}{2}\left[2L + L \right] = \frac{3L}{2}$$

and at $x = L$ the Fourier series will converge to,

$$\frac{1}{2}\left[f\left(L^-\right) + f\left(L^+\right) \right] = \frac{1}{2}\left[2L + L \right] = \frac{3L}{2}$$

Now that we have addressed the convergence of a Fourier series we can briefly turn our attention to the convergence of Fourier sine/cosine series. First, as noted in the previous section the Fourier sine series of an odd function on $-L \le x \le L$ and the Fourier cosine series of an even function on $-L \le x \le L$ are both just special cases of a Fourier series we now know that both of these will have the same convergence as a Fourier series.

Next, if we look at the Fourier sine series of any function, $g(x)$, on $0 \le x \le L$ then we know that this is just the Fourier series of the odd extension of $g(x)$ restricted down to the interval $0 \le x \le L$. Therefore we know that the Fourier series will converge to the odd extension on $-L \le x \le L$ where it is continuous and the average of the limits where the odd extension has a jump discontinuity. However, on $0 \le x \le L$ we know that $g(x)$ and the odd extension are equal and so we can again see that the Fourier sine series will have the same convergence as the Fourier series.

Likewise, we can go through a similar argument for the Fourier cosine series using even extensions to see that Fourier cosine series for a function on $0 \le x \le L$ will also have the same convergence as a Fourier series.

The next topic that we want to briefly discuss here is when will a Fourier series be continuous. From the theorem on the convergence of Fourier series we know that where the function is continuous the Fourier series will converge to the function and hence be continuous at these points. The only places where the Fourier series may not be continuous is if there is a jump discontinuity on the interval $-L \le x \le L$ and potentially at the endpoints as we saw that the periodic extension may introduce a jump discontinuity there.

So, if we're going to want the Fourier series to be continuous everywhere we'll need to make sure that the function does not have any discontinuities in $-L \le x \le L$. Also, in order to avoid having the periodic extension introduce a jump discontinuity we'll need to require that $f(-L) = f(L)$. By doing this the two ends of the graph will match up when we form the periodic extension and hence we will avoid a jump discontinuity at the end points.

Here is a summary of these ideas for a Fourier series.

Suppose $f(x)$ is a piecewise smooth on the interval $-L \le x \le L$. The Fourier series of $f(x)$ will be continuous and will converge to $f(x)$ on $-L \le x \le L$ provided $f(x)$ is continuous on $-L \le x \le L$ and $f(-L) = f(L)$.

Now, how can we use this to get similar statements about Fourier sine/cosine series on $0 \le x \le L$? Let's start with a Fourier cosine series. The first thing that we do is form the even extension of $f(x)$ on $-L \le x \le L$. For the purposes of this discussion let's call the even extension $g(x)$ As we saw when we sketched several even extensions in the Fourier cosine series section that in order for the sketch to be the even extension of the function we must have both,

$$g(0^-) = g(0^+) \qquad\qquad g(-L) = g(L)$$

If one or both of these aren't true then $g(x)$ will not be an even extension of $f(x)$.

So, in forming the even extension we do not introduce any jump discontinuities at $x = 0$ and we get for free that $g(-L) = g(L)$. If we now apply the above theorem to the even extension we see that the Fourier series of the even extension is continuous on $-L \le x \le L$. However, because the even extension and the function itself are the same on $0 \le x \le L$ then the Fourier cosine series of $f(x)$ must also be continuous on $0 \le x \le L$.

Here is a summary of this discussion for the Fourier cosine series.

Suppose $f(x)$ is a piecewise smooth on the interval $0 \le x \le L$. The Fourier cosine series of $f(x)$ will be continuous and will converge to $f(x)$ on $0 \le x \le L$ provided $f(x)$ is continuous on $0 \le x \le L$.

Note that we don't need any requirements on the end points here because they are trivially satisfied when we convert over to the even extension.

For a Fourier sine series we need to be a little more careful. Again, the first thing that we need to do is form the odd extension on $-L \le x \le L$ and let's call it $g(x)$. We know that in order for it to be the odd extension then we know that at all points in $-L \le x \le L$ it must satisfy $g(-x) = -g(x)$ and that is what can lead to problems.

As we saw in the Fourier sine series section it is very easy to introduce a jump discontinuity at $x = 0$ when we form the odd extension. In fact, the only way to avoid forming a jump discontinuity at this point is to require that $f(0) = 0$.

Next, the requirement that at the endpoints we must have $g(-L) = -g(L)$ will practically guarantee that we'll introduce a jump discontinuity here as well when we form the odd extension. Again, the only way to avoid doing this is to require $f(L) = 0$.

So, with these two requirements we will get an odd extension that is continuous and so we know that the Fourier series of the odd extension on $-L \le x \le L$ will be continuous and hence the Fourier sine series will be continuous on $0 \le x \le L$.

Here is a summary of all this for the Fourier sine series.

Suppose $f(x)$ is a piecewise smooth on the interval $0 \le x \le L$. The Fourier sine series of $f(x)$ will be continuous and will converge to $f(x)$ on $0 \le x \le L$ provided $f(x)$ is continuous on $0 \le x \le L$, $f(0) = 0$ and $f(L) = 0$.

The next topic of discussion here is differentiation and integration of Fourier series. In particular we want to know if we can differentiate a Fourier series term by term and have the result be the Fourier series of the derivative of the function. Likewise we want to know if we can integrate a Fourier series term by term and arrive at the Fourier series of the integral of the function.

Note that we'll not be doing much discussion of the details here. All we're really going to be doing is giving the theorems that govern the ideas here so that we can say we've given them.

Let's start off with the theorem for term by term differentiation of a Fourier series.

Given a function $f(x)$ if the derivative, $f'(x)$, is piecewise smooth and the Fourier series of $f(x)$ is continuous then the Fourier series can be differentiated term by term. The result of the differentiation is the Fourier series of the derivative, $f'(x)$.

One of the main condition of this theorem is that the Fourier series be continuous and from above we also know the conditions on the function that will give this. So, if we add this into the theorem to get this form of the theorem,

Suppose $f(x)$ is a continuous function, its derivative $f'(x)$ is piecewise smooth and $f(-L) = f(L)$ then the Fourier series of the function can be differentiated term by term and the result is the Fourier series of the derivative.

For Fourier cosine/sine series the basic theorem is the same as for Fourier series. All that's required is that the Fourier cosine/sine series be continuous and then you can differentiate term by term. The theorems that we'll give here will merge the conditions for the Fourier cosine/sine series to be continuous into the theorem.

Let's start with the Fourier cosine series.

Suppose $f(x)$ is a continuous function and its derivative $f'(x)$ is piecewise smooth then the Fourier cosine series of the function can be differentiated term by term and the result is the Fourier sine series of the derivative.

Next the theorem for Fourier sine series.

Suppose $f(x)$ is a continuous function, its derivative $f'(x)$ is piecewise smooth, $f(0) = 0$ and $f(L) = 0$ then the Fourier sine series of the function can be differentiated term by term and the result is the Fourier cosine series of the derivative.

The theorem for integration of Fourier series term by term is simple so there it is.

Suppose $f(x)$ is piecewise smooth then the Fourier sine series of the function can be integrated term by term and the result is a convergent infinite series that will converge to the integral of $f(x)$.

Note however that the new series that results from term by term integration may not be the Fourier series for the integral of the function.

Partial Differential Equations

Introduction

In this chapter we are going to take a very brief look at one of the more common methods for solving simple partial differential equations. The method we'll be taking a look at is that of Separation of Variables.

We need to make it very clear before we even start this chapter that we are going to be doing nothing more than barely scratching the surface of not only partial differential equations but also of the method of separation of variables. It would take several classes to cover most of the basic techniques for solving partial differential equations. The intent of this chapter is to do nothing more than to give you a feel for the subject and if you'd like to know more taking a class on partial differential equations should probably be your next step.

Also note that in several sections we are going to be making heavy use of some of the results from the previous chapter. That in fact was the point of doing some of the examples that we did there. Having done them will, in some cases, significantly reduce the amount of work required in some of the examples we'll be working in this chapter. When we do make use of a previous result we will make it very clear where the result is coming from.

Here is a brief listing of the topics covered in this chapter.

The Heat Equation – We do a partial derivation of the heat equation in this section as well as a discussion of possible boundary values.

The Wave Equation – Here we do a partial derivation of the wave equation.

Terminology – In this section we take a quick look at some of the terminology used in the method of separation of variables.

Separation of Variables – We take a look at the first step in the method of separation of variables in this section. This first step is really the step that motivates the whole process.

Solving the Heat Equation – In this section we go through the complete separation of variables process and along the way solve the heat equation with three different sets of boundary conditions.

Heat Equation with Non-Zero Temperature Boundaries – Here we take a quick look at solving the heat equation in which the boundary conditions are fixed, non-zero temperature conditions.

Laplace's Equation – We discuss solving Laplace's equation on both a rectangle and a disk in this section.

Vibrating String – Here we solve the wave equation for a vibrating string.

Summary of Separation of Variables – In this final section we give a quick summary of the method of separation of variables.

The Heat Equation

Before we get into actually solving partial differential equations and before we even start discussing the method of separation of variables we want to spend a little bit of time talking about the two main partial differential equations that we'll be solving later on in the chapter. We'll look at the first one in this section and the second one in the next section.

The first partial differential equation that we'll be looking at once we get started with solving will be the heat equation, which governs the temperature distribution in an object. We are going to give several forms of the heat equation for reference purposes, but we will only be really solving one of them.

We will start out by considering the temperature in a 1-D bar of length *L*. What this means is that we are going to assume that the bar starts off at $x = 0$ and ends when we reach $x = L$. We are also going to so assume that at any location, *x* the temperature will be constant an every point in the cross section at that *x*. In other words, temperature will only vary in *x* and we can hence consider the bar to be a 1-D bar. Note that with this assumption the actual shape of the cross section (*i.e.* circular, rectangular, *etc.*) doesn't matter.

Note that the 1-D assumption is actually not all that bad of an assumption as it might seem at first glance. If we assume that the lateral surface of the bar is perfectly insulated (*i.e.* no heat can flow through the lateral surface) then the only way heat can enter or leave the bar as at either end. This means that heat can only flow from left to right or right to left and thus creating a 1-D temperature distribution.

The assumption of the lateral surfaces being perfectly insulated is of course impossible, but it is possible to put enough insulation on the lateral surfaces that there will be very little heat flow through them and so, at least for a time, we can consider the lateral surfaces to be perfectly insulated.

Okay, let's now get some definitions out of the way before we write down the first form of the heat equation.

$$
\begin{aligned}
u(x,t) &= \text{Temperature at any point } x \text{ and any time } t\\
c(x) &= \text{Specific Heat}\\
\rho(x) &= \text{Mass Density}\\
\varphi(x,t) &= \text{Heat Flux}\\
Q(x,t) &= \text{Heat energy generated per unit volume per unit time}
\end{aligned}
$$

We should probably make a couple of comments about some of these quantities before proceeding.

The specific heat, $c(x) > 0$, of a material is the amount of heat energy that it takes to raise one unit of mass of the material by one unit of temperature. As indicated we are going to assume, at least initially, that the specific heat may not be uniform throughout the bar. Note as well that in practice the specific heat depends upon the temperature. However, this will generally only be an issue for large temperature differences (which in turn depends on the material the bar is made out of) and so we're going to assume for the purposes of this discussion that the temperature differences are not large enough to affect our solution.

The mass density, $\rho(x)$, is the mass per unit volume of the material. As with the specific heat we're going to initially assume that the mass density may not be uniform throughout the bar.

The heat flux, $\varphi(x,t)$, is the amount of thermal energy that flows to the right per unit surface area per unit time. The "flows to the right" bit simply tells us that if $\varphi(x,t) > 0$ for some x and t then the heat is flowing to the right at that point and time. Likewise if $\varphi(x,t) < 0$ then the heat will be flowing to the left at that point and time.

The final quantity we defined above is $Q(x,t)$ and this is used to represent any external sources or sinks (*i.e.* heat energy taken out of the system) of heat energy. If $Q(x,t) > 0$ then heat energy is being added to the system at that location and time and if $Q(x,t) < 0$ then heat energy is being removed from the system at that location and time.

With these quantities the heat equation is,

$$c(x)\rho(x)\frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x} + Q(x,t) \qquad \textbf{(2)}$$

While this is a nice form of the heat equation it is not actually something we can solve. In this form there are two unknown functions, u and φ, and so we need to get rid of one of them. With **Fourier's law** we can easily remove the heat flux from this equation.

Fourier's law states that,

$$\varphi(x,t) = -K_0(x)\frac{\partial u}{\partial x}$$

where $K_0(x) > 0$ is the **thermal conductivity** of the material and measures the ability of a given material to conduct heat. The better a material can conduct heat the larger $K_0(x)$ will be. As noted the thermal conductivity can vary with the location in the bar. Also, much like the specific heat the thermal conductivity can vary with temperature, but we will assume that the total temperature change is not so great that this will be an issue and so we will assume for the purposes here that the thermal conductivity will not vary with temperature.

Fourier's law does a very good job of modeling what we know to be true about heat flow. First, we know that if the temperature in a region is constant, *i.e.* $\frac{\partial u}{\partial x} = 0$, then there is no heat flow.

Next, we know that if there is a temperature difference in a region we know the heat will flow from the hot portion to the cold portion of the region. For example, if it is hotter to the right then we know that the heat should flow to the left. When it is hotter to the right then we also know that $\frac{\partial u}{\partial x} > 0$ (*i.e.* the temperature increases as we move to the right) and so we'll have $\varphi < 0$ and so the heat will flow to the left as it should. Likewise, if $\frac{\partial u}{\partial x} < 0$ (*i.e.* it is hotter to the left) then we'll have $\varphi > 0$ and heat will flow to the right as it should.

Finally, the greater the temperature difference in a region (*i.e.* the larger $\frac{\partial u}{\partial x}$ is) then the greater the heat flow.

So, if we plug Fourier's law into (1), we get the following form of the heat equation,

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right) + Q(x,t) \qquad \textbf{(3)}$$

Note that we factored the minus sign out of the derivative to cancel against the minus sign that was already there. We cannot however, factor the thermal conductivity out of the derivative since it is a function of *x* and the derivative is with respect to *x*.

Solving (2) is quite difficult due to the non uniform nature of the thermal properties and the mass density. So, let's now assume that these properties are all constant, *i.e.*,

$$c(x) = c \qquad \rho(x) = \rho \qquad K_0(x) = K_0$$

where *c*, ρ and K_0 are now all fixed quantities. In this case we generally say that the material in the bar is **uniform**. Under these assumptions the heat equation becomes,

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad \textbf{(4)}$$

For a final simplification to the heat equation let's divide both sides by $c\rho$ and define the **thermal diffusivity** to be,

$$k = \frac{K_0}{c\rho}$$

The heat equation is then,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho} \qquad \textbf{(5)}$$

To most people this is what they mean when they talk about the heat equation and in fact it will be the equation that we'll be solving. Well, actually we'll be solving (4) with no external sources, *i.e.* $Q(x,t) = 0$, but we'll be considering this form when we start discussing separation of variables in a couple of sections. We'll only drop the sources term when we actually start solving the heat equation.

Now that we've got the 1-D heat equation taken care of we need to move into the initial and boundary conditions we'll also need in order to solve the problem. If you go back to any of our solutions of ordinary differential equations that we've done in previous sections you can see that the number of conditions required always matched the highest order of the derivative in the equation.

In partial differential equations the same idea holds except now we have to pay attention to the variable we're differentiating with respect to as well. So, for the heat equation we've got a first order time derivative and so we'll need one initial condition and a second order spatial derivative and so we'll need two boundary conditions.

The initial condition that we'll use here is,

$$u\left(x,0\right)=f\left(x\right)$$

and we don't really need to say much about it here other than to note that this just tells us what the initial temperature distribution in the bar is.

The boundary conditions will tell us something about what the temperature and/or heat flow is doing at the boundaries of the bar. There are four of them that are fairly common boundary conditions.

The first type of boundary conditions that we can have would be the **prescribed temperature** boundary conditions, also called **Dirichlet conditions**. The prescribed temperature boundary conditions are,

$$u\left(0,t\right)=g_1\left(t\right) \qquad\qquad u\left(L,t\right)=g_2\left(t\right)$$

The next type of boundary conditions are **prescribed heat flux**, also called **Neumann conditions**. Using Fourier's law these can be written as,

$$-K_0\left(0\right)\frac{\partial u}{\partial x}\left(0,t\right)=\varphi_1\left(t\right) \qquad\qquad -K_0\left(L\right)\frac{\partial u}{\partial x}\left(L,t\right)=\varphi_2\left(t\right)$$

If either of the boundaries are **perfectly insulated,** *i.e.* there is no heat flow out of them then these boundary conditions reduce to,

$$\frac{\partial u}{\partial x}\left(0,t\right)=0 \qquad\qquad \frac{\partial u}{\partial x}\left(L,t\right)=0$$

and note that we will often just call these particular boundary conditions **insulated** boundaries and drop the "perfectly" part.

The third type of boundary conditions use **Newton's law of cooling** and are sometimes called **Robins conditions**. These are usually used when the bar is in a moving fluid and note we can consider air to be a fluid for this purpose.

Here are the equations for this kind of boundary condition.

$$-K_0\left(0\right)\frac{\partial u}{\partial x}\left(0,t\right)=-H\left[u\left(0,t\right)-g_1\left(t\right)\right] \qquad -K_0\left(L\right)\frac{\partial u}{\partial x}\left(L,t\right)=H\left[u\left(L,t\right)-g_2\left(t\right)\right]$$

where H is a positive quantity that is experimentally determined and $g_1\left(t\right)$ and $g_2\left(t\right)$ give the temperature of the surrounding fluid at the respective boundaries.

Note that the two conditions do vary slightly depending on which boundary we are at. At $x=0$ we have a minus sign on the right side while we don't at $x=L$. To see why this is let's first assume that at $x = 0$ we have $u(0,t) > g_1(t)$. In other words the bar is hotter than the surrounding fluid and so at $x = 0$ the heat flow (as given by the left side of the equation) must be to the left, or negative since the heat will flow from the hotter bar into the cooler surrounding liquid. If the heat flow is negative then we need to have a minus sign on the right side of the equation to make sure that it has the proper sign.

If the bar is cooler than the surrounding fluid at $x = 0$, *i.e.* $u(0,t) < g_1(t)$ we can make a similar argument to justify the minus sign. We'll leave it to you to verify this.

If we now look at the other end, $x = L$, and again assume that the bar is hotter than the surrounding fluid or, $u(L,t) > g_2(t)$. In this case the heat flow must be to the right, or be positive, and so in this case we can't have a minus sign. Finally, we'll again leave it to you to verify that we can't have the minus sign at $x = L$ is the bar is cooler than the surrounding fluid as well.

Note that we are not actually going to be looking at any of these kinds of boundary conditions here. These types of boundary conditions tend to lead to boundary value problems such as Example 5 in the Eigenvalues and Eigenfunctions section of the previous chapter. As we saw in that example it is often very difficult to get our hands on the eigenvalues and as we'll eventually see we will need them.

It is important to note at this point that we can also mix and match these boundary conditions so to speak. There is nothing wrong with having a prescribed temperature at one boundary and a prescribed flux at the other boundary for example so don't always expect the same boundary condition to show up at both ends. This warning is more important that it might seem at this point because once we get into solving the heat equation we are going to have the same kind of condition on each end to simplify the problem somewhat.

The final type of boundary conditions that we'll need here are **periodic** boundary conditions. Periodic boundary conditions are,

$$u(-L,t) = u(L,t) \qquad\qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Note that for these kinds of boundary conditions the left boundary tends to be $x = -L$ instead of $x = 0$ as we were using in the previous types of boundary conditions. The periodic boundary conditions will arise very naturally from a couple of particular geometries that we'll be looking at down the road.

We will now close out this section with a quick look at the 2-D and 3-D version of the heat equation. However, before we jump into that we need to introduce a little bit of notation first.

The **del** operator is defined to be,

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \qquad\qquad \nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial k}\vec{k}$$

depending on whether we are in 2 or 3 dimensions. Think of the del operator as a function that takes functions as arguments (instead of numbers as we're used to). Whatever function we "plug" into the operator gets put into the partial derivatives.

So, for example in 3-D we would have,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial k}\vec{k}$$

This of course is also the gradient of the function $f(x, y, z)$.

The del operator also allows us to quickly write down the divergence of a function. So, again using 3-D as an example the divergence of $f(x, y, z)$ can be written as the dot product of the del operator and the function. Or,

$$\nabla \bullet f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial k}$$

Finally, we will also see the following show up in the our work,

$$\nabla \bullet (\nabla f) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial k}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This is usually denoted as,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and is called the **Laplacian**. The 2-D version of course simply doesn't have the third term.

Okay, we can now look into the 2-D and 3-D version of the heat equation and where ever the del operator and or Laplacian appears assume that it is the appropriate dimensional version.

The higher dimensional version of (1) is,

$$c\rho \frac{\partial u}{\partial t} = -\nabla \bullet \varphi + Q \qquad \textbf{(6)}$$

and note that the specific heat, *c*, and mass density, ρ, are may not be uniform and so may be functions of the spatial variables. Likewise, the external sources term, *Q*, may also be a function of both the spatial variables and time.

Next, the higher dimensional version of Fourier's law is,

$$\varphi = -K_0 \nabla u$$

where the thermal conductivity, K_0, is again assumed to be a function of the spatial variables.

If we plug this into (5) we get the heat equation for a non uniform bar (*i.e.* the thermal properties may be functions of the spatial variables) with external sources/sinks,

$$c\rho \frac{\partial u}{\partial t} = \nabla \bullet (K_0 \nabla u) + Q \qquad \textbf{(7)}$$

If we now assume that the specific heat, mass density and thermal conductivity are constant (*i.e.* the bar is uniform) the heat equation becomes,

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \frac{Q}{cp} \qquad \textbf{(8)}$$

where we divided both sides by $c\rho$ to get the thermal diffusivity, k in front of the Laplacian.

The initial condition for the 2-D or 3-D heat equation is,

$$u(x,y,t) = f(x,y) \qquad \text{or} \qquad u(x,y,z,t) = f(x,y,z)$$

depending upon the dimension we're in.

The prescribed temperature boundary condition becomes,

$$u(x,y,t) = T(x,y,t) \qquad \text{or} \qquad u(x,y,z,t) = T(x,y,z,t)$$

where (x,y) or (x,y,z), depending upon the dimension we're in, will range over the portion of the boundary in which we are prescribing the temperature.

The prescribed heat flux condition becomes,

$$-K_0 \nabla u \bullet \vec{n} = \varphi(t)$$

where the left side is only being evaluated at points along the boundary and $\vec{n}$ is the outward unit normal on the surface.

Newton's law of cooling will become,

$$-K_0 \nabla u \bullet \vec{n} = H(u - u_B)$$

where H is a positive quantity that is experimentally determine, u_B is the temperature of the fluid at the boundary and again it is assumed that this is only being evaluated at points along the boundary.

We don't have periodic boundary conditions here as they will only arise from specific 1-D geometries.

We should probably also acknowledge at this point that we'll not actually be solving (7) at any point, but we will be solving a special case of it in the Laplace's Equation section.

The Wave Equation

In this section we want to consider a vertical string of length L that has been tightly stretched between two points at $x = 0$ and $x = L$.

Because the string has been tightly stretched we can assume that the slope of the displaced string at any point is small. So just what does this do for us? Let's consider a point x on the string in its equilibrium position, *i.e.* the location of the point at $t = 0$. As the string vibrates this point will be displaced both vertically and horizontally, however, if we assume that at any point the slope of the string is small then the horizontal displacement will be very small in relation to the vertical displacement. This means that we can now assume that at any point x on the string the displacement will be purely vertical. So, let's call this displacement $u(x,t)$.

We are going to assume, at least initially, that the string is not uniform and so the mass density of the string, $\rho(x)$ may be a function of x.

Next we are going to assume that the string is perfectly flexible. This means that the string will have no resistance to bending. This in turn tells us that the force exerted by the string at any point x on the endpoints will be tangential to the string itself. This force is called the **tension** in the string and its magnitude will be given by $T(x,t)$.

Finally, we will let $Q(x,t)$ represent the vertical component per unit mass of any force acting on the string.

Provided we again assume that the slope of the string is small the vertical displacement of the string at any point is then given by,

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u}{\partial x}\right) + \rho(x)Q(x,t) \qquad \textbf{(9)}$$

This is a very difficult partial differential equation to solve so we need to make some further simplifications.

First, we're now going to assume that the string is perfectly elastic. This means that the magnitude of the tension, $T(x,t)$, will only depend upon how much the string stretches near x. Again, recalling that we're assuming that the slope of the string at any point is small this means that the tension in the string will then very nearly be the same as the tension in the string in its equilibrium position. We can then assume that the tension is a constant value, $T(x,t) = T_0$.

Further, in most cases the only external force that will act upon the string is gravity and if the string light enough the effects of gravity on the vertical displacement will be small and so will also assume that $Q(x,t) = 0$. This leads to

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2}$$

If we know divide by the mass density and define,

$$c^2 = \frac{T_0}{\rho}$$

we arrive at the 1-D **wave equation**,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(10)}$$

In the previous section when we looked at the heat equation he had a number of boundary conditions however in this case we are only going to consider one type of boundary conditions. For the wave equation the only boundary condition we are going to consider will be that of prescribed location of the boundaries or,

$$u(0,t) = h_1(t) \qquad\qquad u(L,t) = h_2(t)$$

The initial conditions (and yes we meant more than one…) will also be a little different here from what we saw with the heat equation. Here we have a 2nd order time derivative and so we'll also need two initial conditions. At any point we will specify both the initial displacement of the string as well as the initial slope of the string. The initial conditions are then,

$$u(x,0) = f(x) \qquad\qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

For the sake of completeness we'll close out this section with the 2-D and 3-D version of the wave equation. We'll not actually be solving this at any point, but since we gave the higher dimensional version of the heat equation (in which we will solve a special case) we'll give this as well.

The 2-D and 3-D version of the wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where ∇^2 is the Laplacian.

Terminology

We've got one more section that we need to take care of before we actually start solving partial differential equations. This will be a fairly short section that will cover some of the basic terminology that we'll need in the next section as we introduce the method of separation of variables.

Let's start off with the idea of an operator. An operator is really just a function that takes a function as an argument instead of numbers as we're used to dealing with in functions. You already know of a couple of operators even if you didn't know that they were operators. Here are some examples of operators.

$$L = \frac{d}{dx} \qquad L = \int dx \qquad L = \int_a^b dx \qquad L = \frac{\partial}{\partial t}$$

Or, if we plug in a function, say $u(x)$, into each of these we get,

$$L(u) = \frac{du}{dx} \qquad L(u) = \int u(x)\,dx \qquad L(u) = \int_a^b u(x)\,dx \qquad L(u) = \frac{\partial u}{\partial t}$$

These are all fairly simple examples of operators but the derivative and integral are operators. A more complicated operator would be the **heat operator**. We get the heat operator from a slight rewrite of the heat equation without sources. The heat operator is,

$$L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$$

Now, what we really want to define here is not an operator but instead a **linear operator**. A linear operator is any operator that satisfies,

$$L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$$

The heat operator is an example of a linear operator and this is easy enough to show using the basic properties of the partial derivative so let's do that.

$$\begin{aligned}
L(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) \\
&= \frac{\partial}{\partial t}(c_1u_1) + \frac{\partial}{\partial t}(c_2u_2) - k\left[\frac{\partial^2}{\partial x^2}(c_1u_1) + \frac{\partial^2}{\partial x^2}(c_2u_2)\right] \\
&= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - k\left[c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2}\right] \\
&= c_1\frac{\partial u_1}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial u_2}{\partial t} - kc_2\frac{\partial^2 u_2}{\partial x^2} \\
&= c_1\left[\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2}\right] + c_2\left[\frac{\partial u_2}{\partial t} - k\frac{\partial^2 u_2}{\partial x^2}\right] \\
&= c_1L(u_1) + c_2L(u_2)
\end{aligned}$$

You might want to verify for yourself that the derivative and integral operators we gave above are also linear operators. In fact, in the process of showing that the heat operator is a linear operator we actually showed as well that the first order and second order partial derivative operators are also linear.

The next term we need to define is a **linear equation**. A linear equation is an equation in the form,

$$L(u) = f \tag{11}$$

where L is a linear operator and f is a known function.

Here are some examples of linear partial differential equations.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla^2 u = 0$$

$$\frac{\partial u}{\partial t} - 4\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} + 8u - g(x,t)$$

The first two from this list are of course the heat equation and the wave equation. The third uses the Laplacian and is usually called **Laplace's Equation**. We'll actually be solving the 2-D version of Laplace's Equation in a few sections. The fourth equation was just made up to give a more complicated example.

Notice as well with the heat equation and the fourth example above that the presence of the $Q(x,t)$ and $g(x,t)$ do not prevent these from being linear equations. The main issue that allows these to be linear equations is the fact that the operator in each is linear.

Now just to be complete here are a couple of examples of nonlinear partial differential equations.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + u^2$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial t} = u + f(x,t)$$

We'll leave it to you to verify that the operators in each of these are not linear however the problem term in the first is the u^2 while in the second the product of the two derivatives is the problem term.

Now, if we go back to (1) and suppose that $f = 0$ then we arrive at,

$$L(u) = 0 \tag{12}$$

We call this a **linear homogeneous equation** (recall that L is a linear operator).

Notice that $u = 0$ will always be a solution to a linear homogeneous equation (go back to what it means to be linear and use $c_1 = c_2 = 0$ with any two solutions and this is easy to verify). We call $u = 0$ the **trivial solution**. In fact this is also a really nice way of determining if an equation is homogeneous. If L is a linear operator and we plug in $u = 0$ into the equation and we get $L(u) = 0$ then we will know that the operator is homogeneous.

We can also extend the ideas of linearity and homogeneous to boundary conditions. If we go back to the various boundary conditions we discussed for the heat equation for example we can also classify them as linear and/or homogeneous.

The prescribed temperature boundary conditions,

$$u(0,t) = g_1(t) \qquad\qquad u(L,t) = g_2(t)$$

are linear and will only be homogenous if $g_1(t) = 0$ and $g_2(t) = 0$.

The prescribed heat flux boundary conditions,

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \varphi_1(t) \qquad\qquad -K_0(L)\frac{\partial u}{\partial x}(L,t) = \varphi_2(t)$$

are linear and will again only be homogeneous if $\varphi_1(t) = 0$ and $\varphi_2(t) = 0$.

Next, the boundary conditions from Newton's law of cooling,

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H\left[u(0,t) - g_1(t)\right] \qquad -K_0(L)\frac{\partial u}{\partial x}(L,t) = H\left[u(L,t) - g_2(t)\right]$$

are again linear and will only be homogenous if $g_1(t) = 0$ and $g_2(t) = 0$.

The final set of boundary conditions that we looked at were the periodic boundary conditions,

$$u(-L,t) = u(L,t) \qquad\qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

and these are both linear and homogeneous.

The final topic in this section is not really terminology but is a restatement of a fact that we've seen several times in these notes already.

Principle of Superposition

If u_1 and u_2 are solutions to a linear homogeneous equation then so is $c_1u_1 + c_2u_2$ for any values of c_1 and c_2.

Now, as stated earlier we've seen this several times this semester but we didn't really do much with it. However this is going to be a key idea when we actually get around to solving partial differential equations. Without this fact we would not be able to solve all but the most basic of partial differential equations.

Separation of Variables

Okay, it is finally time to at least start discussing one of the more common methods for solving basic partial differential equations. The method of **Separation of Variables** cannot always be used and even when it can be used it will not always be possible to get much past the first step in the method. However, it can be used to easily solve the 1-D heat equation with no sources, the 1-D wave equation, and the 2-D version of Laplace's Equation, $\nabla^2 u = 0$.

In order to use the method of separation of variables we must be working with a linear homogenous partial differential equations with linear homogeneous boundary conditions. At this point we're not going to worry about the initial condition(s) because the solution that we initially get will rarely satisfy the initial condition(s). As we'll see however there are ways to generate a solution that will satisfy initial condition(s) provided they meet some fairly simple requirements.

The method of separation of variables relies upon the assumption that a function of the form,

$$u(x,t) = \varphi(x)G(t) \tag{13}$$

will be a solution to a linear homogeneous partial differential equation in x and t. This is called a **product solution** and provided the boundary conditions are also linear and homogeneous this will also satisfy the boundary conditions. However, as noted above this will only rarely satisfy the initial condition, but that is something for us to worry about in the next section.

Now, before we get started on some examples there is probably a question that we should ask at this point and that is : Why? Why did we choose this solution and how do we know that it will work? This seems like a very strange assumption to make. After all there really isn't any reason to believe that a solution to a partial differential equation will in fact be a product of a function of only x's and a function of only t's. This seems more like a hope than a good assumption/guess.

Unfortunately the best answer is that we chose it because it will work. As we'll see it works because it will reduce our partial differential equation down to two ordinary differential equations and provided we can solve those then we're in business and the method will allow us to get a solution to the partial differential equations.

So, let's do a couple of examples to see how this method will reduce a partial differential equation down to two ordinary differential equations.

Example 1 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

So, we have the heat equation with no sources, fixed temperature boundary conditions (that are also homogeneous) and an initial condition. The initial condition is only here because it belongs here, but we will be ignoring it until we get to the next section.

The method of separation of variables tells us to assume that the solution will take the form of the product,

$$u(x,t) = \varphi(x)G(t)$$

so all we really need to do here is plug this into the differential equation and see what we get.

$$\frac{\partial}{\partial t}\left(\varphi(x)G(t)\right)=k\frac{\partial^2}{\partial x^2}\left(\varphi(x)G(t)\right)$$
$$\varphi(x)\frac{dG}{dt}=k\,G(t)\frac{d^2\varphi}{dx^2}$$

As shown above we can factor the $\varphi(x)$ out of the time derivative and we can factor the $G(t)$ out of the spatial derivative. Also notice that after we've factored these out we no longer have a partial derivative left in the problem. In the time derivative we are now differentiating only $G(t)$ with respect to *t* and this is now an ordinary derivative. Likewise, in the spatial derivative we are now only differentiating $\varphi(x)$ with respect to *x* and so we again have an ordinary derivative.

At this point it probably doesn't seem like we've done much to simplify the problem. However, just the fact that we've gotten the partial derivatives down to ordinary derivatives is liable to be good thing even if it still looks like we've got a mess to deal with.

Speaking of that apparent (and yes I said apparent) mess, is it really the mess that it looks like? The idea is to eventually get all the *t*'s on one side of the equation and all the *x*'s on the other side. In other words we want to "separate the variables" and hence the name of the method. In this case let's notice that if we divide both sides by $\varphi(x)G(t)$ we get want we want and we should point out that it won't always be as easy as just dividing by the product solution. So, dividing out gives us,

$$\frac{1}{G}\frac{dG}{dt}=k\frac{1}{\varphi}\frac{d^2\varphi}{dx^2}\qquad\Rightarrow\qquad\frac{1}{kG}\frac{dG}{dt}=\frac{1}{\varphi}\frac{d^2\varphi}{dx^2}$$

Notice that we also divided both sides by *k*. This was done only for convenience down the road. It doesn't have to be done and nicely enough if it turns out to be a bad idea we can always come back to this step and put it back on the right side. Likewise, if we don't do it and it turns out to maybe not be such a bad thing we can always come back and divide it out. For the time being however, please accept our word that this was a good thing to do for this problem. We will discuss the reasoning for this after we're done with this example.

Now, while we said that this is what we wanted it still seems like we've got a mess. Notice however that the left side is a function of only *t* and the right side is a function only of *x* as we wanted. Also notice these two functions must be equal.

Let's think about this for a minute. How is it possible that a function of only *t*'s can be equal to a function of only *x*'s regardless of the choice of *t* and/or *x* that we have? This may seem like an impossibility until you realize that there is one way that this can be true. If both functions (*i.e.* both sides of the equation) were in fact constant and not only a constant, but the same constant then they can in fact be equal.

So, we must have,

$$\frac{1}{kG}\frac{dG}{dt}=\frac{1}{\varphi}\frac{d^2\varphi}{dx^2}=-\lambda$$

where the $-\lambda$ is called the **separation constant** and is arbitrary.

The next question that we should now address is why the minus sign? Again, much like the dividing out the *k* above, the answer is because it will be convenient down the road to have chosen this. The minus sign doesn't have to be there and in fact there are times when we don't want it there.

So how do we know it should be there or not? The answer to that is to proceed to the next step in the process (which we'll see in the next section) and at that point we'll know if would be convenient to have it or not and we can come back to this step and add it in or take it our depending what we chose to do here.

Okay, let's proceed with the process. The next step is to acknowledge that we can take the equation above and split it into the following two ordinary differential equations.

$$\frac{dG}{dt}=-k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2}=-\lambda\varphi$$

Both of these are very simple differential equations, however because we don't know what λ is we actually can't solve the spatial one yet. The time equation however could be solved at this point if we wanted to, although that won't always be the case. At this point we don't want to actually think about solving either of these yet however.

The last step in the process that we'll be doing in this section is to also make sure that our product solution, $u(x,t)=\varphi(x)G(t)$, satisfies the boundary conditions so let's plug it into both of those.

$$u(0,t)=\varphi(0)G(t)=0 \qquad\qquad u(L,t)=\varphi(L)G(t)=0$$

Let's consider the first one for a second. We have two options here. Either $\varphi(0)=0$ or $G(t)=0$ for every *t*. However, if we have $G(t)=0$ for every *t* then we'll also have $u(x,t)=0$, *i.e.* the trivial solution, and as we discussed in the previous section this is definitely a solution to any linear homogeneous equation we would really like a non-trivial solution.

Therefore we will assume that in fact we must have $\varphi(0)=0$. Likewise, from the second boundary condition we will get $\varphi(L)=0$ to avoid the trivial solution. Note as well that we were only able to reduce the boundary conditions down like this because they were homogeneous. Had they not been homogeneous we could not have done this.

So, after applying separation of variables to the given partial differential equation we arrive at a 1st order differential equation that we'll need to solve for $G(t)$ and a 2nd order boundary value problem that we'll need to solve for $\varphi(x)$. The point of this section however is just to get to this point and we'll hold off solving these until the next section.

Let's summarize everything up that we've determined here.

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$
$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

and note that we don't have a condition for the time differential equation and is not a problem. Also note that we rewrote the second one a little.

Okay, so just what have we learned here? By using separation of variables we were able to reduce our linear homogeneous partial differential equation with linear homogeneous boundary conditions down to an ordinary differential equation for one of the functions in our product solution (1), $G(t)$ in this case, and a boundary value problem that we can solve for the other function, $\varphi(x)$ in this case.

Note as well that the boundary value problem is in fact an eigenvalue/eigenfunction problem. When we solve the boundary value problem we will be identifying the eigenvalues, λ, that will generate non-trivial solutions to their corresponding eigenfunctions. Again, we'll look into this more in the next section. At this point all we want to do is identify the two ordinary differential equations that we need to solve to get a solution.

Before we do a couple of other examples we should take a second to address the fact that we made two very arbitrary seeming decisions in the above work. We divided both sides of the equation by *k* at one point and chose to use $-\lambda$ instead of λ as the separation constant.

Both of these decisions were made to simplify the solution to the boundary value problem we got from our work. The addition of the *k* in the boundary value problem would just have complicated the solution process with another letter we'd have to keep track of so we moved it into the time problem where it won't cause as many problems in the solution process. Likewise, we chose $-\lambda$ because we've already solved that particular boundary value problem (albeit with a specific *L*, but the work will be nearly identical) when we first looked at finding eigenvalues and eigenfunctions. This by the way was the reason we rewrote the boundary value problem to make it a little clearer that we have in fact solved this one already.

We can now at least partially answer the question of how do we know to make these decisions. We wait until we get the ordinary differential equations and then look at them and decide of moving things like the *k* or which separation constant to use based on how it will affect the solution of the ordinary differential equations. There is also, of course, a fair amount of experience that comes into play at this stage. The more experience you have in solving these the easier it often is to make these decisions.

Again, we need to make clear here that we're not going to go any farther in this section than getting things down to the two ordinary differential equations. Of course we will need to solve them in order to get a solution to the partial differential equation but that is the topic of the remaining sections in this chapter. All we'll say about it here is that we will need to first solve the boundary value problem, which will tell us what λ must be and then we can solve the other differential equation. Once that is done we can then turn our attention to the initial condition.

Okay, we need to work a couple of other examples and these will go a lot quicker because we won't need to put in all the explanations. After the first example this process always seems like a very long process but it really isn't. It just looked that way because of all the explanation that we had to put into it.

So, let's start off with a couple of more examples with the heat equation using different boundary conditions.

Example 2 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

Solution

In this case we're looking at the heat equation with no sources and perfectly insulated boundaries.

So, we'll start off by again assuming that our product solution will have the form,

$$u(x,t) = \varphi(x)G(t)$$

and because the differential equation itself hasn't changed here we will get the same result from plugging this in as we did in the previous example so the two ordinary differential equations that we'll need to solve are,

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

Now, the point of this example was really to deal with the boundary conditions so let's plug the product solution into them to get,

$$\frac{\partial\big(G(t)\varphi(x)\big)}{\partial x}(0,t) = 0 \qquad \frac{\partial\big(G(t)\varphi(x)\big)}{\partial x}(L,t) = 0$$

$$G(t)\frac{d\varphi}{dx}(0) = 0 \qquad G(t)\frac{d\varphi}{dx}(L) = 0$$

Now, just as with the first example if we want to avoid the trivial solution and so we can't have $G(t) = 0$ for every t and so we must have,

$$\frac{d\varphi}{dx}(0) = 0 \qquad \frac{d\varphi}{dx}(L) = 0$$

Here is a summary of what we get by applying separation of variables to this problem.

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\frac{d\varphi}{dx}(0) = 0 \qquad \frac{d\varphi}{dx}(L) = 0$$

Next, let's see what we get if use periodic boundary conditions with the heat equation.

Example 3 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Solution

First note that these boundary conditions really are homogeneous boundary conditions. If we rewrite them as,

$$u(-L,t) - u(L,t) = 0 \qquad \frac{\partial u}{\partial x}(-L,t) - \frac{\partial u}{\partial x}(L,t) = 0$$

it's a little easier to see.

Now, again we've done this partial differential equation so we'll start off with,

$$u(x,t) = \varphi(x)G(t)$$

and the two ordinary differential equations that we'll need to solve are,

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

Plugging the product solution into the rewritten boundary conditions gives,

$$G(t)\varphi(-L) - G(t)\varphi(L) = G(t)\left[\varphi(-L) - \varphi(L)\right] = 0$$

$$G(t)\frac{d\varphi}{dx}(-L) - G(t)\frac{d\varphi}{dx}(L) = G(t)\left[\frac{d\varphi}{dx}(-L) - \frac{d\varphi}{dx}(L)\right] = 0$$

and we can see that we'll only get non-trivial solution if,

$$\varphi(-L) - \varphi(L) = 0 \qquad \frac{d\varphi}{dx}(-L) - \frac{d\varphi}{dx}(L) = 0$$

$$\varphi(-L) = \varphi(L) \qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L)$$

So, here is what we get by applying separation of variables to this problem.

$$\frac{dG}{dt} = -k\lambda G \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(-L) = \varphi(L) \qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L)$$

Let's now take a look at what we get by applying separation of variables to the wave equation with fixed boundaries.

Example 4 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

Now, as with the heat equation the two initial conditions are here only because they need to be here for the problem. We will not actually be doing anything with them here and as mentioned previously the product solution will rarely satisfy them. We will be dealing with those in a later section when we actually go past this first step. Again, the point of this example is only to get down to the two ordinary differential equations that separation of variables gives.

So, let's get going on that and plug the product solution, $u(x,t) = \varphi(x)h(t)$ (we switched the *G* to an *h* here to avoid confusion with the *g* in the second initial condition) into the wave equation to get,

$$\frac{\partial^2}{\partial t^2}\big(\varphi(x)h(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(\varphi(x)h(t)\big)$$

$$\varphi(x)\frac{d^2 h}{dt^2} = c^2\, h(t)\frac{d^2\varphi}{dx^2}$$

$$\frac{1}{c^2\, h}\frac{d^2 h}{dt^2} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2}$$

Note that we moved the c^2 to the right side for the same reason we moved the *k* in the heat equation. It will make solving the boundary value problem a little easier.

Now that we've gotten the equation separated into a function of only *t* on the left and a function of only *x* on the right we can introduce a separation constant and again we'll use $-\lambda$ so we can arrive at a boundary value problem that we are familiar with. So, after introducing the separation constant we get,

$$\frac{1}{c^2\, h}\frac{d^2 h}{dt^2} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

The two ordinary differential equations we get are then,

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

The boundary conditions in this example are identical to those from the first example and so plugging the product solution into the boundary conditions gives,

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

Applying separation of variables to this problem gives,

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \qquad\qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$
$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

Next, let's take a look at the 2-D Laplace's Equation.

Example 5 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \le x \le L \qquad 0 \le y \le H$$
$$u(0,y) = g(y) \qquad u(L,y) = 0$$
$$u(x,0) = 0 \qquad u(x,H) = 0$$

Solution

This problem is a little (well actually quite a bit in some ways) different from the heat and wave equations. First, we no longer really have a time variable in the equation but instead we usually consider both variables to be spatial variables and we'll be assuming that the two variables are in the ranges shown above in the problems statement. Note that this also means that we no longer have initial conditions, but instead we now have two sets of boundary conditions, one for x and one for y.

Also, we should point out that we have three of the boundary conditions homogeneous and one nonhomogeneous for a reason. When we get around to actually solving this Laplace's Equation we'll see that this is in fact required in order for us to find a solution.

For this problem we'll use the product solution,

$$u(x,y) = h(x)\varphi(y)$$

It will often be convenient to have the boundary conditions in hand that this product solution gives before we take care of the differential equation. In this case we have three homogeneous boundary conditions and so we'll need to convert all of them. Because we've already converted these kind of boundary conditions we'll leave it to you to verify that these will become,

$$h(L) = 0 \qquad \varphi(0) = 0 \qquad \varphi(H) = 0$$

Plugging this into the differential equation and separating gives,

$$\frac{\partial^2}{\partial x^2}\left(h(x)\varphi(y)\right) + \frac{\partial^2}{\partial y^2}\left(h(x)\varphi(y)\right) = 0$$
$$\varphi(y)\frac{d^2h}{dx^2} + h(x)\frac{d^2\varphi}{dy^2} = 0$$
$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\varphi}\frac{d^2\varphi}{dy^2}$$

Okay, now we need to decide upon a separation constant. Note that every time we've chosen the separation constant we did so to make sure that the differential equation

$$\frac{d^2\varphi}{dy^2} + \lambda\varphi = 0$$

would show up. Of course, the letters might need to be different depending on how we defined our product solution (as they'll need to be here). We know how to solve this eigenvalue/eigenfunction problem as we pointed out in the discussion after the first example. However, in order to solve it we need two boundary conditions.

So, for our problem here we can see that we've got two boundary conditions for $\varphi\left(y\right)$ but only one for $h\left(x\right)$ and so we can see that the boundary value problem that we'll have to solve will involve $\varphi\left(y\right)$ and so we need to pick a separation constant that will give use the boundary value problem we've already solved. In this case that means that we need to choose λ for the separation constant. If you're not sure you believe that yet hold on for a second and you'll soon see that it was in fact the correct choice here.

Putting the separation constant gives,

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\varphi}\frac{d^2\varphi}{dy^2} = \lambda$$

The two ordinary differential equations we get from Laplace's Equation are then,

$$\frac{d^2h}{dx^2} = \lambda h \qquad\qquad -\frac{d^2\varphi}{dy^2} = \lambda\varphi$$

and notice that if we rewrite these a little we get,

$$\frac{d^2h}{dx^2} - \lambda h = 0 \qquad\qquad \frac{d^2\varphi}{dy^2} + \lambda\varphi = 0$$

We can now see that the second one does now look like one we've already solved (with a small change in letters of course, but that really doesn't change things).

So, let's summarize up here.

$$\begin{aligned} &\frac{d^2h}{dx^2} - \lambda h = 0 & &\frac{d^2\varphi}{dy^2} + \lambda\varphi = 0 \\ &h\left(L\right) = 0 & &\varphi\left(0\right) = 0 \qquad \varphi\left(H\right) = 0 \end{aligned}$$

So, we've finally seen an example where the constant of separation didn't have a minus sign and again note that we chose it so that the boundary value problem we need to solve will match one we've already seen how to solve so there won't be much work to there.

All the examples worked in this section to this point are all problems that we'll continue in later sections to get full solutions for. Let's work one more however to illustrate a couple of other ideas. We will not however be doing any work with this in later sections however, it is only here to illustrate a couple of points.

Example 6 Use Separation of Variables on the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - u$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad -\frac{\partial u}{\partial x}(L,t) = u(L,t)$$

Solution

Note that this is a heat equation with the source term of $Q(x,t) = -c\rho u$ and is both linear and homogenous. Also note that for the first time we've mixed boundary condition types. At $x = 0$ we've got a prescribed temperature and at $x = L$ we've got a Newton's law of cooling type boundary condition. We should not come away from the first few examples with the idea that the boundary conditions at both boundaries always the same type. Having them the same type just makes the boundary value problem a little easier to solve in many cases.

So we'll start off with,

$$u(x,t) = \varphi(x)G(t)$$

and plugging this into the partial differential equation gives,

$$\frac{\partial}{\partial t}\left(\varphi(x)G(t)\right) = k\frac{\partial^2}{\partial x^2}\left(\varphi(x)G(t)\right) - \varphi(x)G(t)$$

$$\varphi(x)\frac{dG}{dt} = k\,G(t)\frac{d^2\varphi}{dx^2} - \varphi(x)G(t)$$

Now, the next step is to divide by $\varphi(x)G(t)$ and notice that upon doing that the second term on the right will become a one and so can go on either side. Theoretically there is no reason that the one can't be on either side, however from a practical standpoint we again want to keep things a simple as possible so we'll move it to the *t* side as this will guarantee that we'll get a differential equation for the boundary value problem that we've seen before.

So, separating and introducing a separation constant gives,

$$\frac{1}{k}\left(\frac{1}{G}\frac{dG}{dt} + 1\right) = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

The two ordinary differential equations that we get are then (with some rewriting),

$$\frac{dG}{dt} = -(\lambda k + 1)G \qquad \frac{d^2\varphi}{dx^2} = -\lambda\varphi$$

Now let's deal with the boundary conditions.

$$G(t)\varphi(0) = 0$$

$$G(t)\frac{d\varphi}{dx}(L) + G(t)\varphi(L) = G(t)\left[\frac{d\varphi}{dx}(L) + \varphi(L)\right] = 0$$

and we can see that we'll only get non-trivial solution if,

$$\varphi(0) = 0 \qquad \frac{d\varphi}{dx}(L) + \varphi(L) = 0$$

So, here is what we get by applying separation of variables to this problem.

$$\frac{dG}{dt} = -\left(\lambda k + 1\right)G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi\left(0\right) = 0 \qquad \frac{d\varphi}{dx}\left(L\right) + \varphi\left(L\right) = 0$$

On a quick side note we solved the boundary value problem in this example in Example 5 of the Eigenvalues and Eigenfunctions section and that example illustrates why separation of variables is not always so easy to use. As we'll see in the next section to get a solution that will satisfy any sufficiently nice initial condition we really need to get our hands on all the eigenvalues for the boundary value problem. However, as the solution to this boundary value problem shows this is not always possible to do. There are ways (which we won't be going into here) to use the information here to at least get approximations to the solution but we won't ever be able to get a complete solution to this problem.

Okay, that's it for this section. It is important to remember at this point that what we've done here is really only the first step in the separation of variables method for solving partial differential equations. In the upcoming sections we'll be looking at what we need to do to finish out the solution process and in those sections we'll finish the solution to the partial differential equations we started in Example 1 – Example 5 above.

Also, in the Laplace's Equation section the last two examples show pretty much the whole separation of variable process from defining the product solution to getting an actual solution. The only step that's missing from those two examples is the solving of a boundary value problem that will have been already solved at that point and so was not put into the solution given that they tend to be fairly lengthy to solve.

We'll also see a worked example (without the boundary value problem work again) in the Vibrating String section.

Solving the Heat Equation

Okay, it is finally time to completely solve a partial differential equation. In the previous section we applied separation of variables to several partial differential equations and reduced the problem down to needing to solve two ordinary differential equations. In this section we will now solve those ordinary differential equations and use the results to get a solution to the partial differential equation. We will be concentrating on the heat equation in this section and will do the wave equation and Laplace's equation in later sections.

The first problem that we're going to look at will be the temperature distribution in a bar with zero temperature boundaries. We are going to do the work in a couple of steps so we can take our time and see how everything works.

The first thing that we need to do is find a solution that will satisfy the partial differential equation and the boundary conditions. At this point we will not worry about the initial condition. The solution we'll get first will not satisfy the vast majority of initial conditions but as we'll see it can be used to find a solution that will satisfy a sufficiently nice initial condition.

Example 1 Find a solution to the following partial differential equation that will also satisfy the boundary conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

Okay the first thing we technically need to do here is apply separation of variables. Even though we did that in the previous section let's recap here what we did.

First, we assume that the solution will take the form,

$$u(x,t) = \varphi(x)G(t)$$

and we plug this into the partial differential equation and boundary conditions. We separate the equation to get a function of only *t* on one side and a function of only *x* on the other side and then introduce a separation constant. This leaves us with two ordinary differential equations.

We did all of this in Example 1 of the previous section and the two ordinary differential equations are,

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

The time dependent equation can really be solved at any time, but since we don't know what λ is yet let's hold off on that one. Also note that in many problems only the boundary value problem can be solved at this point so don't always expect to be able to solve either one at this point.

The spatial equation is a boundary value problem and we know from our work in the previous chapter that it will only have non-trivial solutions (which we want) for certain values of λ, which we'll recall are called eigenvalues. Once we have those we can determine the non-trivial solutions for each λ, *i.e.* eigenfunctions.

Now, we actually solved the spatial problem,

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$
$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

in Example 1 of the Eigenvalues and Eigenfunctions section of the previous chapter for $L = 2\pi$. So, because we've solved this once for a specific *L* and the work is not all that much different for a general *L* we're not going to be putting in a lot of explanation here and if you need a reminder on how something works or why we did something go back to Example 1 from the Eigenvalues and Eigenfunctions section for a reminder.

We've got three cases to deal with so let's get going.

$\underline{\lambda > 0}$

In this case we know the solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

Now applying the second boundary condition, and using the above result of course, gives,

$$0 = \varphi(L) = c_2 \sin\left(L\sqrt{\lambda}\right)$$

Now, we are after non-trivial solutions and so this means we must have,

$$\sin\left(L\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad L\sqrt{\lambda} = n\pi \quad n = 1, 2, 3, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Note that we don't need the c_2 in the eigenfunction as it will just get absorbed into another constant that we'll be picking up later on.

$\underline{\lambda = 0}$

The solution to the differential equation in this case is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the boundary conditions gives,

$$0 = \varphi(0) = c_1 \qquad 0 = \varphi(L) = c_2 L \qquad \Rightarrow \qquad c_2 = 0$$

So, in this case the only solution is the trivial solution and so $\lambda = 0$ is not an eigenvalue for this boundary value problem.

$\underline{\lambda < 0}$

Here the solution to the differential equation is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \varphi(0) = c_1$$

and applying the second gives,

$$0 = \varphi(L) = c_2 \sinh\left(L\sqrt{-\lambda}\right)$$

So, we are assuming $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$ and this means $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. We therefore we must have $c_2 = 0$ and so we can only get the trivial solution in this case.

Therefore, there will be no negative eigenvalues for this boundary value problem.

The complete list of eigenvalues and eigenfunctions for this problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

Now let's solve the time differential equation,

$$\frac{dG}{dt} = -k\lambda_n G$$

and note that even though we now know λ we're not going to plug it in quite yet to keep the mess to a minimum. We will however now use λ_n to remind us that we actually have an infinite number of possible values here.

This is a simple linear (and separable for that matter) 1st order differential equation and so we'll let you verify that the solution is,

$$G(t) = c\mathbf{e}^{-k\lambda_n t} = c\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Okay, now that we've gotten both of the ordinary differential equations solved we can finally write down a solution. Note however that we have in fact found infinitely many solutions since there are infinitely many solutions (*i.e.* eigenfunctions) to the spatial problem.

Our product solution are then,

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1, 2, 3, \ldots$$

We've denoted the product solution u_n to acknowledge that each value of *n* will yield a different solution. Also note that we've changed the *c* in the solution to the time problem to B_n to denote the fact that it will probably be different for each value of *n* as well and because had we kept the c_2 with the eigenfunction we'd have absorbed it into the *c* to get a single constant in our solution.

So, there we have it. The function above will satisfy the heat equation and the boundary condition of zero temperature on the ends of the bar.

The problem with this solution is that it simply will not satisfy almost every possible initial condition we could possibly want to use. That does not mean however, that there aren't at least a few that it will satisfy as the next example illustrates.

Example 2 Solve the following heat problem for the given initial conditions.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

(a) $f(x) = 6\sin\left(\frac{\pi x}{L}\right)$

(b) $f(x) = 12\sin\left(\frac{9\pi x}{L}\right) - 7\sin\left(\frac{4\pi x}{L}\right)$

Solution

(a) This is actually easier than it looks like. All we need to do is choose $n = 1$ and $B_1 = 6$ in the product solution above to get,

$$u(x,t) = 6\sin\left(\frac{\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{\pi}{L}\right)^2 t}$$

and we've got the solution we need. This is a product solution for the first example and so satisfies the partial differential equation and boundary conditions and will satisfy the initial condition since plugging in $t = 0$ will drop out the exponential.

(b) This is almost as simple as the first part. Recall from the Principle of Superposition that if we have two solutions to a linear homogeneous differential equation (which we've got here) then their sum is also a solution. So, all we need to do is choose *n* and B_n as we did in the first part to get a solution that satisfies each part of the initial condition and then add them up. Doing this gives,

$$u(x,t) = 12\sin\left(\frac{9\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{9\pi}{L}\right)^2 t} - 7\sin\left(\frac{4\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{4\pi}{L}\right)^2 t}$$

We'll leave it to you to verify that this does in fact satisfy the initial condition and the boundary conditions.

So, we've seen that our solution from the first example will satisfy at least a small number of highly specific initial conditions.

Now, let's extend the idea out that we used in the second part of the previous example a little to see how we can get a solution that will satisfy any sufficiently nice initial condition. The Principle of Superposition is, of course, not restricted to only two solutions. For instance the following is also a solution to the partial differential equation.

$$u\left(x,t\right)=\sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and notice that this solution will not only satisfy the boundary conditions but it will also satisfy the initial condition,

$$u\left(x,0\right)=\sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Let's extend this out even further and take the limit as $M \to \infty$. Doing this our solution now becomes,

$$u\left(x,t\right)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

This solution will satisfy any initial condition that can be written in the form,

$$u\left(x,0\right)=f\left(x\right)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This may still seem to be very restrictive, but the series on the right should look awful familiar to you after the previous chapter. The series on the left is exactly the Fourier sine series we looked at in that chapter. Also recall that when we can write down the Fourier sine series for any piecewise smooth function on $0 \le x \le L$.

So, provided our initial condition is piecewise smooth after applying the initial condition to our solution we can determine the B_n as if we were finding the Fourier sine series of initial condition. So we can either proceed as we did in that section and use the orthogonality of the sines to derive them or we can acknowledge that we've already done that work and know that coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L f\left(x\right)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n=1,2,3,\ldots$$

So, we finally can completely solve a partial differential equation.

Example 3 Solve the following BVP.

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$$

$$u\left(x,0\right)=20 \qquad\qquad u\left(0,t\right)=0 \qquad\qquad u\left(L,t\right)=0$$

Solution

There isn't really all that much to do here as we've done most of it in the examples and discussion above.

First, the solution is,

$$u\left(x,t\right)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

The coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L 20\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left(\frac{20L\left(1-\cos\left(n\pi\right)\right)}{n\pi}\right) = \frac{40\left(1-\left(-1\right)^n\right)}{n\pi}$$

If we plug these in we get the solution,

$$u\left(x,t\right) = \sum_{n=1}^{\infty} \frac{40\left(1-\left(-1\right)^n\right)}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

That almost seems anti-climactic. This was a very short problem. Of course some of that came about because we had a really simple constant initial condition and so the integral was very simple. However, don't forget all the work that we had to put into discussing Fourier sine series, solving boundary value problems, applying separation of variables and then putting all of that together to reach this point.

While the example itself was very simple, it was only simple because of all the work that we had to put into developing the ideas that even allowed us to do this. Because of how "simple" it will often be to actually get these solutions we're not actually going to do anymore with specific initial conditions. We will instead concentrate on simply developing the formulas that we'd be required to evaluate in order to get an actual solution.

So, having said that let's move onto the next example. In this case we're going to again look at the temperature distribution in a bar with perfectly insulated boundaries. We are also no longer going to go in steps. We will do the full solution as a single example and end up with a solution that will satisfy any piecewise smooth initial condition.

Example 4 Find a solution to the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u\left(x,0\right) = f\left(x\right) \qquad \frac{\partial u}{\partial x}\left(0,t\right) = 0 \qquad \frac{\partial u}{\partial x}\left(L,t\right) = 0$$

Solution

We applied separation of variables to this problem in Example 2 of the previous section. So, after assuming that our solution is in the form,

$$u\left(x,t\right) = \varphi\left(x\right)G\left(t\right)$$

and applying separation of variables we get the following two ordinary differential equations that we need to solve.

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\frac{d\varphi}{dx}\left(0\right) = 0 \qquad \frac{d\varphi}{dx}\left(L\right) = 0$$

We solved the boundary value problem in Example 2 of the Eigenvalues and Eigenfunctions section of the previous chapter for $L = 2\pi$ so as with the first example in this section we're not going to put a lot of explanation into the work here. If you need a reminder on how this works go back to the previous chapter and review the example we worked there. Let's get going on the three cases we've got to work for this problem.

$\underline{\lambda > 0}$

The solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \frac{d\varphi}{dx}(0) = \sqrt{\lambda}\, c_2 \qquad \Rightarrow \qquad c_2 = 0$$

The second boundary condition gives,

$$0 = \frac{d\varphi}{dx}(L) = -\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right)$$

Recall that $\lambda > 0$ and so we will only get non-trivial solutions if we require that,

$$\sin\left(L\sqrt{\lambda}\right) = 0 \qquad \Rightarrow \qquad L\sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

$\underline{\lambda = 0}$

The general solution is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$0 = \frac{d\varphi}{dx}(0) = c_2$$

Using this the general solution is then,

$$\varphi(x) = c_1$$

and note that this will trivially satisfy the second boundary condition. Therefore $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$\varphi(x) = 1$$

$\underline{\lambda < 0}$

The general solution here is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition gives,

$$0 = \frac{d\varphi}{dx}(0) = \sqrt{-\lambda}\, c_2 \qquad \Rightarrow \qquad c_2 = 0$$

The second boundary condition gives,

$$0 = \frac{d\varphi}{dx}(L) = \sqrt{-\lambda}\, c_1 \sinh\left(L\sqrt{-\lambda}\right)$$

We know that $L\sqrt{-\lambda} \neq 0$ and so $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. Therefore we must have $c_1 = 0$ and so, this boundary value problem will have no negative eigenvalues.

So, the complete list of eigenvalues and eigenfunctions for this problem is then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$
$$\lambda_0 = 0 \qquad \varphi_0(x) = 1$$

and notice that we get the $\lambda_0 = 0$ eigenvalue and its eigenfunction if we allow $n = 0$ in the first set and so we'll use the following as our set of eigenvalues and eigenfunctions.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, 3, \ldots$$

The time problem here is identical to the first problem we looked at so,

$$G(t) = c\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Our product solutions will then be,

$$u_n(x,t) = A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 0, 1, 2, 3, \ldots$$

and the solution to this partial differential equation is,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

If we apply the initial condition to this we get,

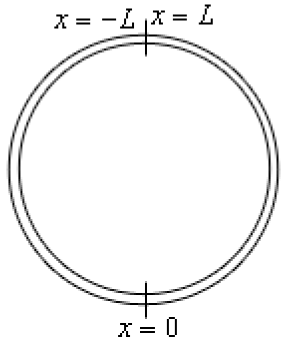
$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

and we can see that this is nothing more than the Fourier cosine series for $f(x)$ on $0 \le x \le L$ and so again we could use the orthogonality of the cosines to derive the coefficients or we could recall that we've already done that in the previous chapter and know that the coefficients are given by,

$$A_n = \begin{cases} \frac{1}{L}\int_0^L f(x)\,dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx & n \neq 0 \end{cases}$$

The last example that we're going to work in this section is a little different from the first two. We are going to consider the temperature distribution in a thin circular ring. We will consider the lateral surfaces to be perfectly insulated and we are also going to assume that the ring is thin enough so that the temperature does not vary with distance from the center of the ring.

So, what does that leave us with? Let's set $x = 0$ as shown below and then let *x* be the arc length of the ring as measured from this point.



We will measure *x* as positive if we move to the right and negative if we move to the left of $x = 0$. This means that at the top of the ring we'll meet where $x = L$ (if we move to the right) and $x = -L$ (if we move to the left). By doing this we can consider this ring to be a bar of length 2*L* and the heat equation that we developed earlier in this chapter will still hold.

At the point of the ring we consider the two "ends" to be in **perfect thermal contact**. This means that at the two ends both the temperature and the heat flux must be equal. In other words we must have,

$$u(-L,t) = u(L,t) \qquad\qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

If you recall from the section in which we derived the heat equation we called these periodic boundary conditions. So, the problem we need to solve to get the temperature distribution in this case is,

Example 5 Find a solution to the following partial differential equation.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad u(-L,t) = u(L,t) \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$$

Solution

We applied separation of variables to this problem in Example 3 of the previous section. So, if we assume the solution is in the form,

$$u(x,t) = \varphi(x)G(t)$$

we get the following two ordinary differential equations that we need to solve.

$$\frac{dG}{dt} = -k\lambda G \qquad\qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0$$

$$\varphi(-L) = \varphi(L) \qquad\qquad \frac{d\varphi}{dx}(-L) = \frac{d\varphi}{dx}(L)$$

As we've seen with the previous two problems we've already solved a boundary value problem like this one back in the Eigenvalues and Eigenfunctions section of the previous chapter, Example 3 to be exact with $L = \pi$. So, if you need a little more explanation of what's going on here go back to this example and you can see a little more explanation.

We again have three cases to deal with here.

$\lambda > 0$

The general solution to the differential equation is,

$$\varphi(x) = c_1 \cos\left(\sqrt{\lambda}\, x\right) + c_2 \sin\left(\sqrt{\lambda}\, x\right)$$

Applying the first boundary condition and recalling that cosine is an even function and sine is an odd function gives us,

$$\begin{aligned} c_1 \cos\left(-L\sqrt{\lambda}\right) + c_2 \sin\left(-L\sqrt{\lambda}\right) &= c_1 \cos\left(L\sqrt{\lambda}\right) + c_2 \sin\left(L\sqrt{\lambda}\right) \\ -c_2 \sin\left(L\sqrt{\lambda}\right) &= c_2 \sin\left(L\sqrt{\lambda}\right) \\ 0 &= 2c_2 \sin\left(L\sqrt{\lambda}\right) \end{aligned}$$

At this stage we can't really say anything as either c_2 or sine could be zero. So, let's apply the second boundary condition and see what we get.

$$\begin{aligned} -\sqrt{\lambda}\, c_1 \sin\left(-L\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(-L\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) + \sqrt{\lambda}\, c_2 \cos\left(L\sqrt{\lambda}\right) \\ \sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) &= -\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) \\ 2\sqrt{\lambda}\, c_1 \sin\left(L\sqrt{\lambda}\right) &= 0 \end{aligned}$$

We get something similar. However notice that if $\sin\left(L\sqrt{\lambda}\right) \neq 0$ then we would be forced to have $c_1 = c_2 = 0$ and this would give us the trivial solution which we don't want.

This means therefore that we must have $\sin\left(L\sqrt{\lambda}\right) = 0$ which in turn means (from work in our previous examples) that the positive eigenvalues for this problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad\qquad n = 1, 2, 3, \ldots$$

Now, there is no reason to believe that $c_1 = 0$ or $c_2 = 0$. All we know is that they both can't be zero and so that means that we in fact have two sets of eigenfunctions for this problem corresponding to positive eigenvalues. They are,

$$\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad\qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

$\underline{\lambda = 0}$

The general solution in this case is,

$$\varphi(x) = c_1 + c_2 x$$

Applying the first boundary condition gives,

$$\begin{aligned} c_1 + c_2(-L) &= c_1 + c_2(L) \\ 2Lc_2 &= 0 \qquad\qquad \Rightarrow \qquad\qquad c_2 = 0 \end{aligned}$$

The general solution is then,

$$\varphi(x) = c_1$$

and this will trivially satisfy the second boundary condition. Therefore $\lambda = 0$ is an eigenvalue for this BVP and the eigenfunctions corresponding to this eigenvalue is,

$$\varphi(x) = 1$$

$\underline{\lambda < 0}$

For this final case the general solution here is,

$$\varphi(x) = c_1 \cosh\left(\sqrt{-\lambda}\, x\right) + c_2 \sinh\left(\sqrt{-\lambda}\, x\right)$$

Applying the first boundary condition and using the fact that hyperbolic cosine is even and hyperbolic sine is odd gives,

$$\begin{aligned} c_1 \cosh\left(-L\sqrt{-\lambda}\right) + c_2 \sinh\left(-L\sqrt{-\lambda}\right) &= c_1 \cosh\left(L\sqrt{-\lambda}\right) + c_2 \sinh\left(L\sqrt{-\lambda}\right) \\ -c_2 \sinh\left(-L\sqrt{-\lambda}\right) &= c_2 \sinh\left(L\sqrt{-\lambda}\right) \\ 2c_2 \sinh\left(L\sqrt{-\lambda}\right) &= 0 \end{aligned}$$

Now, in this case we are assuming that $\lambda < 0$ and so $L\sqrt{-\lambda} \neq 0$. This turn tells us that $\sinh\left(L\sqrt{-\lambda}\right) \neq 0$. We therefore must have $c_2 = 0$.

Let's now apply the second boundary condition to get,

$$\sqrt{-\lambda}\,c_1 \sinh\left(-L\sqrt{-\lambda}\right) = \sqrt{-\lambda}\,c_1 \sinh\left(L\sqrt{-\lambda}\right)$$
$$2\sqrt{-\lambda}\,c_1 \sinh\left(L\sqrt{-\lambda}\right) = 0 \qquad \Rightarrow \qquad c_1 = 0$$

By our assumption on λ we again have no choice here but to have $c_1 = 0$ and so for this boundary value problem there are no negative eigenvalues.

Summarizing up then we have the following sets of eigenvalues and eigenfunctions and note that we've merged the $\lambda = 0$ case into the cosine case since it can be here to simplify things up a little.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 0,1,2,3,\ldots$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1,2,3,\ldots$$

The time problem is again identical to the two we've already worked here and so we have,

$$G(t) = c\,\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Now, this example is a little different from the previous two heat problems that we've looked at. In this case we actually have two different possible product solutions that will satisfy the partial differential equation and the boundary conditions. They are,

$$u_n(x,t) = A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 0,1,2,3,\ldots$$
$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad n = 1,2,3,\ldots$$

The Principle of Superposition is still valid however and so a sum of any of these will also be a solution and so the solution to this partial differential equation is,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

If we apply the initial condition to this we get,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

and just as we saw in the previous two examples we get a Fourier series. The difference this time is that we get the full Fourier series for a piecewise smooth initial condition on $-L \le x \le L$. As noted for the previous two examples we could either rederive formulas for the coefficients using the orthogonality of the sines and cosines or we can recall the work we've already done. There's really no reason at this point to redo work already done so the coefficients are given by,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f\left(x\right)dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f\left(x\right)\cos\left(\tfrac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f\left(x\right)\sin\left(\tfrac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

Note that this is the reason for setting up *x* as we did at the start of this problem. A full Fourier series needs an interval of $-L \le x \le L$ whereas the Fourier sine and cosines series we saw in the first two problems need $0 \le x \le L$.

Okay, we've now seen three heat equation problems solved and so we'll leave this section. You might want to go through and do the two cases where we have a zero temperature on one boundary and a perfectly insulated boundary on the other to see if you've got this process down.

Heat Equation with Non-Zero Temperature Boundaries

In this section we want to expand one of the cases from the previous section a little bit. In the previous section we look at the following heat problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x) \qquad\qquad u(0,t) = 0 \qquad\qquad u(L,t) = 0$$

Now, there is nothing inherently wrong with this problem, but the fact that we're fixing the temperature on both ends at zero is a little unrealistic. The other two problems we looked at, insulated boundaries and the thin ring, are a little more realistic problems, but this one just isn't all that realistic so we'd like to extend it a little.

What we'd like to do in this section is instead look at the following problem.

$$\begin{aligned}&\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\\ &u(x,0) = f(x) \qquad\qquad u(0,t) = T_1 \qquad\qquad u(L,t) = T_2\end{aligned} \tag{14}$$

In this case we'll allow the boundaries to be any fixed temperature, T_1 or T_2. The problem here is that separation of variables will no longer work on this problem because the boundary conditions are no longer homogeneous. Recall that separation of variables will only work if both the partial differential equation and the boundary conditions are linear and homogeneous. So, we're going to need to deal with the boundary conditions in some way before we actually try and solve this.

Luckily for us there is an easy way to deal with them. Let's consider this problem a little bit. There are no sources to add/subtract heat energy anywhere in the bar. Also our boundary conditions are fixed temperatures and so can't change with time and we aren't prescribing a heat flux on the boundaries to continually add/subtract heat energy. So, what this all means is that there will not ever be any forcing of heat energy into or out of the bar and so while some heat energy may well naturally flow into our out of the bar at the end points as the temperature changes eventually the temperature distribution in the bar should stabilize out and no longer depend on time.

Or, in other words it makes some sense that we should expect that as $t \to \infty$ our temperature distribution, $u(x,t)$ should behave as,

$$\lim_{t\to\infty} u(x,t) = u_E(x)$$

where $u_E(x)$ is called the **equilibrium temperature**. Note as well that is should still satisfy the heat equation and boundary conditions. It won't satisfy the initial condition however because it is the temperature distribution as $t \to \infty$ whereas the initial condition is at $t = 0$. So, the equilibrium temperature distribution should satisfy,

$$0 = \frac{d^2 u_E}{dx^2} \qquad\qquad u_E(0) = T_1 \qquad\qquad u_E(L) = T_2 \tag{15}$$

This is a really easy 2nd order ordinary differential equation to solve. If we integrate twice we get,

$$u_E(x) = c_1 x + c_2$$

and applying the boundary conditions (we'll leave this to you to verify) gives us,

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L} x$$

Okay, just what does this have to do with solving the problem given by (1) above? We'll let's define the function,

$$v(x,t) = u(x,t) - u_E(x) \tag{16}$$

where $u(x,t)$ is the solution to (1) and $u_E(x)$ is the equilibrium temperature for (1).

Now let's rewrite this as,

$$u(x,t) = v(x,t) + u_E(x)$$

and let's take some derivatives.

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial u_E}{\partial t} = \frac{\partial v}{\partial t} \qquad\qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

In both of these derivatives we used the fact that $u_E(x)$ is the equilibrium temperature and so is independent of time t and must satisfy the differential equation in (2).

What this tells us is that both $u(x,t)$ and $v(x,t)$ must satisfy the same partial differential equation. Let's see what the initial conditions and boundary conditions would need to be for $v(x,t)$.

$$\begin{aligned} v(x,0) &= u(x,0) - u_E(x) = f(x) - u_E(x) \\ v(0,t) &= u(0,t) - u_E(0) = T_1 - T_1 = 0 \\ v(L,t) &= u(L,t) - u_E(L) = T_2 - T_2 = 0 \end{aligned}$$

So, the initial condition just gets potentially messier, but the boundary conditions are now homogeneous! The partial differential equation that $v(x,t)$ must satisfy is,

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}$$

$$v(x,0) = f(x) - u_E(x) \qquad\qquad v(0,t) = 0 \qquad\qquad v(L,t) = 0$$

We saw how to solve this in the previous section and so we the solution is,

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where the coefficients are given by,

$$B_n = \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

The solution to (1) is then,

$$\begin{aligned} u(x,t) &= u_E(x) + v(x,t) \\ &= T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\mathbf{e}^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

and the coefficients are given above.

Laplace's Equation

The next partial differential equation that we're going to solve is the 2-D Laplace's equation,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A natural question to ask before we start learning how to solve this is does this equation come up naturally anywhere? The answer is a very resounding yes! If we consider the 2-D heat equation,

$$\frac{\partial u}{\partial t} = k\nabla^2 u + \frac{Q}{cp}$$

We can see that Laplace's equation would correspond to finding the equilibrium solution (*i.e.* time independent solution) if there were not sources. So, this is an equation that can arise from physical situations.

How we solve Laplace's equation will depend upon the geometry of the 2-D object we're solving it on. Let's start out by solving it on the rectangle given by $0 \le x \le L$, $0 \le y \le H$. For this geometry Laplace's equation along with the four boundary conditions will be,

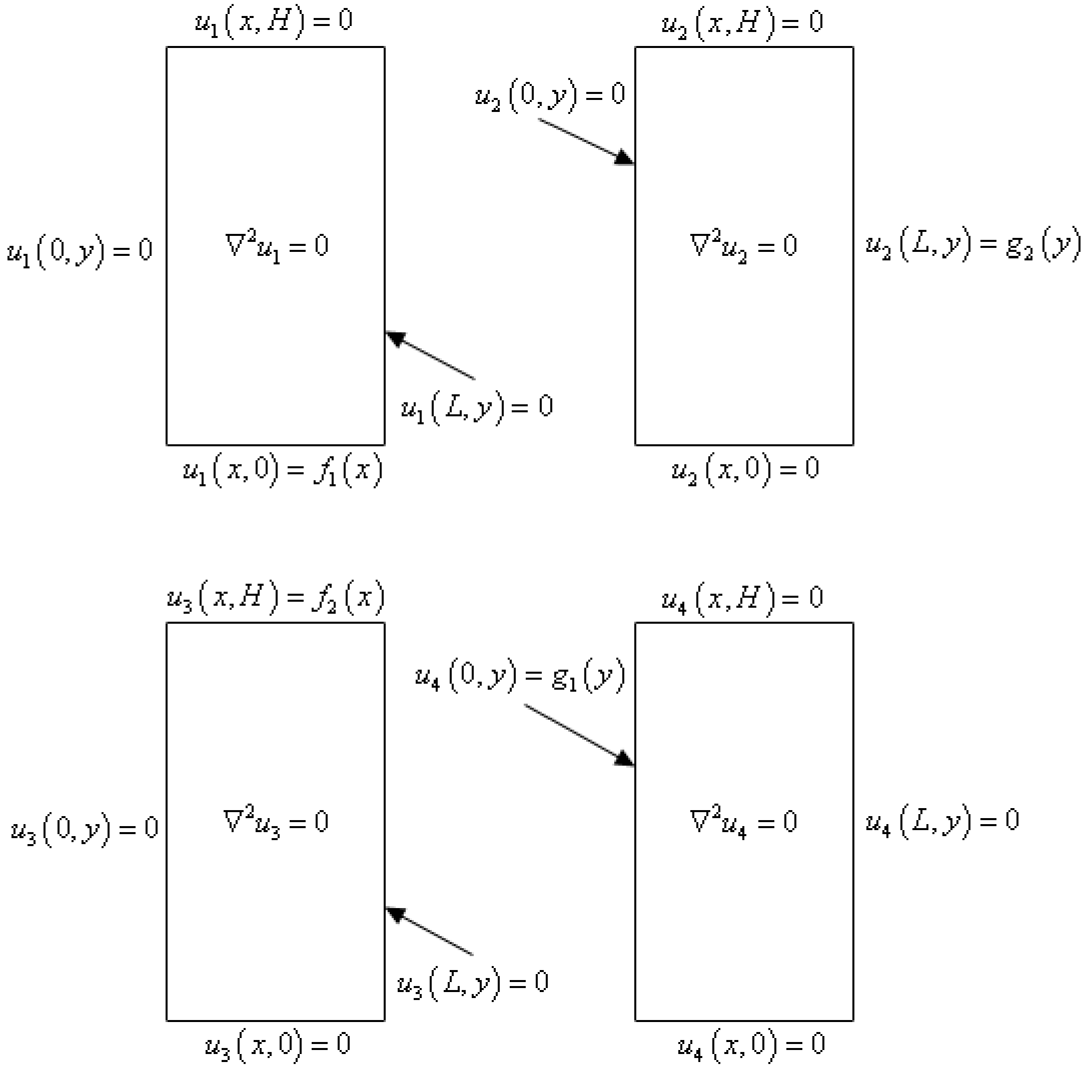
$$\begin{aligned}
&\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\
&u(0,y) = g_1(y) \qquad\qquad u(L,y) = g_2(y) \\
&u(x,0) = f_1(x) \qquad\qquad u(x,H) = f_2(x)
\end{aligned} \tag{17}$$

One of the important things to note here is that unlike the heat equation we will not have any initial conditions here. Both variables are spatial variables and each variable occurs in a 2nd order derivative and so we'll need two boundary conditions for each variable.

Next, let's notice that while the partial differential equation is both linear and homogeneous the boundary conditions are only linear and are not homogeneous. This creates a problem because separation of variables requires homogeneous boundary conditions.

To completely solve Laplace's equation we're in fact going to have to solve it four times. Each time we solve it only one of the four boundary conditions can be nonhomogeneous while the remaining three will be homogeneous.

The four problems are probably best shown with a quick sketch so let's consider the following sketch.



Now, once we solve all four of these problems the solution to our original system, (1), will be,

$$u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$$

Because we know that Laplace's equation is linear and homogeneous and each of the pieces is a solution to Laplace's equation then the sum will also be a solution. Also, this will satisfy each of the four original boundary conditions. We'll verify the first one and leave the rest to you to verify.

$$u(x,0)=u_1(x,0)+u_2(x,0)+u_3(x,0)+u_4(x,0)= f_1(x)+0+0+0= f_1(x)$$

In each of these cases the lone nonhomogeneous boundary condition will take the place of the initial condition in the heat equation problems that we solved a couple of sections ago. We will apply separation of variables to the each problem and find a product solution that will satisfy the differential equation and the three homogeneous boundary conditions. Using the Principle of Superposition we'll find a solution to the problem and then apply the final boundary condition to determine the value of the constant(s) that are left in the problem. The process is nearly identical in many ways to what we did when we were solving the heat equation.

We're going to do two of the cases here and we'll leave the remaining two for you to do.

Example 1 Find a solution to the following partial differential equation.

$$\nabla^2 u_4 = \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$$

$$u_4(0,y) = g_1(y) \qquad u_4(L,y) = 0$$

$$u_4(x,0) = 0 \qquad u_4(x,H) = 0$$

Solution

We'll start by assuming that our solution will be in the form,

$$u_4(x,y) = h(x)\varphi(y)$$

and then recall that we performed separation of variables on this problem (with a small change in notation) back in Example 5 of the Separation of Variables section. So from that problem we know that separation of variables yields the following two ordinary differential equations that we'll need to solve.

$$\frac{d^2 h}{dx^2} - \lambda h = 0 \qquad\qquad \frac{d^2\varphi}{dy^2} + \lambda\varphi = 0$$

$$h(L) = 0 \qquad\qquad \varphi(0) = 0 \qquad \varphi(H) = 0$$

Note that in this case, unlike the heat equation we must solve the boundary value problem first. Without knowing what λ is there is no way that we can solve the first differential equation here with only one boundary condition since the sign of λ will affect the solution.

Let's also notice that we solved the boundary value problem in Example 1 of Solving the Heat Equation and so there is no reason to resolve it here. Taking a change of letters into account the eigenvalues and eigenfunctions for the boundary value problem here are,

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad \varphi_n(y) = \sin\left(\frac{n\pi y}{H}\right) \qquad n = 1, 2, 3, \ldots$$

Now that we know what the eigenvalues are let's write down the first differential equation with λ plugged in.

$$\frac{d^2 h}{dx^2} - \left(\frac{n\pi}{H}\right)^2 h = 0$$

$$h(L) = 0$$

Because the coefficient of $h(x)$ in the differential equation above is positive we know that a solution to this is,

$$h(x) = c_1 \cosh\left(\frac{n\pi x}{H}\right) + c_2 \sinh\left(\frac{n\pi x}{H}\right)$$

However, this is not really suited for dealing with the $h(L) = 0$ boundary condition. So, let's also notice that the following is also a solution.

$$h(x) = c_1 \cosh\left(\frac{n\pi(x-L)}{H}\right) + c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

You should verify this by plugging this into the differential equation and checking that it is in fact a solution. Applying the lone boundary condition to this "shifted" solution gives,

$$0 = h(L) = c_1$$

The solution to the first differential equation is now,

$$h(x) = c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

and this is all the farther we can go with this because we only had a single boundary condition. That is not really a problem however because we now have enough information to form the product solution for this partial differential equation.

A product solution for this partial differential equation is,

$$u_n(x,y) = B_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right) \qquad n = 1, 2, 3, \ldots$$

The Principle of Superposition then tells us that a solution to the partial differential equation is,

$$u_4(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

and this solution will satisfy the three homogeneous boundary conditions.

To determine the constants all we need to do is apply the final boundary condition.

$$u_4(0,y) = g_1(y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

Now, in the previous problems we've done this has clearly been a Fourier series of some kind and in fact it still is. The difference here is that the coefficients of the Fourier sine series are now,

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right)$$

instead of just B_n. We might be a little more tempted to use the orthogonality of the sines to derive formulas for the B_n, however we can still reuse the work that we've done previously to get formulas for the coefficients here.

Remember that a Fourier sine series is just a series of coefficients (depending on *n*) times a sine. We still have that here, except the "coefficients" are a little messier this time that what we saw when we first dealt with Fourier series. So, the coefficients can be found using exactly the same formula from the Fourier sine series section of a function on $0 \le y \le H$ we just need to be careful with the coefficients.

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right) = \frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{2}{H\sinh\left(\frac{n\pi(-L)}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy \qquad n = 1,2,3,\ldots$$

The formulas for the B_n are a little messy this time in comparison to the other problems we've done but they aren't really all that messy.

Okay, let's do one of the other problems here so we can make a couple of points.

Example 2 Find a solution to the following partial differential equation.

$$\nabla^2 u_3 = \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0$$

$$u_3(0,y) = 0 \qquad\qquad u_3(L,y) = 0$$

$$u_3(x,0) = 0 \qquad\qquad u_3(x,H) = f_2(x)$$

Solution

Okay, for the first time we've hit a problem where we haven't previous done the separation of variables so let's go through that. We'll assume the solution is in the form,

$$u_3(x,y) = h(x)\varphi(y)$$

We'll apply this to the homogeneous boundary conditions first since we'll need those once we get reach the point of choosing the separation constant. We'll let you verify that the boundary conditions become,

$$h(0) = 0 \qquad\qquad h(L) = 0 \qquad\qquad \varphi(0) = 0$$

Next, we'll plug the product solution into the differential equation.

$$\frac{\partial^2}{\partial x^2}\left(h(x)\varphi(y)\right) + \frac{\partial^2}{\partial y^2}\left(h(x)\varphi(y)\right) = 0$$

$$\varphi(y)\frac{d^2h}{dx^2} + h(x)\frac{d^2\varphi}{dy^2} = 0$$

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\varphi}\frac{d^2\varphi}{dy^2}$$

Now, at this point we need to choose a separation constant. We've got two homogeneous boundary conditions on *h* so let's choose the constant so that the differential equation for *h* yields a familiar boundary value problem so we don't need to redo any of that work. In this case, unlike the u_4 case, we'll need $-\lambda$.

This is a good problem in that is clearly illustrates that sometimes you need λ as a separation constant and at other times you need $-\lambda$. Not only that but sometimes all it takes is a small change in the boundary conditions it force the change.

So, after adding in the separation constant we get,

$$\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\varphi}\frac{d^2\varphi}{dy^2} = -\lambda$$

and two ordinary differential equations that we get from this case (along with their boundary conditions) are,

$$\frac{d^2h}{dx^2} + \lambda h = 0 \qquad\qquad \frac{d^2\varphi}{dy^2} - \lambda\varphi = 0$$
$$h(0) = 0 \quad h(L) = 0 \qquad\qquad \varphi(0) = 0$$

Now, as we noted above when we were deciding which separation constant to work with we've already solved the first boundary value problem. So, the eigenvalues and eigenfunctions for the first boundary value problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad h_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

The second differential equation is then,

$$\frac{d^2\varphi}{dx^2} - \left(\frac{n\pi}{L}\right)^2 \varphi = 0$$
$$\varphi(0) = 0$$

Because the coefficient of the φ is positive we know that a solution to this is,

$$\varphi(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

In this case, unlike the previous example, we won't need to use a shifted version of the solution because this will work just fine with the boundary condition we've got for this. So, applying the boundary condition to this gives,

$$0 = \varphi(0) = c_1$$

and this solution becomes,

$$\varphi(y) = c_2 \sinh\left(\frac{n\pi y}{L}\right)$$

The product solution for this case is then,

$$u_n(x,y) = B_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

The solution to this partial differential equation is then,

$$u_3(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Finally, let's apply the nonhomogeneous boundary condition to get the coefficients for this solution.

$$u_3(x,H) = f_2(x) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

As we've come to expect this is again a Fourier sine (although it won't always be a sine) series and so using previously done work instead of using the orthogonality of the sines to we see that,

$$B_n \sinh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

$$B_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

Okay, we've worked two of the four cases that would need to be solved in order to completely solve (1). As we've seen each case was very similar and yet also had some differences. We saw the use of both separation constants and that sometimes we need to use a "shifted" solution in order to deal with one of the boundary conditions.

Before moving on let's note that we used prescribed temperature boundary conditions here, but we could just have easily used prescribed flux boundary conditions or a mix of the two. No matter what kind of boundary conditions we have they will work the same.

As a final example in this section let's take a look at solving Laplace's equation on a disk of radius a and a prescribed temperature on the boundary. Because we are now on a disk it makes sense that we should probably do this problem in polar coordinates and so the first thing we need to so do is write down Laplace's equation in terms of polar coordinates.

Laplace's equation in terms of polar coordinates is,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Okay, this is a lot more complicated than the Cartesian form of Laplace's equation and it will add in a few complexities to the solution process, but it isn't as bad as it looks. The main problem that we've got here really is that fact that we've got a single boundary condition. Namely,

$$u(a,\theta) = f(\theta)$$

This specifies the temperature on the boundary of the disk. We are clearly going to need three more conditions however since we've got a 2nd derivative in both r and θ.

When we solved Laplace's equation on a rectangle we used conditions at the end points of the range of each variable and so it makes some sense here that we should probably need the same kind of conditions here as well. The range on our variables here are,

$$0 \le r \le a \qquad\qquad -\pi \le \theta \le \pi$$

Note that the limits on θ are somewhat arbitrary here and are chosen for convenience here. Any set of limits that covers the complete disk will work, however as we'll see with these limits we will get another familiar boundary value problem arising. The best choice here is often not known until the separation of variables is done. At that point you can go back and make your choices.

Okay, we now need conditions for $r = 0$ and $\theta = \pm\pi$. First, note that Laplace's equation in terms of polar coordinates is singular at $r = 0$ (*i.e.* we get division by zero). However, we know from physical considerations that the temperature must remain finite everywhere in the disk and so let's impose the condition that,

$$\left|u(0,\theta)\right| < \infty$$

This may seem like an odd condition and it definitely doesn't conform to the other boundary conditions that we've seen to this point, but it will work out for us as we'll see.

Now, for boundary conditions for θ we'll do something similar to what we did for the 1-D head equation on a thin ring. The two limits on θ are really just different sides of a line in the disk and so let's use the periodic conditions there. In other words,

$$u(-\pi,t) = u(\pi,t) \qquad\qquad \frac{\partial u}{\partial r}(-\pi,t) = \frac{\partial u}{\partial r}(\pi,t)$$

With all of this out of the way let's solve Laplace's equation on a disk of radius *a*.

Example 3 Find a solution to the following partial differential equation.

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$$

$$\left|u(0,\theta)\right| < \infty \qquad\qquad u(a,\theta) = f(\theta)$$

$$u(-\pi,t) = u(\pi,t) \qquad\qquad \frac{\partial u}{\partial r}(-\pi,t) = \frac{\partial u}{\partial r}(\pi,t)$$

Solution
In this case we'll assume that the solution will be in the form,

$$u(\theta,r) = \varphi(\theta)G(r)$$

Plugging this into the periodic boundary conditions gives,

$$\varphi(-\pi) = \varphi(\pi) \qquad\qquad \frac{d\varphi}{d\theta}(-\pi) = \frac{d\varphi}{d\theta}(\pi)$$

$$\left|G(0)\right| < \infty$$

Now let's plug the product solution into the partial differential equation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\big(\varphi(\theta)G(r)\big)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\big(\varphi(\theta)G(r)\big) = 0$$

$$\varphi(\theta)\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + G(r)\frac{1}{r^2}\frac{d^2\varphi}{d\theta^2} = 0$$

This is definitely more of a mess that we've seen to this point when it comes to separating variables. In this case simply dividing by the product solution, while still necessary, will not be sufficient to separate the variables. We are also going to have to multiply by r^2 to completely separate variables. So, doing all that, moving each term to one side of the equal sign and introduction a separation constant gives,

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\varphi}\frac{d^2\varphi}{d\theta^2} = \lambda$$

We used λ as the separation constant this time to get the differential equation for φ to match up with one we've already done.

The ordinary differential equations we get are then,

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = 0 \qquad \frac{d^2\varphi}{d\theta^2} + \lambda\varphi = 0$$

$$\left|G(0)\right| < \infty \qquad \varphi(-\pi) = \varphi(\pi) \qquad \frac{d\varphi}{d\theta}(-\pi) = \frac{d\varphi}{d\theta}(\pi)$$

Now, we solved the boundary value problem above in Example 3 of the Eigenvalues and Eigenfunctions section of the previous chapter and so there is no reason to redo it here. The eigenvalues and eigenfunctions for this problem are,

$$\lambda_n = n^2 \qquad \varphi_n(\theta) = \sin(n\theta) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = n^2 \qquad \varphi_n(\theta) = \cos(n\theta) \qquad n = 0, 1, 2, 3, \ldots$$

Plugging this into the first ordinary differential equation and using the product rule on the derivative we get,

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - n^2 G = 0$$

$$r\left(r\frac{d^2G}{dr^2} + \frac{dG}{dr}\right) - n^2 G = 0$$

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0$$

This is an Euler differential equation and so we know that solutions will be in the form $G(r) = r^p$ provided p is a root of,

$$p(p-1) + p - n^2 = 0$$

$$p^2 - n^2 = 0 \qquad \Rightarrow \qquad p = \pm n \qquad n = 0, 1, 2, 3, \ldots$$

So, because the $n = 0$ case will yield a double root, versus two real distinct roots if $n \neq 0$ we have two cases here. They are,

$$G(r) = c_1 + c_2 \ln r \qquad n = 0$$

$$G(r) = \overline{c}_1 r^n + \overline{c}_2 r^{-n} \qquad n = 1, 2, 3, \ldots$$

Now we need to recall the condition that $\left|G(0)\right| < \infty$. Each of the solutions above will have $G(r) \to \infty$ as $r \to 0$ Therefore in order to meet this boundary condition we must have $c_2 = \overline{c}_2 = 0$.

Therefore, the solution reduces to,

$$G(r) = c_1 r^n \qquad\qquad n = 0, 1, 2, 3, \ldots$$

and notice that with the second term gone we can combine the two solutions into a single solution.

So, we have two product solutions for this problem. They are,

$$u_n(\theta, r) = A_n\, r^n \cos(n\theta) \qquad\qquad n = 0, 1, 2, 3, \ldots$$
$$u_n(\theta, r) = B_n\, r^n \sin(n\theta) \qquad\qquad n = 1, 2, 3, \ldots$$

Our solution is then the sum of all these solutions or,

$$u(\theta, r) = \sum_{n=0}^{\infty} A_n\, r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n\, r^n \sin(n\theta)$$

Applying our final boundary condition to this gives,

$$u(a, \theta) = f(\theta) = \sum_{n=0}^{\infty} A_n\, a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n\, a^n \sin(n\theta)$$

This is a full Fourier series for $f(\theta)$ on the interval $-\pi \le \theta \le \pi$, *i.e.* $L = \pi$. Also note that once again the "coefficients" of the Fourier series are a little messier than normal, but not quite as messy as when we were working on a rectangle above. We could once again use the orthogonality of the sines and cosines to derive formulas for the A_n and B_n or we could just use the formulas from the Fourier series section with $L = \pi$ to get,

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \qquad\qquad n = 1, 2, 3, \ldots$$
$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \qquad\qquad n = 1, 2, 3, \ldots$$

Upon solving for the coefficients we get,

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$
$$A_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta \qquad\qquad n = 1, 2, 3, \ldots$$
$$B_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta \qquad\qquad n = 1, 2, 3, \ldots$$

Prior to this example most of the separation of variable problems tended to look very similar and it is easy to fall in to the trap of expecting everything to look like what we'd seen earlier. With this example we can see that the problems can definitely be different on occasion so don't get too locked into expecting them to always work in exactly the same way.

Before we leave this section let's briefly talk about what you'd need to do on a partial disk. The periodic boundary conditions above were only there because we had a whole disk. What if we only had a disk between say $\alpha \le \theta \le \beta$.

When we've got a partial disk we now have two new boundaries that we not present in the whole disk and the periodic boundary conditions will no longer make sense. The periodic boundary conditions are only used when we have the two "boundaries" in contact with each other and that clearly won't be the case with a partial disk.

So, if we stick with prescribed temperature boundary conditions we would then have the following conditions

$$\begin{aligned} &\left|u(0,\theta)\right| < \infty && \\ &u(a,\theta) = f(\theta) && \alpha \le \theta \le \beta \\ &u(r,\alpha) = g_1(r) && 0 \le r \le a \\ &u(r,\beta) = g_2(r) && 0 \le r \le a \end{aligned}$$

Also note that in order to use separation of variables on these conditions we'd need to have $g_1(r) = g_2(r) = 0$ to make sure they are homogeneous.

As a final note we could just have easily used flux boundary conditions for the last two if we'd wanted to. The boundary value problem would be different, but outside of that the problem would work in the same manner.

We could also use a flux condition on the $r = a$ boundary but we haven't really talked yet about how to apply that kind of condition to our solution. Recall that this is the condition that we apply to our solution to determine the coefficients. It's not difficult to use we just haven't talked about this kind of condition yet. We'll be doing that in the next section.

Vibrating String

This will be the final partial differential equation that we'll be solving in this chapter. In this section we'll be solving the 1-D wave equation to determine the displacement of a vibrating string. There really isn't much in the way of introduction to do here so let's just jump straight into the example.

Example 1 Find a solution to the following partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(0,t) = 0 \qquad u(L,t) = 0$$

Solution

One of the main differences here that we're going to have to deal with is the fact that we've now got two initial conditions. That is not something we've seen to this point, but will not be all that difficult to deal with when the time rolls around.

We've already done the separation of variables for this problem, but let's go ahead and redo it here so we can say we've got another problem almost completely worked out.

So, let's start off with the product solution.

$$u(x,t) = \varphi(x)h(t)$$

Plugging this into the two boundary conditions gives,

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

Plugging the product solution into the differential equation, separating and introducing a separation constant gives,

$$\frac{\partial^2}{\partial t^2}\big(\varphi(x)h(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(\varphi(x)h(t)\big)$$

$$\varphi(x)\frac{d^2 h}{dt^2} = c^2 h(t)\frac{d^2\varphi}{dx^2}$$

$$\frac{1}{c^2 h}\frac{d^2 h}{dt^2} = \frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\lambda$$

We moved the c^2 to the left side for convenience and chose $-\lambda$ for the separation constant so the differential equation for φ would match a known (and solved) case.

The two ordinary differential equations we get from separation of variables are then,

$$\frac{d^2 h}{dt^2} + c^2\lambda h = 0 \qquad \frac{d^2\varphi}{dx^2} + \lambda\varphi$$

$$\varphi(0) = 0 \qquad \varphi(L) = 0$$

We solved the boundary value problem above in Example 1 of the Solving the Heat Equation section of this chapter and so the eigenvalues and eigenfunctions for this problem are,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

The first ordinary differential equation is now,

$$\frac{d^2h}{dt^2} + \left(\frac{n\pi c}{L}\right)^2 h = 0$$

and because the coefficient of the h is clearly positive the solution to this is,

$$h(t) = c_1 \cos\left(\frac{n\pi c\,t}{L}\right) + c_2 \sin\left(\frac{n\pi c\,t}{L}\right)$$

Because there is no reason to think that either of the coefficients above are zero we then get two product solutions,

$$\begin{aligned} u_n(x,t) &= A_n \cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\ u_n(x,t) &= B_n \sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \end{aligned} \qquad n = 1, 2, 3, \ldots$$

The solution is then,

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

Now, in order to apply the second initial condition we'll need to differentiate this with respect to t so,

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L} A_n \sin\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \frac{n\pi c}{L} B_n \cos\left(\frac{n\pi c\,t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]$$

If we now apply the initial conditions we get,

$$u(x,0) = f(x) = \sum_{n=1}^{\infty}\left[A_n \cos(0)\sin\left(\frac{n\pi x}{L}\right) + B_n \sin(0)\sin\left(\frac{n\pi x}{L}\right)\right] = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Both of these are Fourier sine series. The first is for $f(x)$ on $0 \le x \le L$ while the second is for $g(x)$ on $0 \le x \le L$ with a slightly messy coefficient. As in the last few sections we're faced with the choice of either using the orthogonality of the sines to derive formulas for A_n and B_n or we could reuse formula from previous work.

It's easier to reuse formulas so using the formulas form the Fourier sine series section we get,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

$$\frac{n\pi c}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

Upon solving the second one we get,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1, 2, 3, \ldots$$

So, there is the solution to the 1-D wave equation and with that we've solved the final partial differential equation in this chapter.

Summary of Separation of Variables

Throughout this chapter we've been talking about and solving partial differential equations using the method of separation of variables. However, the one thing that we've not really done is completely work an example from start to finish showing each and every step.

Each partial differential equation that we solved made use somewhere of the fact that we'd done at least part of the problem in another section and so it makes some sense to have a quick summary of the method here.

Also note that each of the partial differential equations only involved two variables. The method can often be extended out to more than two variables, but the work in those problems can be quite involved and so we didn't cover any of that here.

So with all of that out of the way here is a quick summary of the method of separation of variables for partial differential equations in two variables.

1. Verify that the partial differential equation is linear and homogeneous.

2. Verify that the boundary conditions are in proper form. Note that this will often depend on what is in the problem. So,
   a. If you have initial conditions verify that all the boundary conditions are linear and homogeneous.
   b. If there are no initial conditions (such as Laplace's equation) the verify that all but one of the boundary conditions are linear and homogeneous.
   c. In some cases (such as we saw with Laplace's equation on a disk) a boundary condition will take the form of requiring that the solution stay finite and in these cases we just need to make sure the boundary condition is met.

3. Assume that solutions will be a product of two functions each a function in only one of the variables in the problem. This is called a product solution.

4. Plug the product solution into the partial differential equation, separate variables and introduce a separation constant. This will produce two ordinary differential equations.

5. Plug the product solution into the homogeneous boundary conditions. Note that often it will be better to do this prior to doing the differential equation so we can use these to help us chose the separation constant.

6. One of the ordinary differential equations will be a boundary value problem. Solve this to determine the eigenvalues and eigenfunctions for the problem.

   Note that this is often very difficult to do and in some cases it will be impossible to completely find all eigenvalues and eigenfunctions for the problem. These cases can be dealt with to get at least an approximation of the solution, but that is beyond the scope of this quick introduction.

7. Solve the second ordinary differential equation using any remaining homogeneous boundary conditions to simplify the solution if possible.

8. Use the Principle of Superposition and the product solutions to write down a solution to the partial differential equation that will satisfy the partial differential equation and homogeneous boundary conditions.

9. Apply the remaining conditions (these may be initial condition(s) or a single nonhomogeneous boundary condition) and use the orthogonality of the eigenfunctions to find the coefficients.

Note that in all of our examples the eigenfunctions were sines and/or cosines however they won't always be sines and cosines. If the boundary value problem is sufficiently nice (and that's beyond the scope of this quick introduction to the method) we can always guarantee that the eigenfunctions will be orthogonal regardless of what they are.